Math 307, Section 103, Quiz 2

Name: Student ID#:

Problem 1. (6 points.) Lagrange Interpolation

Suppose that we would like to fit a degree 3 polynomial $p(x) = a_1x^3 + a_2x^2 + a_3x + a_4$ through the following data points. For each set of data points below, write down in *matrix form* the equations that a_1, a_2, a_3, a_4 must satisfy so that $p(x)$ passes through each data point. State whether each of these linear systems has (i) a unique solution, (ii) infinitely many solutions, or (iii) no solution at all (no need to explain your answer).

(a) $\{(0,1),(1,1),(2,2),(3,3)\}$

$p(0) = 1 \qquad a_4 = 1$

$p(1) = 1 \qquad a_1 + a_2 + a_3 + a_4 = 1$

$p(2) = 2 \qquad 2^3a_1 + 2^2a_2 + 2a_3 + a_4 = 2$

$p(3) = 3 \qquad 3^3a_1 + 3^2a_2 + 3a_3 + a_4 = 3$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 27 & 9 & 3 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

unique solution.

(b) $\{(0,1),(1,1),(2,2)\}$

$p(0) = 1$

$p(1) = 1$

$p(2) = 2$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

infinitely many solutions.

(c) $\{(0,1),(1,1),(2,2),(2,3)\}$

$p(0) = 1$

$p(1) = 1$

X $\left[\begin{array}{l} p(2) = 2 \\ p(2) = 3 \end{array}\right.$

$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 \\ 8 & 4 & 2 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

no solution!

Problem 2. (4 points) We are looking for an interpolation function $f(x)$ of the form

$$f(x) = \begin{cases} p_1(x) = a_1x^3 + b_1x^2 + c_1x + d_1, & 0 \le x \le 1, \\ p_2(x) = b_2x^2 + c_2x + d_2, & 1 \le x \le 2. \end{cases}$$

(a) Write down the equations that $a_1, b_1, b_2, c_1, c_2, d_1, d_2$ must satisfy to ensure that $f(x)$ satisfies the following three conditions.

- $f(x)$ is continuous and its graph passes through $(x_0, y_0) = (0, 2), (x_1, y_1) = (1, 4), (x_2, y_2) = (2, 1)$,
- $f'(x) = 0$ at the end points x_0 and x_2,
- $f'(x)$ is continuous at the midpoint x_1.

$p_1(0) = 2 \qquad d_1 = 2$

$p_1(1) = 4 \qquad a_1 + b_1 + c_1 + d_1 = 4$

$p_2(1) = 4 \qquad b_2 + c_2 + d_2 = 4$

$p_2(2) = 1 \qquad 4b_2 + 2c_2 + d_2 = 1$

$p_1'(x) = 3a_1x^2 + 2b_1x + c_1 \quad \Rightarrow \quad p_1'(0) = c_1 = 0$

$p_2'(x) = 2b_2x + c_2 \quad \Rightarrow \quad p_2'(2) = 4b_2 + c_2 = 0$

$p_1'(1) = p_2'(1) \quad \Rightarrow \quad 3a_1 + 2b_1 + c_1 = 2b_2 + c_2$

$3a_1 + 2b_1 + c_1 - 2b_2 - c_2 = 0$

(b) Let the vector $\mathbf{a}$ be defined as $\mathbf{a} := [a_1, b_1, c_1, d_1, b_2, c_2, d_2]^T$. Express the system of equations in part (a) as a matrix equation.

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 1 & 0 \\ 3 & 2 & 1 & 0 & -2 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ b_1 \\ c_1 \\ d_1 \\ b_2 \\ c_2 \\ d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$