Math 307 Quiz 1

Name: Student ID#:

Problem 1. (6 points.)

For each linear system below, determine whether it has (i) a unique solution, (ii) infinitely many solutions, or (iii) no solution et all. Explain your answers. **NO NEED TO SOLVE THE SYSTEMS.**

(a) $\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ $\det\begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = 2\cdot(-2)\cdot 3 = -12 \neq 0$

$\Rightarrow$ unique solution.

(b) $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}$ $\det\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 6 \\ 0 & 0 & 0 \end{bmatrix} = \cancel{0}1\cdot 1\cdot 0 = 0.$

Last equation is $0\cdot x_1 + 0 x_2 + 0 x_3 = 1$

$\Rightarrow$ No solution

(c) $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 0 \end{bmatrix}$ $\det\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = 1\cdot 1\cdot 0 = 0.$

~~Last~~ equation for free variable is $0\cdot x_3 = 0 \Rightarrow x_3 = t \in \mathbb{R}$ and we have infinitely many solutions.

Problem 2. (4 points) Compute the following:

(a) $\left\| \begin{bmatrix} 5 \\ 4 \\ -2 \\ 2 \end{bmatrix} \right\|_2 = \sqrt{25+16+4+4} = \sqrt{49} = 7.$

(b) $\left\| \begin{bmatrix} 5 \\ -4 \\ -2 \\ 2 \end{bmatrix} \right\|_1 = |5| + |-4| + |-2| + |2| = 13.$

(c) $\left\| \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & -3 \end{bmatrix} \right\|_{HS} = \sqrt{1+1+1+4+9} = \sqrt{16} = 4.$

Problem 3. (5 points.) Let $\mathbf{A}$ be an invertible matrix with $\mathbf{A}\begin{bmatrix}2\\0\\0\end{bmatrix} = \begin{bmatrix}-3\\0\\0\end{bmatrix}$ and $\mathbf{A}\begin{bmatrix}0\\3\\0\end{bmatrix} = \begin{bmatrix}0\\7\\0\end{bmatrix}$. Determine whether the following statements are true or false. Justify your answer.

(a) The operator norm of $\mathbf{A}$ is at least $\frac{7}{3}$. TRUE

$A \cdot \begin{bmatrix}0\\3\\0\end{bmatrix} = \begin{bmatrix}0\\7\\0\end{bmatrix}$. $\|A\|_{op} = \max_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}$. If $x = \begin{bmatrix}0\\3\\0\end{bmatrix}$, we have $\frac{\|Ax\|_2}{\|x\|_2} = \frac{7}{3}$. $\Rightarrow \|A\|_{op} \geq \frac{7}{3}$.

(b) The operator norm of $\mathbf{A}^{-1}$ is at least $\frac{2}{3}$. TRUE

$\begin{bmatrix}2\\0\\0\end{bmatrix} = A^{-1}\begin{bmatrix}-3\\0\\0\end{bmatrix}$. $\|A^{-1}\|_{op} = \max_{x \neq 0} \frac{\|A^{-1}x\|_2}{\|x\|_2}$. If $x = \begin{bmatrix}-3\\0\\0\end{bmatrix}$, we have $\frac{\|A^{-1}x\|_2}{\|x\|_2} = \frac{2}{3}$. $\Rightarrow \|A^{-1}\|_{op} \geq \frac{2}{3}$.

(c) The condition number of $\mathbf{A}$, denoted by $\text{cond}(\mathbf{A})$, is at most 3/2. FALSE

$\text{cond}(A) = \|A\|_{op}\|A^{-1}\|_{op} \geq \frac{7}{3} \cdot \frac{2}{3} = \frac{14}{9} > \frac{3}{2}$.

(d) The condition number of the matrix $\begin{bmatrix}-\frac{3}{2} & 0 & 0\\0 & \frac{7}{3} & 0\\0 & 0 & \frac{1}{3}\end{bmatrix}$ is 7. Let $A = \begin{bmatrix}-3/2 & & \\ & 7/3 & \\ & & 1/3\end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix}-\frac{2}{3} & & \\ & \frac{3}{7} & \\ & & 3\end{bmatrix}$. $\text{cond}(A) = \|A\|_{op}\|A^{-1}\|_{op} = \frac{7}{3} \cdot 3 = 7$.

$\Rightarrow$ TRUE.