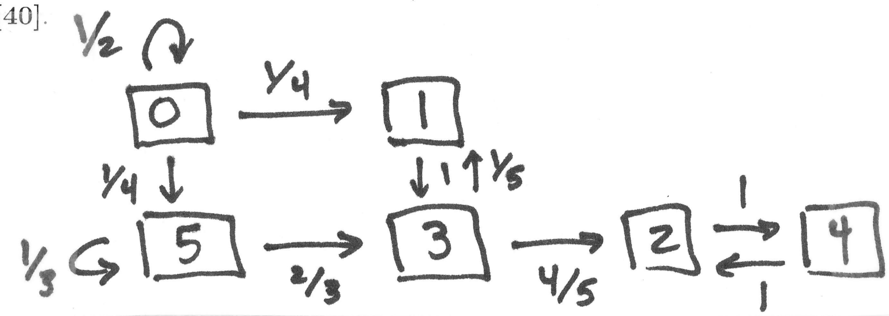
Math 303 Solutions to Test #1

February 8, 2017

Total marks = [40].

1. (a) .



[1]

   (b) The four classes are $\{0\}$, $\{1,3\}$, $\{2,4\}$, $\{5\}$. [3]

   (c) Transient: 0,1,3,5. Recurrent: 2,4. [2]

   (d) Aperiodic: 0,5. Period 2: 1,3; 2,4. [2]

   (e) Since 0 is a transient state, it will be visited only finitely often, so the long run proportion of time spent in state 0 is zero.
   Eventually the recurrent class $\{2,4\}$ will be entered, and then the states $2,4$ simply alternate, so the proportion of time spent in state 4 is $\frac{1}{2}$. [2]

2. (a) .

$\mathbf{P} =$

	C	A
C	0.9	0.1
A	1	0

[2]

   (b) $\pi = [\frac{10}{11}, \frac{1}{11}]$ [2]

   (c) $\pi_C = \frac{10}{11}$ [1]

   (d) $\pi_C P_{CA} = \frac{1}{11}$ [2]

   (e) $8\pi_C P_{CC} + 12(\pi_C P_{CA} + \pi_A P_{AC}) = 8 \times \frac{9}{11} + 12 \times (\frac{1}{11} + \frac{1}{11}) = \frac{96}{11}$ [3]

3. (a) Let $f_i = P(X_n = i \text{ for some } n \geq 1 | X_0 = i)$.
   Then i is recurrent if $f_i = 1$ and i is transient if $f_i < 1$. [2]

   (b) For $k \geq 1$, the event that $T = k$ occurs precisely when there are $k-1$ steps to the right followed by a step to 0. Thus $P(T = k) = p^{k-1}(1-p)$, for $k \geq 1$. [3]

   (c) The random variable T is geometric with parameter $1-p$, so it is finite with probability 1 and hence 0 is recurrent. Since $ET = \frac{1}{1-p} < \infty$, the state 0 is positive recurrent. [3]

   (d) There are exactly two trajectories that go from 2 to 2 in six steps, namely 2012012 and 2345012. The first has probability $(1-p)pp(1-p)pp$ and the second has probability $ppp(1-p)pp$, so the total probability is the sum, namely $(1-p)p^4$.
   CORRECTION: The above is wrong, as there are multiple additional trajectories, such as 2000012, 2010012, etc. This makes the problem much more complicated than anticipated, and it is not a good question for the test. Because of this, full marks will be awarded for a partial solution. [2]

4. (a) The state changes from i to $i-1$ when a white ball is chosen and replaced by a black ball. This happens with probability $P_{i,i-1} = \frac{i}{100}(1-p)$. [1]
   The state changes from i to $i+1$ when a black ball is chosen from the urn and replaced by a white ball. This happens with probability $P_{i,i+1} = \frac{100-i}{100}p$. [1]
   The only other possibility is a transition from state i to state i, which happens with probability $P_{i,i} = 1 - P_{i,i-1} - P_{i,i+1} = \frac{i}{100}p + \frac{100-i}{100}(1-p)$. [1]

(b) Guess $\pi \sim \mathrm{Bin}(100, p)$, i.e., [3]

$$\pi_i = \binom{100}{i} p^i (1-p)^{100-i} \quad (1 = 0, 1, \ldots, 100).$$

To verify the guess, we check that $\pi_i P_{ij} = \pi_j P_{ji}$ and it suffices to do this for $j = i+1$. This equation becomes

$$\binom{100}{i} p^i (1-p)^{100-i} \frac{100-i}{100} p = \binom{100}{i+1} p^{i+1} (1-p)^{100-i-1} \frac{i+1}{100} (1-p),$$

which holds if

$$\binom{100}{i} (100-i) = \binom{100}{i+1} (i+1),$$

and this indeed holds. [2]

(c) $\frac{1}{\pi_0} = (1-p)^{-100}$ [2]