MATH 303 Test 1

Wednesday, February 8, 2017

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Please show all work and calculations. Calculators or other aids are not permitted. Each question is worth 10 marks, for a total of 40.

1. Consider the Markov chain with state space $\{0, 1, 2, 3, 4, 5\}$ and transition matrix

$$\mathbf{P} = \begin{array}{c|cccccc|} & 0 & 1 & 2 & 3 & 4 & 5 \\ \hline 0 & \frac{1}{2} & \frac{1}{4} & 0 & 0 & 0 & \frac{1}{4} \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 & 0 \\ 3 & 0 & \frac{1}{5} & \frac{4}{5} & 0 & 0 & 0 \\ 4 & 0 & 0 & 1 & 0 & 0 & 0 \\ 5 & 0 & 0 & 0 & \frac{2}{3} & 0 & \frac{1}{3} \end{array}$$

   (a) (1 marks) Draw the transition diagram showing the six states with arrows indicating possible transitions and their probabilities.

   (b) (3 marks) Determine all the communicating classes of this Markov chain.

   (c) (2 marks) Determine which states are recurrent and which are transient.

   (d) (2 marks) Determine the period of each state.

   (e) (2 marks) In the long run, what fraction of time does the Markov chain spend in state 0? What fraction does it spend in state 4? (The answers can be determined by inspection, calculation is not required.)

2. Smith drives a taxi that serves the city and the airport. A trip that originates in the city has a destination in the city with probability 0.9, and has the airport as destination with probability 0.1. A trip that originates in the airport always goes to the city.

   (a) (2 marks) Let X_n denote Smith's location (city or airport) after his n^{th} trip. This defines a Markov chain. What is its transition matrix?

   (b) (2 marks) Determine the stationary distribution of the Markov chain.

   (c) (1 mark) What fraction of trips originate in the city, in the long run?

   (d) (2 marks) What fraction of all trips are trips from the city to the airport?

   (e) (3 marks) Smith makes an average profit of \$8 for trips that remain in the city, and an average profit of \$12 for trips that involve the airport. What is his overall average profit per trip?

3. Consider the Markov chain on the non-negative integers $\{0, 1, 2, 3, \ldots\}$ with transition probabilities $P_{n,n+1} = p$ and $P_{n,0} = 1 - p$, for $n \geq 0$. Here $0 < p < 1$.

   (a) (2 marks) In general, for a Markov chain, what exactly does it mean to say that a state i is a recurrent state? What exactly does it mean to say that a state i is a transient state?

   (b) (3 marks) Suppose that the Markov chain is initially in state 0. Let T be the number of steps until the first return to state 0. Determine the probability mass function of T.

   (c) (3 marks) Is 0 a transient state, a positive recurrent state, or a null recurrent state? Explain your reasoning.

   (d) (2 marks) Determine the 6-step transition probability $P^6_{2,2}$.

4. One hundred balls, some of them black and some of them white, are in an urn. At each time step, a ball is chosen from the urn uniformly at random, and is replaced by a white ball with probability p and by a black ball with probability $1 - p$ (independently for each time). Here $0 < p < 1$. Let X_n denote the number of white balls after the n^{th} replacement.

   (a) (3 marks) Find the transition probabilities for this Markov chain.

   (b) (5 marks) Using any method, determine the stationary distribution of the Markov chain.

   (c) (2 marks) Suppose that there are initially only black balls in the urn. How long will it take, on average, until there are again only black balls in the urn?