RECURRENCE RELATIONS

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1. Homogeneous Linear Recurrence Relations

A homogeneous linear recurrence relation has the form

$$f_{n+1} = a_0 f_n + a_1 f_{n-1} + \cdots + a_k f_{n-k},$$

where $a_0, \ldots, a_k$ are constants. The aim is to find a closed-form formula for f_n.

Problem 1. Consider the relation $a_{n+1} = 2a_n$, $a_0 = 1$. What is a_n?

Problem 2 (The Fibonacci Sequence)**.** The Fibonacci sequence is given by

$$f_0 = 0, \quad f_1 = 1, \quad f_{n+1} = f_n + f_{n-1}, \forall n \geq 1.$$

What is f_{10}? How about f_{2020}? Find a closed-form formula for f_n.

Problem 3. Let $a_{n+1} = 5a_n - 6a_{n-1}$, $a_0 = 1, a_1 = 2$. Find a closed-form formula for a_n.

Problem 4. Let $a_{n+1} = 4a_n - 4a_{n-1}$, $a_0 = 1, a_1 = 2$. Find a closed-form formula for a_n.

Problem 5. Let $a_{n+1} = 2a_n - 2a_{n-1}$, $a_0 = 1, a_1 = 2$. Find a closed-form formula for a_n.

Problem 6. Let $a_{n+1} = 4a_n - a_{n-1} - 6a_{n-2}$, $a_0 = 1, a_1 = 2, a_2 = 3$. Find a closed-form formula for a_n.

Problem 7 (The Gambler's Ruin Problem)**.** Smith has \$$n$ at the beginning of the day, and starts playing the following gambling game. At each step he tosses a coin, which comes up Heads with probability $\frac{1}{2}$, and Tails with probability $\frac{1}{2}$. If the coin comes up Heads, Smith gains \$1, and if it comes up Tails, he loses \$1. The game ends if either Smith has a total of \$$N$, where $N > n$, or if he has no money left. Find the probability q_n of Smith winning (i.e. having \$$N$) if he starts the day with \$$n$.

Date: January 23, 2020.

2. Non-homogeneous Linear Recurrence Relations

A non-homogeneous linear recurrence relation has the form

$$f_{n+1} = a_0 f_n + a_1 f_{n-1} + \cdots + a_k f_{n-k} + g(n),$$

where $a_0, \ldots, a_k$ are constants, and $g(n)$ is a function that depends on n. The aim, again, is to find a closed-form formula for the n-th term f_n.

The general algorithm for solving such a relation is to first find a *particular solution*, x_n. Then, the sequence $(f_n - x_n)$ satisfies the homogeneous recurrence relation:

$$(f_{n+1} - x_{n+1}) = a_0(f_n - x_n) + a_1(f_{n-1} - x_{n-1}) + \cdots + a_k(f_{n-k} - x_{n-k}),$$

and, therefore, we can solve it using the tools we learned above.

Problem 8. Solve the recurrence relation

$$a_{n+1} = 3a_n + 1, \quad a_0 = 0.$$

Problem 9. Find all solutions to the recurrence relation

$$a_{n+1} = 3a_n + 4a_{n-1} + 3.$$

Problem 10 (The Towers of Hanoi)**.** Suppose we have 3 pegs, and there are n disks of increasing size on one of the pegs. The goal is to move all n disks to one of the other 2 pegs. We are only allowed to move one disk at a time, and cannot put a larger disk on top of a smaller one. Let H_n be the number of moves it takes to move the n disks. Show that H_n satisfies the recurrence relation

$$H_n = 2H_{n-1} + 1, \quad H_0 = 0,$$

and then solve this relation.

Problem 11 (The Binary Search Algorithm)**.** Suppose we are given n ordered real numbers $a_1 < a_2 < \cdots < a_n$, and another real number b. How many times do we have to check whether $b_j < a_j$ for some j in order to find the unique $i \in \{0, 1, \ldots, n\}$ so that $a_i \leq b < a_{i+1}$? *It might be easier to assume that n is a power of 2.*

Problem 12. Find all solutions to the recurrence relation

$$a_{n+1} = 2a_n + n, \quad a_0 = 0.$$

Now, try solving all of the problems above using the method of generating functions!

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1. $a_{n+1} = 2a_n, \quad a_0 = 1.$

$1, 2, 4, 8, 16, \ldots$

$a_n = 2^n$

In general, if

$$a_{n+1} = c \cdot a_n$$

then

$$a_n = c \cdot a_{n-1}$$
$$= c \cdot (c \cdot a_{n-2}) = c^2 a_{n-2}$$
$$= c^3 a_{n-3}$$
$$\vdots$$
$$= c^n \cdot a_0$$

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2. $f_0 = 0, \; f_1 = 1 \quad (**)$

$f_{n+1} = f_n + f_{n-1} \quad (*)$

$0, 1, 1, 2, 3, 5, 8, 13, 21, \ldots$

Characteristic equation:

plug in x^n for f_n in the recurrence:

$$x^{n+1} = x^n + x^{n-1}$$
$$x^{n+1} - x^n - x^{n-1} = 0$$
$$\cancel{x^{n-1}}(x^2 - x - 1) = 0$$

$$x^2 - x - 1 = 0$$
$$x_{1,2} = \frac{1 \pm \sqrt{5}}{2}$$

$f_n = x_1^n$, $f_n = x_2^n$ are solutions to $(*)$

Then,

$$f_n = \alpha \cdot x_1^n + \beta \cdot x_2^n$$

for some $\alpha, \beta \in \mathbb{R}$.

Plug in $f_0 = 0$ and $f_1 = 1$:

$$\begin{cases} f_0 = \alpha + \beta = 0 \\ f_1 = \alpha x_1 + \beta x_2 = 1 \end{cases}$$

$$\begin{cases} \beta = 1 - \alpha \\ \alpha x_1 + (1-\alpha) x_2 = 1 \end{cases}$$

$$\begin{cases} \beta = 1 - \alpha \\ \alpha (x_1 - x_2) = 1 - x_2 \end{cases}$$

$$\begin{cases} \beta = 1 - \alpha \\ \alpha \cancel{\sqrt{5}} \underbrace{\left(\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2} \right)}_{\sqrt{5}} = \underbrace{1 - \frac{1-\sqrt{5}}{2}}_{\frac{1+\sqrt{5}}{2}} \end{cases}$$

$$\alpha = \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{1}{\sqrt{5}} x_1$$

$$\beta = 1 - \frac{1+\sqrt{5}}{2\sqrt{5}} = \frac{\sqrt{5}-1}{2\sqrt{5}} = -\frac{x_2}{\sqrt{5}}$$

$$\Rightarrow f_n = \frac{x_1}{\sqrt{5}} x_1^n - \frac{x_2}{\sqrt{5}} x_2^n$$

$$= \frac{x_1^{n+1}}{\sqrt{5}} - \frac{x_2^{n+1}}{\sqrt{5}},$$

where $x_1 = \frac{1+\sqrt{5}}{2}$, $x_2 = \frac{1-\sqrt{5}}{2}$

$$\boxed{f_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.}$$

$$f_{n+1} = f_n + f_{n-1}$$

Set $v_n = \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix}$. Then,

$$v_{n+1} = \begin{bmatrix} f_{n+1} \\ f_n \end{bmatrix} = \begin{bmatrix} f_n + f_{n-1} \\ f_n \end{bmatrix}$$

$$v_{n+1} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_n \\ f_{n-1} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} v_n$$

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$$\Rightarrow v_{n+1} = \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}}_{A} v_n$$

$$v_n = A^{n-1} v_1$$

$$A = S \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} S^{-1}$$

diagonalization

$$A^{n-1} = S \begin{bmatrix} \lambda_1^{n-1} & \\ & \lambda_2^{n-1} \end{bmatrix} S^{-1}$$

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3.

$$a_{n+1} = 5a_n - 6a_{n-1}$$

$$a_0 = 1, \quad a_1 = 2$$

$$a_n = x^n$$

$$x^{n+1} = 5x^n - 6x^{n-1}$$

$$\cancel{x^{n-1}} \underbrace{(x^2 - 5x + 6)}_{} = 0$$

$$(x-2)(x-3) = 0$$

$$x_1 = 2, \quad x_2 = 3$$

$$\Rightarrow a_n = \alpha \cdot 2^n + \beta \cdot 3^n$$

$$a_0 = 1 = \alpha + \beta \qquad \alpha = 1$$

$$a_1 = 2 = 2\alpha + 3\beta \qquad \beta = 0$$

$$\Rightarrow \boxed{a_n = 2^n}$$

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4. $a_{n+1} = 4a_n - 4a_{n-1}$

$a_0 = 1, \quad a_1 = 2$

Characteristic equation:

$x^{n+1} - 4x^n + 4x^{n-1} = 0$

$x^2 - 4x + 4 = 0$

$(x-2)^2 = 0$

$x_1 = x_2 = 2$ repeated root!

Then, $\boxed{a_n = \alpha \cdot x_1^n + \beta \cdot n \cdot x_1^n}$

$= \alpha \cdot 2^n + \beta \cdot n \cdot 2^n$

<u>Check:</u> 2^n and $n \cdot 2^n$ are both solutions to the recurrence relation.

Plug in a_0 and a_1:

$$\begin{cases} a_0 = 1 = \alpha \\ a_1 = 2 = \alpha \cdot 2 + \beta \cdot 1 \cdot 2^1 \end{cases}$$

$\Rightarrow \alpha = 1, \beta = 0; \Rightarrow \boxed{a_n = 2^n}$

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$a_{n+1} = 4a_n - 4a_{n-1}$

$a_0 = 1, \quad a_1 = 3$

$a_n = \alpha \cdot 2^n + \beta \cdot n \cdot 2^n$

$$\begin{cases} a_0 = \alpha = 1 \\ a_1 = \alpha \cdot 2 + \beta \cdot 1 \cdot 2 = 3 \end{cases}$$

$$\begin{cases} \alpha = 1 \\ 2\beta = 1 \end{cases} \qquad \begin{cases} \alpha = 1 \\ \beta = \frac{1}{2} \end{cases}$$

$\Rightarrow \boxed{a_n = 2^n + \frac{1}{2} \cdot n \cdot 2^n.}$

5. $a_{n+1} = 2a_n - 2a_{n-1}$

$a_0 = 1, \; a_1 = 2$

Set $a_n = x^n$ and plug in:

$$x^{n+1} = 2x^n - 2x^{n-1}$$

$$x^{n-1}\underbrace{(x^2 - 2x + 2)} = 0$$

$b^2 - 4ac < 0$

$$x_{1,2} = 1 \pm \underbrace{\sqrt{-1}}_{i}$$

$\Rightarrow$ no real solutions

$i := \sqrt{-1}$ has the property:

$i^2 = -1$

Complex numbers $\mathbb{C}$

$$\mathbb{C} = \{a + ib : a, b \in \mathbb{R}\}$$

$$x_1 = 1 + i, \quad x_2 = 1 - i.$$

$$x_{1,2} = re^{\pm i\theta}$$

$$\Rightarrow a_n = \alpha \cdot x_1^n + \beta \cdot x_2^n$$

Plug in: $\left| \begin{array}{l} a_0 = 1 = \alpha + \beta \\ a_2 = 2 = \alpha x_1 + \beta x_2 \end{array} \right.$

$$\left| \begin{array}{l} \beta = 1 - \alpha \\ \alpha(1+i) + (1-\alpha)(1-i) = 2 \end{array} \right.$$

$$\alpha(i + i + 1) + 1 - i = 2$$

$$\alpha(2i + 1) = i + 1$$

$$\alpha = \frac{i+1}{2i+1}$$

$$\beta = 1 - \alpha = \frac{i}{2i+1}.$$

$$a_n = \frac{i+1}{2i+1}(1+i)^n + \frac{i}{2i+1}(1-i)^n.$$

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Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Every complex number can be written as

$$\underset{\mathbb{R}}{r}\cdot e^{i\overset{\mathbb{R}}{\theta}} = r(\cos\theta + i\sin\theta)$$

$$x_1 = 1+i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{2}\, e^{i\frac{\pi}{4}}$$

$$x_2 = 1-i = \sqrt{2}\left(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}\right)$$

$$= \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$$

$$= \sqrt{2}\, e^{i\left(-\frac{\pi}{4}\right)}$$

$$a_n = \alpha x_1^n + \beta x_2^n$$

$$x_1^n = \left(\sqrt{2}\, e^{i\frac{\pi}{4}}\right)^n = \sqrt{2}^n e^{in\frac{\pi}{4}}$$

$$x_1^n = \sqrt{2}^n \underbrace{\left(\cos\left(n\frac{\pi}{4}\right) + i\sin\left(n\frac{\pi}{4}\right)\right)}_{(*)}$$

Similarly,

$$x_2^n = \sqrt{2}^n \underbrace{\left(\cos\left(n\frac{\pi}{4}\right) - i\sin\left(n\frac{\pi}{4}\right)\right)}_{(**)}$$

$$a_n = \alpha(*) + \beta(**)$$

$a_n = \dfrac{x_1^n + x_2^n}{2}$ is a solution

$a_n = \dfrac{x_1^n - x_2^n}{2i}$ is a solution

$$\frac{x_1^n + x_2^n}{2} = \sqrt{2}^n \cos\left(n\frac{\pi}{4}\right)$$

$$\frac{x_1^n - x_2^n}{2i} = \sqrt{2}^n \sin\left(n\frac{\pi}{4}\right)$$

General sol:

$$a_n = \gamma\cdot\sqrt{2}^n\cos\left(n\frac{\pi}{4}\right) + \delta\sqrt{2}^n\sin\left(n\frac{\pi}{4}\right)$$

$$a_n = \gamma\cdot r^n\cos(n\theta) + \delta r^n\sin(n\theta)$$

Plug in (13)

$$\begin{cases} a_0 = 1 = \gamma \\ a_1 = 2 = \gamma\cdot\sqrt{2}\underbrace{\cos\frac{\pi}{4}}_{\frac{1}{\sqrt{2}}} + \delta\sqrt{2}\sin\frac{\pi}{4} \end{cases}$$

$$\begin{cases} \gamma = 1 \\ 2 = \gamma + \delta \end{cases} \Rightarrow \delta = 1$$

$$\boxed{a_n = \sqrt{2}^n\cos\left(n\frac{\pi}{4}\right) + \sqrt{2}^n\sin\left(n\frac{\pi}{4}\right)}$$

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6. (14)

$$a_{n+1} = 4a_n - a_{n-1} - 6a_{n-2}$$

$$a_0 = 1,\ a_1 = 2,\ a_2 = 3$$

$$a_n = x^n$$

$$x^{n+1} = 4x^n - x^{n-1} - 6x^{n-2}$$

$$x^{n-2}\underbrace{(x^3 - 4x^2 + x + 6)}_{\text{characteristic equation.}} = 0$$

$$(x+1)(x-2)(x-3) = 0$$

$x_1 = -1$ $\quad (-1)^n$

$x_2 = 2$ $\quad 2^n$

$x_3 = 3$ $\quad 3^n$

$$\boxed{a_n = \alpha(-1)^n + \beta\cdot 2^n + \gamma 3^n}$$

$a_0 = 1 = \alpha + \beta + \gamma$

$a_1 = 2 = -\alpha + 2\beta + 3\gamma$

$a_2 = 3 = \alpha + 4\beta + 9\gamma$

$$\begin{cases} \alpha + \beta + \gamma = 1 \\ -\alpha + 2\beta + 3\gamma = 2 \\ \alpha + 4\beta + 9\gamma = 3 \end{cases}$$

$$\begin{cases} 3\beta + 4\gamma = 3 \\ 6\beta + 12\gamma = 5 \\ \alpha = 1 - \beta - \gamma \end{cases} \qquad \begin{cases} 4\gamma = -1 \\ \gamma = -\frac{1}{4} \\ 3\beta - 1 = 3 \\ \beta = \frac{4}{3} \end{cases}$$

$$\alpha = 1 - \frac{4}{3} + \frac{1}{4} = -\frac{1}{3} + \frac{1}{4} = -\frac{1}{12}$$

$$\boxed{a_n = -\frac{1}{12}(-1)^n + \frac{4}{3}\cdot 2^n - \frac{1}{4}3^n}$$

7. The Gambler's Ruin Problem

q_n = probability Smith wins the game if he starts with \$n.

$p \neq \frac{1}{2}$

$$q_0 = 0, \quad q_N = 1.$$

$1 \le n \le N$

$$q_n = \frac{1}{2} q_{n-1} + \frac{1}{2} q_{n+1}$$

$$x^n = \frac{1}{2} x^{n-1} + \frac{1}{2} x^{n+1}$$

$$\frac{1}{2} x^{n-1} \underbrace{(x^2 - 2x + 1)} = 0$$

$$(x-1)^2 = 0$$

$$x_1 = x_2 = 1$$

$$q_n = \alpha \cdot 1^n + \beta \cdot n \cdot 1^n$$

$$\Rightarrow \boxed{q_n = \alpha + n\beta}$$

$$q_0 = 0 = \alpha \qquad \alpha = 0$$

$$q_N = 1 = \alpha + N\beta \qquad \beta = \frac{1}{N}$$

$$\Rightarrow \boxed{q_n = \frac{n}{N}}$$

8. $a_{n+1} = 3a_n + 1$

$$a_0 = 0$$

Guess $a_n = c$

$$c = 3c + 1$$

$$c = -\frac{1}{2}$$

$$\cancel{a_{n+1} - \frac{1}{2} = 3\left(a_n - \frac{1}{2}\right)}$$

Set $b_n = a_n - \left(-\frac{1}{2}\right) = a_n + \frac{1}{2}$

$$b_{n+1} = a_{n+1} + \frac{1}{2} = 3a_n + 1 + \frac{1}{2}$$

$$= 3\left(a_n + \frac{1}{2}\right)$$

$$= 3 b_n$$

$$\Rightarrow \boxed{b_{n+1} = 3b_n}, \quad b_0 = a_0 + \frac{1}{2} = \frac{1}{2}$$

$$\boxed{b_n = \frac{1}{2} \cdot 3^n} \Rightarrow \boxed{a_n = \frac{1}{2} 3^n - \frac{1}{2}}.$$

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9. $a_{n+1} = 3a_n + 4a_{n-1} + 3$

set $a_n = c$

$$c = 3c + 4c + 3$$

$$c = -\frac{1}{2}$$

Set $b_n = a_n - c = a_n + \frac{1}{2}$

Then:

$$b_{n+1} = 3b_n + 4b_{n-1}$$

Characteristic equation:

$$x^{n-1}(x^2 - 3x - 4) = 0$$

$$x_1 = 4, \quad x_2 = -1$$

$$b_n = \alpha \cdot 4^n + \beta \cdot (-1)^n$$

$$\Rightarrow a_n = b_n - \frac{1}{2} = \boxed{\alpha \cdot 4^n + \beta \cdot (-1)^n - \frac{1}{2}}$$

for all $\alpha, \beta \in \mathbb{R}$.

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12.

$$a_{n+1} = 2a_n + n, \quad a_0 = 0.$$

$$a_{n+1} + n + 1 = 2a_n + n + n + 1$$

$$\underbrace{(a_{n+1} + n + 1)}_{b_{n+1}} = 2\underbrace{(a_n + n)}_{b_n} + 1$$

$$b_{n+1} = 2b_n + 1$$

Then solve via Problem 10.

<u>Method of generating functions</u>

$$a_{n+1} = 2a_n + n, \quad a_0 = 0$$

Define $f(x) = \sum_{n=0}^{\infty} a_n x^n$

Then, find $f(x)$.

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$$f(x) = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots$$

$$\frac{f(x) - a_0}{x} = \sum_{n=0}^{\infty} a_{n+1} x^n$$

$$= \sum_{n=0}^{\infty} (2a_n + n) x^n$$

$$= 2 \underbrace{\sum_{n=0}^{\infty} a_n x^n}_{f(x)} + \sum_{n=0}^{\infty} n x^n$$

$$\frac{f(x) - a_0}{x} = 2f(x) + \underbrace{\sum_{n=0}^{\infty} n x^n}_{= S}$$

$$S - xS = \sum_{n=0}^{\infty} n x^n - \sum_{n=1}^{\infty} (n-1) x^n$$

$$= \sum_{n=1}^{\infty} x^n = \frac{1}{1-x} - 1$$

$$= \frac{x}{1-x}$$

$$S = \frac{x}{(1-x)^2}$$

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$$S = \sum_{n=0}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1}$$

$$= x \left(\sum_{n=0}^{\infty} x^n \right)'$$

$$= x \left(\frac{1}{1-x} \right)' = \frac{x}{(1-x)^2}$$

$$\frac{f(x) - \overset{=0}{a_0}}{x} = 2f(x) + \frac{x}{(1-x)^2}$$

$$(1-2x) f(x) = \frac{x^2}{(1-x)^2}$$

$$\Rightarrow f(x) = x^2 \underbrace{\frac{1}{(1-2x)(1-x)^2}}_{\frac{A}{1-2x} + \frac{B}{1-x} + \frac{C}{(1-x)^2}}$$

$$f(x) = \cancel{x} \left(-\frac{1}{(1-x)^2} + \frac{2}{1-2x} \right)$$

$$= \cancel{x} \left(-\underbrace{(1+x+x^2+\cdots)^2}_{(1+x+x^2+\cdots)(1+x+x^2+\cdots)} + 2\underbrace{(1 + 2x + (2x)^2 + \cdots)}_{2^n x^n} \right)$$

$$f(x) = \cancel{}\left(-\sum_{n=0}^{\infty}(n+1)x^n + 2\sum_{n=0}^{\infty}2^n x^n\right)$$

$$= \sum_{n=0}^{\infty}(2^{n+1}-n-1)x^n$$

$$\Rightarrow \boxed{a_n = 2^{n+1}-n-1}.$$

10. $H_n = 2H_{n-1}+1, \quad H_0=0.$

$$f(x) = \sum_{n=0}^{\infty} H_n x^n$$

$$\frac{f(x)-\overset{=0}{H_0}}{x} = \sum_{n=0}^{\infty} H_{n+1}x^n$$

$$= \sum_{n=0}^{\infty}(2H_n+1)x^n$$

$$= 2\left(\sum_{n=0}^{\infty} H_n x^n\right) + \sum_{n=0}^{\infty} x^n$$

$$= 2f(x) + \frac{1}{1-x}$$

$$\frac{f(x)}{x} = 2f(x) + \frac{1}{1-x}$$

$$f(x)(1-2x) = \frac{x}{1-x}$$

$$f(x) = \frac{x}{(1-x)(1-2x)}$$

$$= \frac{A}{1-x} + \frac{B}{1-2x} =$$

$$\vdots$$

$$= \frac{2x}{1-2x} - \frac{x}{1-x}$$

$$= 2x\left(1+(2x)+(2x)^2+\dots\right)$$

$$-x\left(1+x+x^2+\dots\right)$$

$$= \sum_{n=1}^{\infty}(2^n-1)x^n$$

$$\boxed{a_n = 2^n-1}.$$

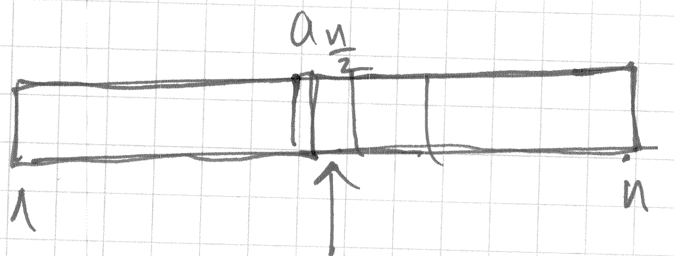
11.

$a_1 < a_2 < \dots < a_n$

b

Find $\quad a_i \leq b < a_{i+1}$

$a_j \leq b$



$$f_n = f_{\frac{n}{2}} + 1$$

$$= 1 + 1 + f_{\lceil \frac{n}{4} \rceil}$$

$$\vdots$$

$$= \lceil \log_2 n \rceil.$$