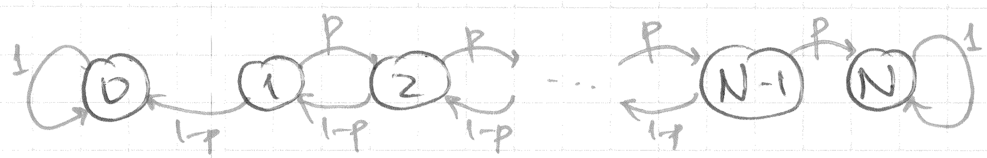
Last time: Gambler's ruin problem:



Q: What is $P(\text{lose} \mid X_0 = n) =: \bar{p}(n)$

We found a recurrence relation:

$$\boxed{\begin{aligned} &\bar{p}(n) = \bar{p}(n+1)\cdot p + \bar{p}(n-1)\cdot(1-p) \\ &\left.\begin{aligned}\bar{p}(0) &= 1 \\ \bar{p}(N) &= 0\end{aligned}\right\} \text{ initial/boundary conditions.}\end{aligned}}$$

Relation: $\bar{p}(n+1) = \frac{1}{p}\bar{p}(n) - \left(\frac{1-p}{p}\right)\bar{p}(n-1)$

Characteristic equation:

$$x^2 - \frac{1}{p}x + \frac{1-p}{p} = 0$$

Solutions: $y_1 = 1$, $y_2 = \frac{1-p}{p}$.

Case 1: $y_1 \neq y_2$, i.e. $\frac{1-p}{p} \neq 1$, i.e. $p \neq \frac{1}{2}$

Then, $\boxed{\bar{p}(n) = a\cdot 1^n + b\cdot\left(\frac{1-p}{p}\right)^n}$

Recurrence relations:

$$X_{n+1} = \alpha X_n + \beta X_{n-1}$$

To solve, form characteristic equation:

$$x^2 - \alpha x - \beta = 0$$

Solutions y_1 and y_2.

Case 1: $y_1 \neq y_2$

Then $X_n = a\cdot y_1^n + b\cdot y_2^n$ for some $a, b \in \mathbb{R}$

Case 2: $y_1 = y_2$

Then, $X_n = a\cdot y_1^n + b n y_1^n$. for some $a, b \in \mathbb{R}$

→ Plug in initial conditions:

$$\left.\begin{aligned}\bar{p}(0) &= 1 \\ \bar{p}(N) &= 0\end{aligned}\right\} \Leftrightarrow \left\{\begin{aligned} &a + b = 1 \\ &a + b\left(\frac{1-p}{p}\right)^N = 0\end{aligned}\right.$$

$$\left.\begin{aligned} &b = 1 - a \\ &a + (1-a)\left(\frac{1-p}{p}\right)^N = 0\end{aligned}\right\} \qquad a = \frac{\left(\frac{1-p}{p}\right)^N}{\left(\frac{1-p}{p}\right)^N - 1} \qquad b = \frac{-1}{\left(\frac{1-p}{p}\right)^N - 1}$$

$$\Rightarrow \bar{p}(n) = \frac{\left(\frac{1-p}{p}\right)^N}{\left(\frac{1-p}{p}\right)^N - 1} - \frac{\left(\frac{1-p}{p}\right)^n}{\left(\frac{1-p}{p}\right)^N - 1}$$ (3)

$$\boxed{\bar{p}(n) = \frac{\left(\frac{1-p}{p}\right)^N - \left(\frac{1-p}{p}\right)^n}{\left(\frac{1-p}{p}\right)^N - 1}}$$

Case 2: $y_1 = y_2 = 1$, i.e., $p = \frac{1}{2}$

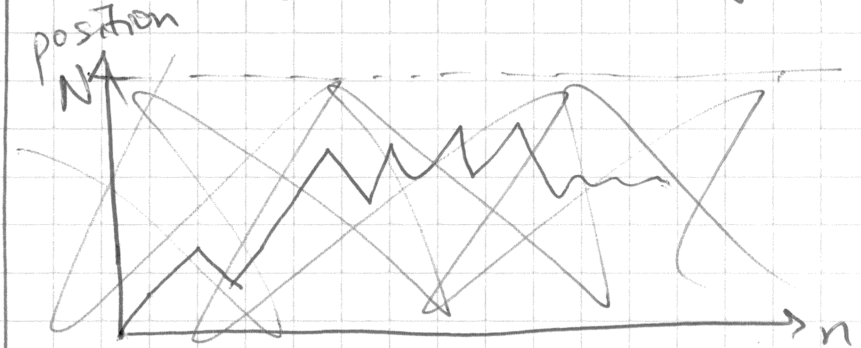
$$\bar{p}(n) = a \cdot 1^n + b \cdot n \cdot 1^n$$
$$= a + bn$$

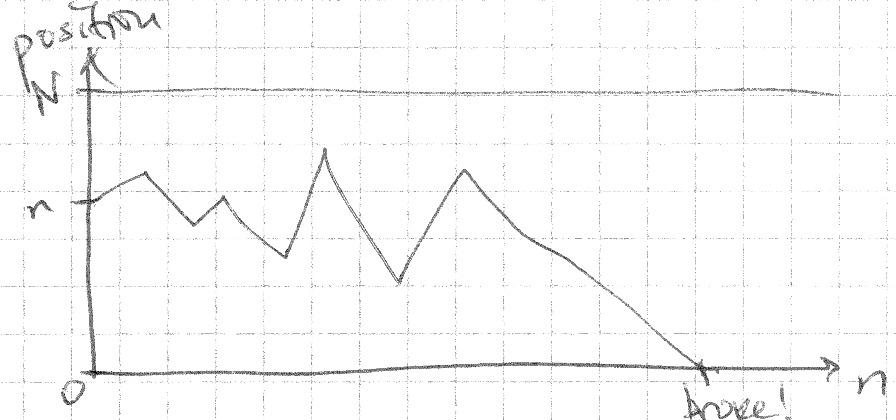
Plugging in the initial conditions:

$$\bar{p}(0) = 1 \quad \Leftrightarrow \quad a = 1$$
$$\bar{p}(N) = 0 \qquad\quad a + bN = 0$$

$$\begin{cases} a = 1 \\ b = -\frac{1}{N} \end{cases} \Rightarrow \boxed{\bar{p}(n) = 1 - \frac{n}{N}}.$$

Remark: Can think of Gambler's ruin problem as a 1-D R.W. s.t. we want to find P(hit 0 before we hit N) (4)

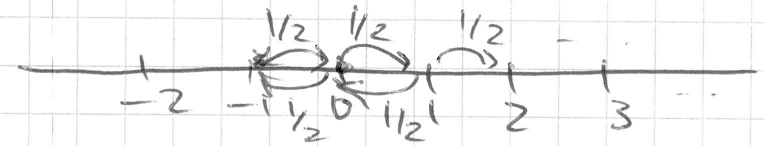




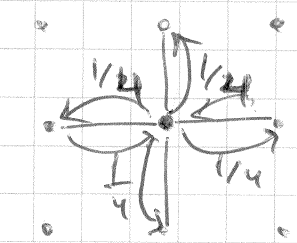
(B). Random walk in $\mathbb{Z}^d$ (5)

We consider the uniform r.w. in $\mathbb{Z}^d$

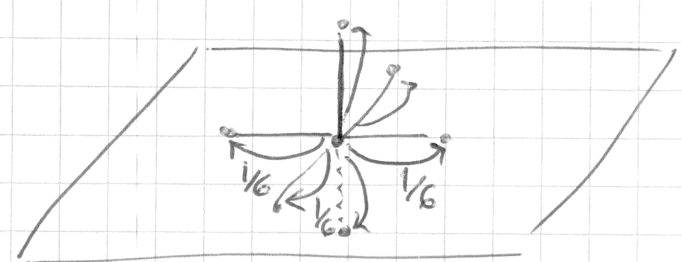
d=1:



d=2:



d=3:



Q: Is the process recurrent or transient?

~~def~~ We will study $\boxed{\sum_{n=0}^{\infty} p_{00}^{(n)} \overset{?}{=} \infty}$

(6)

d=1: Starting from 0, we can only come back in an even number of steps.

$p_{00}^{(2n)}$ = prob of moving n times forward and n times backward.

$$= \left(\frac{1}{2}\right)^{2n} \binom{2n}{n}$$

$\underbrace{FF\cdots F}_{n}\ \underbrace{BB\cdots B}_{n}$ BFBF ⋯ BF

$B\cdots B\ F\cdots F$

$$\Rightarrow \sum_{n=0}^{\infty} p_{00}^{(2n)} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{2n} \binom{2n}{n}$$

$$= \sum_{n=0}^{\infty} \frac{(2n)!}{(n!)^2\, 2^{2n}}$$

Stirling's approximation:

$$n! \sim \left(\frac{n}{e}\right)^n \sqrt{2\pi n} \qquad n^{n+\frac{1}{2}}$$

$$\frac{(2n)!}{(n!)^2\, 2^{2n}} \approx \cancel{\left(\frac{2n}{e}\right)^{2n}} \sqrt{2\pi \cdot 2n} \cdot \frac{1}{\cancel{\left(\frac{n}{e}\right)^{2n}} (2\pi n)\, \cancel{2^{2n}}}$$

$$= \frac{1}{\sqrt{\cdots}}$$

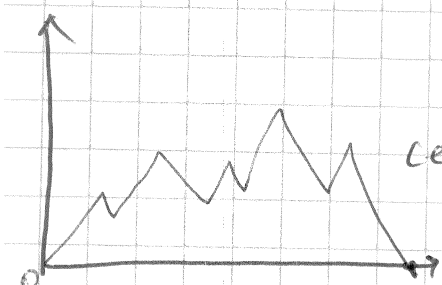
(7)

$$\sum_{n=0}^{\infty} p_{00}^{(2n)} \approx \sum_{n=0}^{\infty} \frac{1}{\sqrt{\pi n}}$$

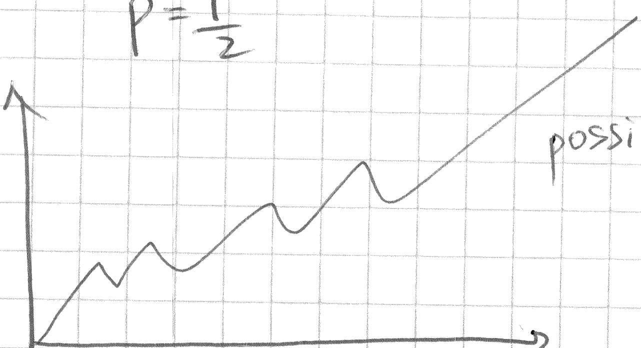
$$= \frac{1}{\sqrt{\pi}} \sum_{n=0}^{\infty} \frac{1}{n^{1/2}} = +\infty.$$

$\Rightarrow$ The $\boxed{\text{R.W. is recurrent!}}$ when $p = \frac{1}{2}$

Remarks: • When $d=1$, $p \neq \frac{1}{2}$, we can show that R.W. is transient!



$p = \frac{1}{2}$



$p > \frac{1}{2}$

(8)

• $d \geq 2$: we are going to use a different approach to check $\sum_{n=0}^{\infty} p_{00}^{(n)} \overset{?}{=} \infty$

characteristic functions.

Notation: unit vetors in $\mathbb{Z}^d$

$$\vec{e}_j = (0, \dots, 0, \underset{\substack{\uparrow \\ j^{th} \text{ position}}}{1}, 0, \dots, 0)$$

(e.g., $d=3$: $\vec{e}_1 = (1,0,0)$, $\vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$)

$\vec{X}_i$: the stop we take at time i

$$P(\vec{X}_i = \vec{e}_j) = P(\vec{X}_i = -\vec{e}_j) = \frac{1}{2d}$$

$\vec{S}_0$ = initial position $= \vec{0}$

$\vec{S}_n$ = postion at time n

$= \vec{X}_1 + \vec{X}_2 + \dots + \vec{X}_n$, $n \geq 1$.

$$\vec{S}_{n+1} = \vec{S}_n + \vec{X}_{n+1}$$

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$$P(\vec{S}_{n+1} = \vec{x} + \vec{e}_j \mid \vec{S}_n = \vec{x}) =$$

$$= P(\vec{X}_{n+1} = \vec{e}_j) = \frac{1}{2d}$$

Similarly,

$$P(\vec{S}_{n+1} = \vec{x} - \vec{e}_j \mid \vec{S}_n = \vec{x}) =$$

$$= P(\vec{X}_{n+1} = -\vec{e}_j) = \frac{1}{2d}.$$

This defines our transition probabilities.

For $\vec{k} \in \mathbb{R}^d$, $\vec{k} = (k_1, \dots, k_d)$, we define $\Phi_n(\vec{k})$ to be the characteristic function of $\vec{S}_n$ evaluated at $\vec{k}$:

$$\Phi_n(\vec{k}) = \mathbb{E}\left[e^{i\vec{k}\cdot\vec{S}_n}\right]$$

where $\vec{k}\cdot\vec{S}_n$ is the dot product:

$$\vec{x}\cdot\vec{y} = x_1 y_1 + \dots + x_n y_n.$$