Outline

Week 9: complex numbers; complex exponential and polar form

Course Notes: 5.1, 5.2, 5.3, 5.4

Goals:
Fluency with arithmetic on complex numbers
Using matrices with complex entries: finding determinants and inverses, solving systems, etc.
Visualizing complex numbers in coordinate systems
We use $i$ (as in "imaginary") to denote the number whose square is $-1$. 

\[ i^2 = -1 \]
\[ i^3 = -i \]
\[ i^4 = 1 \]
Complex Arithmetic

We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

$i^2 = -1$
Complex Arithmetic

We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

\[ i^2 = -1 \quad (-i)^2 = \]
Complex Arithmetic

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$$i^2 = -1 \quad (-i)^2 = -1$$
Complex Arithmetic

We use $i$ (as in “imaginary”) to denote the number whose square is $-1$.

\[ i^2 = -1 \quad (-i)^2 = -1 \quad i^3 = \]
Complex Arithmetic

We use \( i \) (as in "imaginary") to denote the number whose square is \(-1\).

\[
\begin{align*}
i^2 &= -1 \\
(-i)^2 &= -1 \\
i^3 &= -i
\end{align*}
\]
We use $i$ (as in "imaginary") to denote the number whose square is $-1$.  

\[ i^2 = -1 \quad (-i)^2 = -1 \quad i^3 = -i \quad i^4 = \]
Complex Arithmetic

$i$

We use $i$ (as in ”imaginary”) to denote the number whose square is $-1$.

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i^2 = -1 \quad (-i)^2 = -1 \quad i^3 = -i \quad i^4 = 1
\]
Complex Arithmetic

$i$

We use $i$ (as in "imaginary") to denote the number whose square is $-1$.

\[
i^2 = -1 \quad (\neg i)^2 = -1 \quad i^3 = -i \quad i^4 = 1
\]

When we talk about "complex numbers," we allow numbers to have real parts and imaginary parts:

\[
2 + 3i \quad -1 \quad 2i
\]
Complex Arithmetic

\[ 2 + 3i - 1 \]

imaginary

real
Complex Arithmetic

2 + 3i

−1

2i

real

imaginary

2

3

2 + 3i
Complex Arithmetic

2 + 3i

-1

2i

imaginary

3

2 + 3i

real

-1

2
Complex Arithmetic

\[ 2 + 3i - \frac{1}{2}i \]

Diagram: Points on the complex plane with real and imaginary axes.

- Point at \( 2 + 3i \)
- Point at \( -1 \)
- Point at \( 2i \)
Complex Arithmetic

Addition happens component-wise, just like with vectors or polynomials.
Complex Arithmetic

Addition happens component-wise, just like with vectors or polynomials.

\[(2 + 3i) + (3 - 4i) = \]

\[
\begin{align*}
\text{imaginary} & \\
\text{real} & \\
2 + 3i & \rightarrow
\end{align*}
\]

\[
\begin{align*}
\text{imaginary} & \\
\text{real} & \\
3 - 4i & \rightarrow
\end{align*}
\]
Complex Arithmetic

Addition happens component-wise, just like with vectors or polynomials.

\[(2 + 3i) + (3 - 4i) = 5 - i\]
Complex Arithmetic

Multiplication is similar to polynomials.
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) =\]
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]

\[= 6 + 9i - 8i + 12\]
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i) = 6 + 9i - 8i + 12 = 18 + i\]
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]
\[= 6 + 9i - 8i + 12 = 18 + i\]

A: \((-4 + 3i) + (1 - i)\)

B: \(i(2 + 3i)\)

C: \((i + 1)(i - 1)\)

D: \((2i + 3)(i + 4)\)

I: 0

II: -1

III: -2

IV: 2i + 12

V: -3 + 2i

VI: 3 + 2i

VII: 10 + 11i
Complex Arithmetic

Multiplication is similar to polynomials.

\[(2 + 3i)(3 - 4i) = 2 \cdot 3 + 3i \cdot 3 + (2)(-4i) + (3i)(-4i)\]
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## Complex Arithmetic

### Modulus

The **modulus** of \((x + yi)\) is:

\[
|x + yi| = \sqrt{x^2 + y^2}
\]

like the norm/length/magnitude of a vector.
Complex Arithmetic

Modulus

The **modulus** of \((x + yi)\) is:

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|x + yi| = \sqrt{x^2 + y^2}
\]

like the norm/length/magnitude of a vector.

Complex Conjugate

The **complex conjugate** of \((x + yi)\) is:

\[
\overline{x + yi} = x - yi
\]

the reflection of the vector over the real \((x)\) axis.
Complex Arithmetic

$$|x + yi| = \sqrt{x^2 + y^2}$$

$$\overline{x + yi} = x - yi$$
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \overline{z} \)
- \( z + \overline{z} \)
- \( z\overline{z} - |z|^2 \)
- \( \overline{zw} - (\overline{z})(\overline{w}) \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \quad \quad \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \overline{z} = 2yi \) \quad \( y \) is called the imaginary part of \( z \)
- \( z + \overline{z} \)
- \( z\overline{z} - |z|^2 \)
- \( \overline{zw} - (\overline{z})(\overline{w}) \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \bar{z} = 2yi \) \( y \) is called the imaginary part of \( z \)
- \( z + \bar{z} = 2x \) \( x \) is called the real part of \( z \)
- \( z\bar{z} - |z|^2 \)
- \( \bar{z}w - (\bar{z})(\bar{w}) \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \overline{x + yi} = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \overline{z} = 2yi \)  
  \( y \) is called the imaginary part of \( z \)
- \( z + \overline{z} = 2x \)  
  \( x \) is called the real part of \( z \)
- \( z\overline{z} - |z|^2 = 0 \)  
  So, \( z\overline{z} = |z|^2 \)
- \( \overline{zw} - (\overline{z})(\overline{w}) \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \quad \quad x + yi = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

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- \( z\bar{z} - |z|^2 = 0 \) \quad So, \( z\bar{z} = |z|^2 \)
- \( \bar{z}w - (\bar{z})(\bar{w}) = 0 \) \quad So, \( \bar{z}w = z \bar{w} \)
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \overline{x + yi} = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

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- \( \overline{zw} - (\overline{z})(\overline{w}) = 0 \) \quad So, \( \overline{zw} = \overline{z} \overline{w} \)

Division

\[
\frac{z}{w} = \quad \text{Division}
\]
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \overline{x + yi} = x - yi \]

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- \( z\overline{z} - |z|^2 = 0 \) \quad So, \( z\overline{z} = |z|^2 \)
- \( \overline{zw} - (\overline{z})(\overline{w}) = 0 \) \quad So, \( \overline{zw} = \overline{z} \overline{w} \)

Division

\[
\frac{z}{w} = \frac{z}{w} \cdot \frac{\overline{w}}{\overline{w}}
\]
Complex Arithmetic

\[ |x + yi| = \sqrt{x^2 + y^2} \quad \overline{x + yi} = x - yi \]

Suppose \( z = x + yi \) and \( w = a + bi \). Calculate the following.

- \( z - \overline{z} = 2yi \) \( y \) is called the imaginary part of \( z \)
- \( z + \overline{z} = 2x \) \( x \) is called the real part of \( z \)
- \( z\overline{z} - |z|^2 = 0 \) So, \( z\overline{z} = |z|^2 \)
- \( \overline{zw} - (\overline{z})(\overline{w}) = 0 \) So, \( \overline{zw} = \overline{z} \overline{w} \)

Division

\[
\frac{z}{w} = \frac{z}{w} \cdot \frac{\overline{w}}{\overline{w}} = \frac{zw}{|w|^2}
\]
Complex Arithmetic

\[ \frac{z}{w} = \frac{zw}{|w|^2} \]
Complex Arithmetic

\[ \frac{z}{w} = \frac{zw}{|w|^2} \]

Compute:

- \( \frac{2+3i}{3+4i} \)
- \( \frac{1+3i}{1-3i} \)
- \( \frac{2}{1+i} \)
- \( \frac{5}{i} \)
Complex Arithmetic

\[
\frac{z}{w} = \frac{zw}{|w|^2}
\]

Compute:

\[
\frac{2+3i}{3+4i} = \frac{18}{25} + \frac{1}{25}i
\]

\[
\frac{1+3i}{1-3i}
\]

\[
\frac{2}{1+i}
\]

\[
\frac{5}{i}
\]
Complex Arithmetic

\[
\frac{z}{w} = \frac{zw}{|w|^2}
\]

Compute:

\[
\begin{align*}
\text{• } \frac{2+3i}{3+4i} & = \frac{18}{25} + \frac{1}{25}i \\
\text{• } \frac{1+3i}{1-3i} & = \frac{-4}{5} + \frac{3}{5}i \\
\text{• } \frac{2}{1+i} & \\
\text{• } \frac{5}{i}
\end{align*}
\]
**Complex Arithmetic**

\[
\frac{z}{w} = \frac{zw}{|w|^2}
\]

Compute:

- \[
\frac{2+3i}{3+4i} = \frac{18}{25} + \frac{1}{25}i
\]
- \[
\frac{1+3i}{1-3i} = \frac{-4}{5} + \frac{3}{5}i
\]
- \[
\frac{2}{1+i} = 1 - i
\]
- \[
\frac{5}{i}
\]
Complex Arithmetic

\[
\frac{z}{w} = \frac{z \overline{w}}{|w|^2}
\]

Compute:

- \[
\frac{2+3i}{3+4i} = \frac{18}{25} + \frac{1}{25}i
\]

- \[
\frac{1+3i}{1-3i} = \frac{-4}{5} + \frac{3}{5}i
\]

- \[
\frac{2}{1+i} = 1 - i
\]

- \[
\frac{5}{i} = -5i \quad \text{(dividing by } i \text{ is the same as multiplying by } -i)\]
Polynomial Factorizations

Fundamental Theorem of Algebra

Every polynomial can be factored completely over the complex numbers.
Polynomial Factorizations

**Fundamental Theorem of Algebra**

Every polynomial can be factored completely over the complex numbers.

Example: \( x^2 + 1 = \)
Polynomial Factorizations

Fundamental Theorem of Algebra

Every polynomial can be factored completely over the complex numbers.

Example: \( x^2 + 1 = x^2 - (-1) = x^2 - i^2 \)
Polynomial Factorizations

Fundamental Theorem of Algebra
Every polynomial can be factored completely over the complex numbers.

Example: \( x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - i)(x + i) \)
Polynomial Factorizations

**Fundamental Theorem of Algebra**

Every polynomial can be factored completely over the complex numbers.

Example: \(x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - i)(x + i)\)

\(f(x) = x^2 + 1\) has no real roots, but it has two complex roots.
Polynomial Factorizations

Fundamental Theorem of Algebra

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Example: \( x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - i)(x + i) \)

\( f(x) = x^2 + 1 \) has no real roots, but it has two complex roots. It is not factorable over \( \mathbb{R} \), but it is factorable over \( \mathbb{C} \)
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Example: \( x^2 + 2x + 10 = \)
Fundamental Theorem of Algebra

Every polynomial can be factored completely over the complex numbers.

Example: $x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - i)(x + i)$

$f(x) = x^2 + 1$ has no real roots, but it has two complex roots. It is not factorable over $\mathbb{R}$, but it is factorable over $\mathbb{C}$

Example: $x^2 + 2x + 10 = (x + 1 + 3i)(x + 1 - 3i)$
Fundamental Theorem of Algebra

Every polynomial can be factored completely over the complex numbers.

Example: \( x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - i)(x + i) \)
\( f(x) = x^2 + 1 \) has no real roots, but it has two complex roots. It is not factorable over \( \mathbb{R} \), but it is factorable over \( \mathbb{C} \)

Example: \( x^2 + 2x + 10 = (x + 1 + 3i)(x + 1 - 3i) \)
If a quadratic equation has roots \( a \) and \( b \), then it can be written as \( c(x - a)(x - b) \)
Polynomial Factorizations

**Fundamental Theorem of Algebra**

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Example: \( x^2 + 2x + 10 = (x + 1 + 3i)(x + 1 - 3i) \)

If a quadratic equation has roots \( a \) and \( b \), then it can be written as \( c(x - a)(x - b) \)

Example: \( x^2 + 4x + 5 = \)
Fundamental Theorem of Algebra

Every polynomial can be factored completely over the complex numbers.

Example: \( x^2 + 1 = x^2 - (-1) = x^2 - i^2 = (x - i)(x + i) \)

\( f(x) = x^2 + 1 \) has no real roots, but it has two complex roots.
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Example: \( x^2 + 2x + 10 = (x + 1 + 3i)(x + 1 - 3i) \)
If a quadratic equation has roots \( a \) and \( b \), then it can be written as \( c(x - a)(x - b) \)

Example: \( x^2 + 4x + 5 = (x + 2 + i)(x + 2 - i) \)
Calculating Determinants

We calculate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

\[ \begin{vmatrix} 1 + i & 1 - i \\ i & 2 \end{vmatrix} = (1 + i)(2) - (1 - i)(i) = -3 + 3i \]

\[ \begin{vmatrix} 1 & 2 & 3 \\ i & 4 & 3 \\ 1 + i & -i & 5 \end{vmatrix} = 2 - 16i \]
Calculating Determinants

We calculate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

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\begin{vmatrix}
1 + i & 1 - i \\
2 & i
\end{vmatrix}
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i & 4 & 3i \\
1 + i & 2 - i & 5
\end{vmatrix}
\]
Calculating Determinants

We calculate the determinant of a matrix with complex entries in the same way we calculate the determinant of a matrix with real entries.

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\[
\det \begin{bmatrix} 1 & 2 & 3 \\ i & 4 & 3i \\ 1 + i & 2 - i & 5 \end{bmatrix} = 2 - 16i
\]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[ \begin{align*}
ix_1 & \quad + \quad x_2 & \quad + \quad 2x_3 & \quad = \quad 0 \\
ix_2 & \quad + \quad 3x_3 & \quad = \quad 0 \\
2ix_1 & \quad + \quad (2 - i)x_2 & \quad + \quad x_3 & \quad = \quad 0
\end{align*} \]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[ \begin{align*}
ix_1 &+ x_2 + 2x_3 = 0 \\
ix_2 &+ 3x_3 = 0 \\
2ix_1 &+ (2 - i)x_2 + x_3 = 0
\end{align*} \]

Solve the following system of equations:

\[ \begin{align*}
ix_1 &+ 2x_2 = 9 \\
3x_1 &+ (1 + i)x_2 = 5 + 8i
\end{align*} \]
Give a parametric equation for all solutions to the homogeneous system:

\[ \begin{align*}
ix_1 + x_2 + 2x_3 &= 0 \\
i x_2 + 3x_3 &= 0 \\
2ix_1 + (2 - i)x_2 + x_3 &= 0
\end{align*} \]

Solve the following system of equations:

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ix_1 + 2x_2 &= 9 \\
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Find the inverse of the matrix

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\begin{bmatrix}
i & 1 \\2 & 3i
\end{bmatrix}
\]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

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ix_1 & + x_2 + 2x_3 = 0 \\
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\end{align*} \]

\[ [x_1, x_2, x_3] = s[-3 + 2i, 3i, 1] \]

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Solve the following system of equations:

\[
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ix_1 + 2x_2 &= 9 \\
3x_1 + (1 + i)x_2 &= 5 + 8i
\end{align*}
\]

\[x_1 = i, \ x_2 = 5\]

Find the inverse of the matrix \[
\begin{bmatrix} i & 1 \\ 2 & 3i \end{bmatrix}
\]
Gaussian Elimination

Give a parametric equation for all solutions to the homogeneous system:

\[ ix_1 + x_2 + 2x_3 = 0 \]
\[ ix_2 + 3x_3 = 0 \]
\[ 2ix_1 + (2 - i)x_2 + x_3 = 0 \]

\[ [x_1, x_2, x_3] = s[-3 + 2i, 3i, 1] \]

Solve the following system of equations:

\[ ix_1 + 2x_2 = 9 \]
\[ 3x_1 + (1 + i)x_2 = 5 + 8i \]

\[ x_1 = i, \quad x_2 = 5 \]

Find the inverse of the matrix

\[
\begin{bmatrix}
i & 1 \\
2 & 3i
\end{bmatrix}
\]
\[
\begin{bmatrix}
\frac{-3i}{5} & \frac{1}{5}i \\
\frac{2}{5} & -\frac{1}{5}i
\end{bmatrix}
\]
Exponentials

What to do when $i$ is the power of a function?
Exponentials

What to do when $i$ is the power of a function?

Maclaurin (Taylor) Series: *(you won't be assessed on this explanation)*

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

We know how to do the operations on the right
What to do when $i$ is the power of a function?

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$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

We know how to do the operations on the right

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$
Exponentials

What to do when \( i \) is the power of a function?
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\]

We know how to do the operations on the right

\[
e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots
\]

\[
= 1 + ix - \frac{x^2}{2!} - i \frac{x^3}{3!} + \frac{x^4}{4!} + i \frac{x^5}{5!} - \frac{x^6}{6!} \cdots
\]
What to do when $i$ is the power of a function?

Maclaurin (Taylor) Series: (you won’t be assessed on this explanation)

\[ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \]

\[ e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots \]

\[ = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} \cdots \]

\[ = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right) \]
Exponentials

What to do when $i$ is the power of a function?

Maclaurin (Taylor) Series: (you won’t be assessed on this explanation)

\[
e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots
\]

\[
\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots
\]

\[
\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots
\]

\[
e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots
\]

\[
= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} + \cdots
\]

\[
= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)
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Exponentials

What to do when $i$ is the power of a function?

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$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots$$

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$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!} + \cdots$$

$$= 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} \cdots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + i \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \cos x + isin x$$
Does that even make sense?

\[ e^{i\theta} = \cos \theta + i \sin \theta \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = a e^{ax};
\]
Does that even make sense?

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Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = ae^{ax}; \\
\frac{d}{dx}[e^{ix}] = \frac{d}{dx}[\cos x + i \sin x] \\
= -\sin x + i \cos x = i^2 \sin x + i \cos x = i(\cos x + i \sin x) = ie^{ix}
\]
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\[ e^{x+y} = e^x e^y; \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = ae^{ax}; \\
\frac{d}{dx}[e^{ix}] = \frac{d}{dx} [\cos x + i \sin x] \\
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\]

\[ e^{x+y} = e^x e^y; \]
\[ e^{ix+iy} = \]
5.1: Complex Arithmetic
5.2: Complex Matrices and Linear Systems
5.3: Complex Exponential
5.4: Polar Representation

Does that even make sense?

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\frac{d}{dx} [e^{ax}] = ae^{ax};
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\]

\[ e^{x+y} = e^x e^y; \]
\[ e^{i(x+y)} = e^{i(x+y)} = \cos(x + y) + i \sin(x + y) \]
Does that even make sense?

\[ e^{ix} = \cos x + i \sin x \]

\[
\frac{d}{dx}[e^{ax}] = ae^{ax};
\]
\[
\frac{d}{dx}[e^{ix}] = \frac{d}{dx}[\cos x + i \sin x]
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\[ e^{ix+iy} = e^{i(x+y)} = \cos(x + y) + i \sin(x + y) \]

\[= \cos x \cos y - \sin x \sin y + i[\sin x \cos y + \cos x \sin y] \]
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\[ e^{x+y} = e^x e^y; \]
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\[ = \cos x \cos y - \sin x \sin y + i[\sin x \cos y + \cos x \sin y] \]
\[ = (\cos x + i \sin y)(\cos y + i \sin x) = e^{ix} e^{iy} \]


\[ e^{ix} = \cos x + i \sin x \]

Evaluate:

\[
e^{\frac{\pi i}{2}}
\]

\[
e^{2+i}
\]

\[
\sqrt{2} e^{\frac{\pi i}{4}}
\]

\[
2^i
\]

\[
e^{\pi i} + 1
\]

\[ |e^{xi}|, \text{ where } x \text{ is any real number.} \]
Computation Practice

\[ e^{ix} = \cos x + i \sin x \]

Evaluate:

\[ e^{\pi i/2} = i \]

\[ e^{2 + i} \]

\[ \sqrt{2} e^{\pi i/4} \]

\[ 2^i \]

\[ e^{\pi i} + 1 \]

\[ |e^{xi}|, \text{ where } x \text{ is any real number.} \]
Computation Practice

\[ e^{ix} = \cos x + i \sin x \]

Evaluate:

\[ e^{\frac{\pi i}{2}} = i \]

\[ e^{2+i} = e^{2} (\cos 1 + i \sin 1) \]

\[ \sqrt{2} e^{\frac{\pi i}{4}} \]

\[ 2^i \]

\[ e^{\pi i} + 1 \]

\[ |e^{xi}|, \text{ where } x \text{ is any real number.} \]
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\[ 2^i = e^{i \ln 2} = \cos(\ln 2) + i \sin(\ln 2) \]

\[ e^{\pi i} + 1 \]

\[ |e^{xi}|, \text{ where } x \text{ is any real number.} \]
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\[ e^{\pi i} + 1 = 0 \text{ (Euler’s Identity)} \]

\[ |e^{xi}|, \text{ where } x \text{ is any real number.} \]
Computation Practice

\[ e^{ix} = \cos x + i \sin x \]

Evaluate:

\[ e^{\frac{\pi i}{2}} = i \]

\[ e^{2+i} = e^2 (\cos 1 + i \sin 1) \]

\[ \sqrt{2} e^{\frac{\pi i}{4}} = i + 1 \]

\[ 2^i = e^{i \ln 2} = \cos(\ln 2) + i \sin(\ln 2) \]

\[ e^{\pi i} + 1 = 0 \text{ (Euler’s Identity)} \]

\[ |e^{xi}|, \text{ where } x \text{ is any real number.} = 1 \]
Let $x$ be a real number.
True or False?

(1) $e^x = \cos x$

(2) $e^{ix} = e^{i(x+2\pi)}$

(3) $e^{ix} = -e^{i(x+\pi)}$

(4) $e^{ix} + e^{-ix}$ is a real number
Complex exponentiation: \( e^{ix} = \cos x + i \sin x \)

Let \( x \) be a real number.
True or False?

(1) \( e^x = \cos x \) False

Remember these are real numbers: \( e^x \) is unbounded, \( \cos x \) stays between \(-1\) and \(1\).

(2) \( e^{ix} = e^{i(x+2\pi)} \)

(3) \( e^{ix} = -e^{i(x+\pi)} \)

(4) \( e^{ix} + e^{-ix} \) is a real number
Complex exponentiation: $e^{ix} = \cos x + i \sin x$

Let $x$ be a real number.
True or False?

(1) $e^x = \cos x$    \quad False
Remember these are real numbers: $e^x$ is unbounded, $\cos x$ stays between $-1$ and $1$.

(2) $e^{ix} = e^{i(x+2\pi)}$    \quad True
For real numbers, a larger exponent gives a larger $e^x$; complex numbers, not necessarily: $e^{ix} = a + bi$ where $|a|, |b| \leq 1$.

(3) $e^{ix} = -e^{i(x+\pi)}$

(4) $e^{ix} + e^{-ix}$ is a real number
Complex exponentiation: $e^{ix} = \cos x + i \sin x$

Let $x$ be a real number.
True or False?

(1) $e^x = \cos x$  
False
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(2) $e^{ix} = e^{i(x+2\pi)}$  
True
For real numbers, a larger exponent gives a larger $e^x$; complex numbers, not necessarily: $e^{ix} = a + bi$ where $|a|, |b| \leq 1$.

(3) $e^{ix} = -e^{i(x+\pi)}$  
True
$\cos x = -\cos(x + \pi)$; $\sin x = -\sin(x + \pi)$

(4) $e^{ix} + e^{-ix}$ is a real number
Complex exponentiation: $e^{ix} = \cos x + i \sin x$

Let $x$ be a real number.
True or False?

(1) $e^x = \cos x$     False
Remember these are real numbers: $e^x$ is unbounded, $\cos x$ stays between $-1$ and $1$.

(2) $e^{ix} = e^{i(x+2\pi)}$     True
For real numbers, a larger exponent gives a larger $e^x$; complex numbers, not necessarily: $e^{ix} = a + bi$ where $|a|, |b| \leq 1$.

(3) $e^{ix} = -e^{i(x+\pi)}$     True
$\cos x = -\cos(x + \pi)$; $\sin x = -\sin(x + \pi)$

(4) $e^{ix} + e^{-ix}$ is a real number     True
using even and odd symmetry of cosine and sine, $e^{ix} + e^{-ix} = 2 \cos x$
Coordinates Revisited

A complex number can be expressed in polar form as:

\[ z = r \cos \theta + i r \sin \theta \]

where:
- \( r \) is the magnitude (or absolute value) of the complex number.
- \( \theta \) is the argument (or angle) of the complex number.

This representation allows us to visualize complex numbers on a plane with the real part on the horizontal axis (Real) and the imaginary part on the vertical axis (Im).

[Diagram of the complex plane showing a vector from the origin to a point representing a complex number.]
Coordinates Revisited

Complex number: \( r \cos \theta + i \sin \theta = re^{i\theta} \)
Coordinates Revisited

Complex number: \( r \cos \theta + ir \sin \theta = re^{i \theta} \)
Coordinates Revisited

Complex number: \( r (\cos \theta + i \sin \theta) = re^{i\theta} \)
Coordinates Revisited

Complex number: \[ r(\cos \theta + i \sin \theta) = re^{i\theta} \]
\[ \sqrt{3} + i = 2 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right) = 2e^{\frac{\pi}{6}i} \]
Coordinates and Exponentials

\[ \sqrt{3} + i \]

\[ 2 \cos\left(\frac{\pi}{6}\right) + i \sin\left(\frac{\pi}{6}\right) = 2e^{i\pi/6} \]
\[ \sqrt{3} + i = 2 \left( \cos \left( \frac{\pi}{6} \right) + i \sin \left( \frac{\pi}{6} \right) \right) = 2 e^{\frac{\pi}{6} i} \]
\[
\sqrt{3} + i = 2(\cos(\pi/6) + i \sin(\pi/6)) = 2e^{\pi/6}i
\]
-1 + i
Coordinates and Exponentials

\[-1 + i\]

\[\sqrt{2} \quad \text{Im} \]

\[\text{Real} \quad -1 + i\]


Coordinates and Exponentials

\[ -1 + i = \sqrt{2} \cos \left( \frac{3\pi}{4} \right) + i \sin \left( \frac{3\pi}{4} \right) = \sqrt{2} e^{3\pi i / 4} \]
Coordinates and Exponentials

\[-1 + i = \sqrt{2}(\cos(3\pi/4) + i \sin(3\pi/4)) = \sqrt{2}e^{3\pi/4i}\]
Coordinates and Multiplication

Geometric interpretation of multiplication of two complex numbers:
add the angles, multiply the lengths (moduli).
Coordinates and Multiplication

\[ re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)} \]
Coordinates and Multiplication

Geometric interpretation of multiplication of two complex numbers: add the angles, multiply the lengths (moduli).

\[ re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)} \]
Coordinates and Multiplication

Geometric interpretation of multiplication of two complex numbers: add the angles, multiply the lengths (moduli).

\[ re^{i\theta} \cdot se^{i\phi} = (rs)e^{i(\theta+\phi)} \]
Roots of Unity
Roots of Unity

\[(re^{i\theta})^3 = 1\]
Roots of Unity

$e^{i \frac{2\pi}{3}}$

$e^{i \frac{4\pi}{3}}$

$(re^{i\theta})^3 = 1$
Roots of Unity

\[ (re^{i\theta})^5 = 1 \]
Roots of Unity

\[(re^{i\theta})^5 = 1\]
Roots of Unity

\[ (re^{i\theta})^{12} = 1 \]
Roots of Unity

\[
\left( re^{i\theta} \right)^{12} = 1
\]
Find all complex numbers $z$ such that $z^3 = 8$.

Find all complex numbers $z$ such that $z^3 = 27e^{i\pi/2}$.

Find all complex numbers $z$ such that $z^4 = 81e^{2i}$.
$z^3 = 8$

$2e^{\frac{2\pi i}{3}}$
$z^3 = 8$
$z^3 = 8$

The diagram shows the complex numbers $2e^{\frac{2\pi i}{3}}$, $4e^{\frac{4\pi i}{3}}$, and $8e^{\frac{6\pi i}{3}}$ in the complex plane, with the real and imaginary axes labeled as Re and Im, respectively.
$z^3 = 8$

Diagram showing a complex number $2e^{\frac{4\pi i}{3}}$ on the complex plane.
$z^3 = 8$
\[ z^3 = 8 \]
$z^3 = 8$
$z^3 = 8$
$z^3 = 8$
Roots

Find all complex numbers $z$ such that $z^3 = 8$.
$2, e^{\frac{2\pi i}{3}}, 2e^{\frac{4\pi i}{3}}$

Find all complex numbers $z$ such that $z^3 = 27e^{\frac{i\pi}{2}}$.

Find all complex numbers $z$ such that $z^4 = 81e^{2i}$. 
We solve \((re^{i\theta}) = 27e^{\frac{i\pi}{2}}\). That is, \(r^3e^{i3\theta} = 27e^{\frac{i\pi}{2}}\)

- The modulus of our answer is 27; the modulus of \(re^{i\theta}\) is \(r\).
- So, we need \(r^3 = 27\), so \(r = 3\).
- That leaves us with \(e^{3i\theta} = e^{\frac{i\pi}{2}}\).
  - There are going to be three distinct answers (since there are three roots of unity)
  - We write \(e^{\frac{i\pi}{2}}\) three ways: \(e^{\frac{i\pi}{2}} = e^{i\left(\frac{\pi}{2}+2\pi\right)} = e^{i\left(\frac{\pi}{2}+4\pi\right)}\).
    - \(e^{3i\theta} = e^{\frac{i\pi}{2}} \implies 3\theta = \frac{\pi}{2} \implies \theta = \frac{\pi}{6}\)
    - \(e^{3i\theta} = e^{i\left(\frac{\pi}{2}+2\pi\right)} \implies 3\theta = \frac{\pi}{2} + 2\pi \implies \theta = \frac{5\pi}{6}\)
    - \(e^{3i\theta} = e^{i\left(\frac{\pi}{2}+4\pi\right)} \implies 3\theta = \frac{\pi}{2} + 4\pi \implies \theta = \frac{3\pi}{2}\)
- So, our solutions are \(3e^{\frac{\pi i}{6}}, 3e^{\frac{5\pi i}{6}}, 3e^{\frac{3\pi i}{2}}\)
Roots

Find all complex numbers \( z \) such that \( z^3 = 8 \).

\[ 2, e^{\frac{2\pi i}{3}}, 2e^{\frac{4\pi i}{3}} \]

Find all complex numbers \( z \) such that \( z^3 = 27e^{\frac{i\pi}{2}} \).

\[ 3e^{\frac{\pi i}{6}}, 3e^{\frac{5\pi i}{6}}, 3e^{\frac{3\pi i}{2}} \]

Find all complex numbers \( z \) such that \( z^4 = 81e^{2i} \).
Find all complex numbers $z$ such that $z^3 = 8$.

$2, e^{\frac{2\pi i}{3}}, 2e^{\frac{4\pi i}{3}}$

Find all complex numbers $z$ such that $z^3 = 27e^{\frac{i\pi}{2}}$.

$3e^{\frac{\pi i}{6}}, 3e^{\frac{5\pi i}{6}}, 3e^{\frac{3\pi i}{2}}$

Find all complex numbers $z$ such that $z^4 = 81e^{2i}$.

$3e^{\frac{i}{2}}, 3e^{\frac{(1+\pi)i}{2}}, 3e^{\frac{(1+2\pi)i}{2}}, 3e^{\frac{(1+3\pi)i}{2}}$