COLLISION OF ORBITS FOR A ONE-PARAMETER FAMILY OF DRINFELD MODULES

DRAGOS GHIOCA

Abstract. We prove a result (see Theorem 1.1) regarding unlikely intersections of orbits for a given 1-parameter family of Drinfeld modules. We also advance a couple of general conjectures regarding unlikely intersections for algebraic families of Drinfeld modules (see Conjectures 1.3 and 2.3).

1. Introduction

1.1. **General setup for Drinfeld modules.** Throughout this paper, we let $\mathbb{F}_q$ be a finite field and (unless otherwise noted) we let $K := \overline{\mathbb{F}_q}(t)$ be the rational function field (of transcendence degree 1) over the algebraic closure of $\mathbb{F}_q$; also, we let $\overline{K}$ be a fixed algebraic closure of K.

Let L be a field extension of K. A Drinfeld module (of generic characteristic) defined over L is a ring homomorphism $\Phi : \mathbb{F}_q[T] \longrightarrow \mathrm{End}_L(\mathbb{G}_a)$; in particular, each Drinfeld module is uniquely determined by $\Phi_T := \Phi(T)$, which is a separable, additive polynomial of degree larger than 1. Furthermore, our Drinfeld module action is always assumed to be $\mathbb{F}_q$-linear; in other words, there exists an integer $r \geq 1$ and there exist $a_1, \dots, a_r \in L$ such that

$$\Phi_T(x) = tx + \sum_{i=1}^{r} a_i x^{q^i}.$$

For each $P \in \mathbb{F}_q[T]$, we write $\Phi_P := \Phi(P)$. Given finitely many points $\alpha_1, \dots, \alpha_k \in \overline{L}$, their *orbit* under (the action of) Φ is the set of all

$$\sum_{i=1}^{k} \Phi_{P_i}(\alpha_i), \tag{1.1.1}$$

as we vary the polynomials $P_i \in \mathbb{F}_q[T]$, not all equal to 0. We say that a point $\alpha \in \overline{L}$ is *torsion* if 0 lies in its orbit under Φ, i.e., there exists $Q \in \mathbb{F}_q[T] \setminus \{0\}$ such that $\Phi_Q(\alpha) = 0$.

The endomorphism ring $\mathrm{End}(\Phi)$ of the Drinfeld module Φ consists of all additive polynomials $f \in \overline{L}[x]$ with the property that $f \circ \Phi_T = \Phi_T \circ f$. We say that the points $\alpha_1, \dots, \alpha_k \in \overline{L}$ are *linearly independent over* $\mathrm{End}(\Phi)$ if whenever

$$f_1(\alpha_1) + \cdots + f_k(\alpha_k) = 0,$$

for some $f_1, \dots, f_k \in \mathrm{End}(\Phi)$, then we must have $f_1 = \cdots = f_k = 0$; otherwise, we say that the points $\alpha_1, \dots, \alpha_k$ are $\mathrm{End}(\Phi)$-linearly dependent. For more details on Drinfeld modules, we refer the reader to [Gos96]; see also [Ghi07a] for a study of the Φ-module structure induced on L.

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1.2. **Our main result.** We prove the following.

Theorem 1.1. *Let $r \geq 2$ be an integer, let $\mathbb{F}_q$ be a finite field and let $K = \overline{\mathbb{F}_q}(t)$. Let $\Phi^{(\lambda)} : \mathbb{F}_q[T] \longrightarrow \mathrm{End}_{K(\lambda)}(\mathbb{G}_a)$ be the 1-parameter family of Drinfeld modules (parameterized by $\lambda \in \overline{K}$) for which $\Phi_T^{(\lambda)}(x) = tx + \lambda x^q + x^{q^r}$. We let $\mathbf{a}, \mathbf{b} \in K^*$ and let $\mathbf{c} \in \overline{K}$. Then there exist infinitely many $\lambda \in \overline{K}$ with the property that the orbit of $\mathbf{a}$ under $\Phi^{(\lambda)}$ meets the orbit of $\mathbf{b}$ under $\Phi^{(\lambda)}$ at the point $\mathbf{c}$, i.e., there exist nonzero polynomials $P_\lambda, Q_\lambda \in \mathbb{F}_q[T]$ such that*

$$\Phi_{P_\lambda}^{(\lambda)}(\mathbf{a}) = \Phi_{Q_\lambda}^{(\lambda)}(\mathbf{b}) = \mathbf{c}$$

if and only if at least one of the following two conditions are met:

(i) *there exists $v \in \mathbb{F}_q^*$ such that either $v \cdot \mathbf{a} = \mathbf{c}$ or $v \cdot \mathbf{b} = \mathbf{c}$.*

(ii) *there exists $u \in \mathbb{F}_q^*$ such that $u \cdot \mathbf{a} = \mathbf{b}$. Furthermore, in this case, we have that for each $\lambda \in \overline{K}$ and for each $P \in \mathbb{F}_q[T]$, then $\Phi_P^{(\lambda)}(\mathbf{b}) = \mathbf{c}$ if and only if $\Phi_{u \cdot P}^{(\lambda)}(\mathbf{a}) = \mathbf{c}$.*

Remark 1.2. In Theorem 1.1, it makes sense to exclude the possibility that $\mathbf{a} = 0$ (or $\mathbf{b} = 0$) since the orbit of 0 under any Drinfeld module action consists only of the point 0 itself, thus making our question trivial.

On the other hand, the case $\mathbf{c} = 0$ is quite nontrivial because this time we are asking for which $\lambda \in \overline{K}$ we have that both $\mathbf{a}$ and $\mathbf{b}$ are torsion for the same Drinfeld module $\Phi^{(\lambda)}$. Furthermore, taking $\mathbf{c} = 0$ in Theorem 1.1, we see that conclusion (i) is impossible (because $\mathbf{a}, \mathbf{b} \neq 0$) and so, we are left with conclusion (ii), thus showing that in this special case, we rediscover the result from [GH13, Theorem 1.5].

1.3. **Interpretation of our result.** We discuss next our result in the larger context of unlikely intersections in arithmetic dynamics; for more details on the unlikely intersection principle in arithmetic geometry, we refer the reader to the excellent book of Zannier [Zan12].

With the notation as in Theorem 1.1, there always exist infinitely many $\lambda \in \overline{K}$ with the property that the orbits of $\mathbf{a}$ and of $\mathbf{b}$ intersect; one simply needs to solve various equations (in λ):

$$\Phi_{P(T)}^{(\lambda)}(\mathbf{a}) = \Phi_{Q(T)}^{(\lambda)}(\mathbf{b}) \tag{1.3.1}$$

and a similar argument as in Corollary 4.5 would yield that there exist infinitely many solutions λ for the equation (1.3.1), as we vary $P, Q \in \mathbb{F}_q[T]$. However, if we impose the extra condition that the intersection of the two orbits under $\Phi^{(\lambda)}$ occurs precisely at the point $\mathbf{c}$, then this becomes quite *unlikely* and so, it is reasonable to expect that in this case, there will be *only* finitely many $\lambda \in \overline{K}$ with the property that the orbits under $\Phi^{(\lambda)}$ of $\mathbf{a}$ and of $\mathbf{b}$ meet precisely at $\mathbf{c}$, *unless* there is a *global dynamical relation*. Conclusion (i) from Theorem 1.1 says that $\mathbf{c}$ is either always in the orbit of $\mathbf{a}$, or always in the orbit of $\mathbf{b}$. Assuming $\mathbf{c}$ is always in the orbit of $\mathbf{a}$ (say), then all one needs is to ensure that for infinitely many $\lambda \in \overline{K}$, we would have that $\mathbf{c}$ lies in the orbit under $\Phi^{(\lambda)}$ of $\mathbf{b}$; this is exactly what is proven in Corollary 4.5. Similarly, if conclusion (ii) in Theorem 1.1 holds, then this means that the orbits of $\mathbf{a}$ and $\mathbf{b}$ are identical for *all* λ and then, once again, all one needs to ensure is that for infinitely many $\lambda \in \overline{K}$, the point $\mathbf{c}$ lies in the orbit under $\Phi^{(\lambda)}$ of $\mathbf{b}$.

This principle of unlikely intersections in arithmetic dynamics led to numerous results in the past 15 years, starting with the work of Masser and Zannier [MZ10, MZ12] regarding torsion points in families of elliptic curves. A similar outstanding result was later obtained by Baker-DeMarco [BD11] (see also [BD13, GHT13] for further extensions), who proved that if there exist infinitely many $\lambda \in \mathbb{C}$ such that the given complex numbers a and b are preperiodic under the action of $z \mapsto z^d + \lambda$ (for a given integer $d \geq 2$), then $a^d = b^d$. In other words, if the *unlikely event* that both a and b have finite orbit under the action of $z \mapsto z^d + \lambda$ occurs *infinitely often*, then there must be a *global dynamical relation*, which is (in this case) that for *each* λ, both a and b are mapped to the same point after one iteration of our map $z \mapsto z^d + \lambda$. Despite the large number of unlikely intersection results for dynamical systems over fields of characteristic 0 proven in the past years, very few results were obtained in characteristic $p > 0$. Using [GH13] as a starting point, we propose here a couple of questions (see Conjectures 1.3 and 2.3) regarding the unlikely intersection principle in the context of Drinfeld modules.

First of all, it is natural to ask whether the conclusion of Theorem 1.1 holds more generally, both in terms of an arbitrary family of Drinfeld modules, but also in terms of arbitrary *starting points* $\mathbf{a}$ and $\mathbf{b}$ and *target point* $\mathbf{c}$; we believe such an extension should hold. However, in this case, the appropriate conclusions (i) and (ii) will necessarily be more complex since there are more possibilities for the orbits of $\mathbf{a}$ and $\mathbf{b}$ (along with the target point $\mathbf{c}$) to be globally related dynamically. Furthermore, we believe that the principle of unlikely intersections generalizes to orbits of finitely many points (see (1.1.1)); we formulate the following conjecture.

Conjecture 1.3. *Let $r \geq 2$ be an integer and let $\left\{\Phi^{(\lambda)} : \mathbb{F}_q[T] \longrightarrow \mathrm{End}(\mathbb{G}_a)\right\}_\lambda$ be an algebraic family of Drinfeld modules parameterized by $\lambda \in \overline{K}$, i.e., there exist polynomials $f_1, \dots, f_{r-1} \in \overline{K}[u]$ (not all constant) such that*

$$\Phi_T^{(\lambda)}(x) := tx + \sum_{i=1}^{r-1} f_i(\lambda) \cdot x^{q^i} + x^{q^r}. \tag{1.3.2}$$

Let $L := \overline{K}(u)$ be the rational function field in one variable over $\overline{K}$, and we denote by $\mathbf{\Phi} : \mathbb{F}_q[T] \longrightarrow \mathrm{End}_L(\mathbb{G}_a)$ the generic action of our Drinfeld modules family, i.e.,

$$\mathbf{\Phi}_T(x) := tx + \sum_{i=1}^{r-1} f_i(u) \cdot x^{q^i} + x^{q^r}.$$

We let $\mathbf{c} \in \overline{K}[u]$, and for some integers $k, \ell \geq 1$, we let $\mathbf{a}_1, \dots, \mathbf{a}_k \in \overline{K}[u]$ be $\mathrm{End}_{\overline{L}}(\mathbf{\Phi})$-linearly independent points, and also, we let $\mathbf{b}_1, \dots, \mathbf{b}_\ell \in \overline{K}[u]$ be $\mathrm{End}_{\overline{L}}(\mathbf{\Phi})$-linearly independent points. If there exist infinitely many $\lambda \in \overline{K}$ such that $\mathbf{c}(\lambda)$ lies both in the orbit of $\mathbf{a}_1(\lambda), \dots, \mathbf{a}_k(\lambda)$ under $\Phi^{(\lambda)}$ and in the orbit of $\mathbf{b}_1(\lambda), \dots, \mathbf{b}_\ell(\lambda)$ under $\Phi^{(\lambda)}$ then at least one of the following two statements must hold:

(i) $\mathbf{c}$ *lies either in the orbit of* $\mathbf{a}_1, \dots, \mathbf{a}_k$ *under* $\mathbf{\Phi}$*, or in the orbit of* $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ *under* $\mathbf{\Phi}$*.*
(ii) *the points* $\mathbf{a}_1, \dots, \mathbf{a}_k, \mathbf{b}_1, \dots, \mathbf{b}_\ell$ *are* $\mathrm{End}(\mathbf{\Phi})$*-linearly dependent.*

1.4. **Plan for our paper.** In Section 2 we discuss in-depth Conjecture 1.3, especially explaining the difficulties one would encounter when trying to extend our proof of Theorem 1.1 to the more general setting of Conjecture 1.3. Also, in Section 2, we formulate Conjecture 2.3 which asks for an extension of Theorem 1.1 for 2-parameter families of Drinfeld modules. In Section 3, we introduce the canonical heights associated to Drinfeld modules; the main goal of Section 3 is proving Proposition 3.1, which is the key ingredient for Proposition 4.4. Finally, in Section 4 we gather the remaining necessary technical ingredients for our proofs and then proceed to proving Theorem 1.1.

2. Unlikely intersections for Drinfeld modules

We start with a discussion of Conjecture 1.3.

2.1. **Collision of orbits generated by multiple points.** We note that one can easily formulate Conjecture 1.3 using families of Drinfeld modules parameterized by $\overline{K}$-points on arbitrary curves (not necessarily $\mathbb{A}^1$); however, we prefer a more concrete formulation of this already very general question. Also, we prefer to normalize our family of Drinfeld modules so that each $\Phi_T^{(\lambda)}$ is a monic polynomial (see (1.3.2)) because this allows for cleaner statements for arithmetic questions regarding Drinfeld modules and also, there is no loss in generality assuming the Drinfeld modules are normalized as such (see also [Ghi07a, Ghi07b]).

We continue with a discussion of the conditions (i)-(ii) in Conjecture 1.3.

Remark 2.1. We note that the conditions (i)-(ii) are indeed necessary when dealing with the unlikely intersection problem formulated in Conjecture 1.3. Condition (i) is easily seen necessary since for general families of Drinfeld modules and general starting points $\mathbf{a}_1, \dots, \mathbf{a}_k$ and general target point $\mathbf{c}$, it could be that $\mathbf{c}$ equals any given $\mathbf{\Phi}(\mathbb{F}_q[T])$-linear combination of $\mathbf{a}_1, \dots, \mathbf{a}_k$, in which case we would then have that for *each* $\lambda \in \overline{K}$, the target $\mathbf{c}(\lambda)$ lies in the orbit under $\Phi^{(\lambda)}$ of the starting points $\mathbf{a}_1(\lambda), \dots, \mathbf{a}_k(\lambda)$. Then a generalization of Corollary 4.5 should prove that for *infinitely many* $\lambda \in \overline{K}$, we have that $\mathbf{c}(\lambda)$ lies in the orbit of $\mathbf{b}_1(\lambda), \dots, \mathbf{b}_\ell(\lambda)$ under $\Phi^{(\lambda)}$.

Condition (ii) from Conjecture 1.3 is more subtle. Indeed, assume $f \in \mathrm{End}_{\overline{L}}(\mathbf{\Phi})$ and that $\mathbf{b}_1 = f(\mathbf{a}_1)$ (where $k = \ell = 1$ and $\mathbf{a}_1$ is a non-torsion point for $\mathbf{\Phi}$). Then working with the target $\mathbf{c} = 0$, we see that whenever 0 lies in the orbit of $\mathbf{a}_1(\lambda)$ under $\Phi^{(\lambda)}$, then it also lies in the orbit of $\mathbf{b}_1(\lambda)$ under $\Phi^{(\lambda)}$. On the other hand, assuming now that there is no $P \in \mathbb{F}_q[T]$ such that the endomorphism f equals $\mathbf{\Phi}_P$ and also assuming the target point $\mathbf{c}$ along with $\mathbf{a}_1$ are $\mathrm{End}(\mathbf{\Phi})$-linearly independent, then working with the starting points $\mathbf{a}_1$ and $\mathbf{b}_1 = f(\mathbf{a}_1)$ should produce only finitely many $\lambda \in \overline{K}$ such that $\mathbf{c}(\lambda)$ lies in both orbits of $\mathbf{a}_1(\lambda)$ and of $\mathbf{b}_1(\lambda)$ under the action of $\Phi^{(\lambda)}$. This is the reason why our Conjecture 1.3 does not ask for an equivalence of two statements.

Remark 2.2. In Conjecture 1.3, the points $\mathbf{a}_1, \dots, \mathbf{a}_k$, respectively $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ were assumed to be $\mathrm{End}_{\overline{L}}(\mathbf{\Phi})$-linearly independent in order to guarantee that there exist infinitely many λ such that $\mathbf{c}(\lambda)$ lies in the orbit of both $\mathbf{a}_1(\lambda), \dots, \mathbf{a}_k(\lambda)$ and of $\mathbf{b}_1(\lambda), \dots, \mathbf{b}_\ell(\lambda)$ under $\Phi^{(\lambda)}$ (this is similar to the concept of stability discussed in [DeM16] for complex dynamics). Indeed, if each $\mathbf{a}_i$ were torsion for $\mathbf{\Phi}$, while $\mathbf{c}$ is not $\mathbf{\Phi}$-torsion, then there would be only finitely many $\lambda \in \overline{K}$ for which $\mathbf{c}(\lambda)$ were in the orbit of $\mathbf{a}_1(\lambda), \dots, \mathbf{a}_k(\lambda)$ under $\Phi^{(\lambda)}$.

One could weaken the assumptions on the points $\mathbf{a}_1, \dots, \mathbf{a}_k$ (resp. $\mathbf{b}_1, \dots, \mathbf{b}_\ell$) by asking that at least one $\mathbf{a}_i$ and at least one $\mathbf{b}_j$ is not $\mathbf{\Phi}$-torsion. But then we would also need to modify appropriately conclusion (ii) in Conjecture 1.3 by asking that the orbits of $\mathbf{a}_1, \dots, \mathbf{a}_k$ and of $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ under $\mathrm{End}(\mathbf{\Phi})$ meet in a point which is *not* $\mathbf{\Phi}$-torsion. Since very little is earned in terms of generality for our problem by having the sets of points $\mathbf{a}_1, \dots, \mathbf{a}_k$, respectively $\mathbf{b}_1, \dots, \mathbf{b}_\ell$ be $\mathrm{End}_{\overline{L}}(\mathbf{\Phi})$-linearly dependent, we prefer to have the cleaner statement from Conjecture 1.3.

We discuss next the strategy for our proof of Theorem 1.1 and in particular, we will point out the technical difficulties one would have to overcome in order to extend our current result to the more general setting from Conjecture 1.3.

2.2. **The strategy for proving Theorem 1.1.** We work with the notation as in Theorem 1.1. First, we note that the converse implication from Theorem 1.1 (i.e., assuming either conclusion (i) or (ii) holds and then prove that there exist infinitely many $\lambda \in \overline{K}$ with the property that $\mathbf{c}$ is in the orbit under $\Phi^{(\lambda)}$ of both $\mathbf{a}$ and $\mathbf{b}$) is the easier of the two implications; essentially, this is proven in Corollary 4.5.

Now, for the direct implication in Theorem 1.1, there are two main steps in our proof. So, we assume there exists an infinite sequence $\{\lambda_n\}$ in $\overline{K}$ such that $\mathbf{c}$ is in the orbit of both $\mathbf{a}$ and of $\mathbf{b}$ under $\Phi^{(\lambda_n)}$; also, we assume conclusion (i) from Theorem 1.1 does not hold. Then using the technical results from Section 3 regarding heights associated to Drinfeld modules (especially, see Proposition 3.1), we derive the fact that the canonical heights of both $\mathbf{a}$ and of $\mathbf{b}$ under $\Phi^{(\lambda_n)}$ must tend to 0 (see Proposition 4.4). Then we employ [GH13, Theorem 1.5] (see also our Proposition 4.3) to derive that conclusion (ii) must hold in Theorem 1.1.

The same strategy can be followed in attacking Conjecture 1.3 when $k = \ell = 1$; however, there are significant technical difficulties to overcome (for example, see the discussion from Remark 2.1 regarding the alteration of condition (ii)). The main obstacle is the fact that [GH13, Theorem 1.5] does not have an extension beyond its current hypotheses. For example, consider just the slight weakening of the hypotheses from Theorem 1.1 in which we assume $\mathbf{a}, \mathbf{b} \in \overline{K}$ (instead of asking that $\mathbf{a}, \mathbf{b} \in K$). Propositions 3.1 and 4.4 would still hold with the same proof to derive that the corresponding canonical heights $\widehat{h}_{\lambda_n}$ tend to 0, i.e.,

$$\lim_{n\to\infty} \widehat{h}_{\lambda_n}(\mathbf{a}) = \lim_{n\to\infty} \widehat{h}_{\lambda_n}(\mathbf{b}) = 0 \tag{2.2.1}$$

(see Section 3 for the setup regarding the canonical height $\widehat{h}_\lambda$ associated to the Drinfeld module $\Phi^{(\lambda)}$). Now, the arguments from [GH13] (employing the powerful equidistribution theorem of Baker-Rumely [BR10] for points of small height) allow one to deduce from (2.2.1) the fact that

$$\widehat{h}_\lambda(\mathbf{a}) = \widehat{h}_\lambda(\mathbf{b}) \text{ for each } \lambda \in \overline{K}. \tag{2.2.2}$$

However, it is *very difficult* to go from (2.2.2) to the desired conclusion that $\mathbf{a}/\mathbf{b} \in \mathbb{F}_q^*$. Using [GH13, Theorem 2.6], one obtains only that $\mathbf{a}/\mathbf{b} \in \overline{\mathbb{F}_q}^*$; however, if $\mathbf{a}/\mathbf{b} \notin \mathbb{F}_q$, one cannot expect that there will be infinitely many $\lambda \in \overline{K}$ such that $\mathbf{c}$ lies in the orbits of both $\mathbf{a}$ and $\mathbf{b}$ under $\Phi^{(\lambda)}$ because the linear maps $x \mapsto \mu \cdot x$ do not commute with the Drinfeld module action when $\mu \notin \mathbb{F}_q$.

It is worth pointing out that the great difficulty in obtaining the *precise relation* between $\mathbf{a}$ and $\mathbf{b}$ comes from the fact that one lacks the complex potential theory that in [BD11] was used at an archimedean place in order to derive the relation between the starting points a and b for the aforementioned unlikely intersection problem from Section 1.3 involving the family of polynomials $z \mapsto z^d + \lambda$. Actually, in all of the unlikely intersection results obtained in characteristic 0 (see also [BD13, GHT13, GHT15]), the use of complex analysis for studying the Böttcher's coordinate for the polynomials in the given algebraic family was essential for deriving the precise relation between the starting points. Since in positive characteristic we are lacking this last ingredient from complex analysis and we are also missing the precise description of invariant plane curves under the coordinatewise action of one-variable polynomials (as established in the seminal paper of Medvedev-Scanlon [MS14], which was then used in [BD13] for finding the exact relation between starting points sharing an unlikely dynamical property), then Conjecture 1.3 is significantly difficult.

When one deals in Conjecture 1.3 with more general families of Drinfeld modules and also with non-constant starting points $\mathbf{a}, \mathbf{b}$ and non-constant target point $\mathbf{c}$ (which all depend on λ), then even after establishing a similar conclusion as in (2.2.1), [GH13, Theorem 2.6] only says that the canonical heights $\widehat{h}_\lambda(\mathbf{a}(\lambda))$ and $\widehat{h}_\lambda(\mathbf{b}(\lambda))$ are proportional; proving a more precise statement as in conclusion (ii) from Conjecture 1.3 seems *very difficult*. Finally, when $k, \ell > 1$ in Conjecture 1.3, even the first part of the above strategy will have to be significantly altered and perhaps a different relation than equation (2.2.1) needs to be proven.

2.3. **Further generalization for our problem of unlikely intersections.** It is also natural to look for a generalization of our Theorem 1.1 in which we consider a 2-parameter family of Drinfeld modules. Our question could be formulated even more generally (involving families of Drinfeld modules parameterized by points on varieties of arbitrary dimension), but once again, for the sake of concreteness, we prefer to formulate our conjecture as follows.

Conjecture 2.3. *Let $K = \overline{\mathbb{F}_q}(t)$, let $1 < s < r$ be integers and consider the family of Drinfeld modules $\Phi^{(\lambda,\mu)} : \mathbb{F}_q[T] \longrightarrow \mathrm{End}_{K(\lambda,\mu)}(\mathbb{G}_a)$ given by*

$$\Phi_T^{(\lambda,\mu)}(x) = tx + \lambda x^q + \mu x^{q^s} + x^{q^r},$$

parameterized by $\lambda, \mu \in \overline{K}$. Let $\mathbf{a}, \mathbf{b} \in \overline{K}^*$ and let $\mathbf{c}_1, \mathbf{c}_2 \in \overline{K}$. Then the set of points $(\lambda, \mu) \in \mathbb{A}^2(\overline{K})$ for which both $\mathbf{c}_1$ and $\mathbf{c}_2$ are contained in the orbits of both $\mathbf{a}$ under $\Phi^{(\lambda,\mu)}$ and of $\mathbf{b}$ under $\Phi^{(\lambda,\mu)}$ is Zariski dense in $\mathbb{A}^2$ if and only if at least one of the following two conditions must hold:

(i) *there exist $v_1, v_2 \in \mathbb{F}_q^*$ such that either $\mathbf{c}_1 = v_1 \cdot \mathbf{a}$ and $\mathbf{c}_2 = v_2 \cdot \mathbf{b}$, or $\mathbf{c}_1 = v_1 \cdot \mathbf{b}$ and $\mathbf{c}_2 = v_2 \cdot \mathbf{a}$.*

(ii) *there exists $u \in \mathbb{F}_q^*$ such that either $\mathbf{b} = u \cdot \mathbf{a}$, or $\mathbf{c}_2 = u \cdot \mathbf{c}_1$.*

Remark 2.4. When dealing with a 2-parameter family of Drinfeld modules as in Conjecture 2.3, in order to have an *unlikely event*, one needs to ask that *two* points lie in the orbits of $\mathbf{a}$ and of $\mathbf{b}$ under the action of $\Phi^{(\lambda,\mu)}$. Indeed, if we were to ask only that the same point $\mathbf{c}$ lies in the two orbits, then for each nonzero $P, Q \in \mathbb{F}_q[T]$, when we solve the two equations

$$\Phi_P^{(\lambda,\mu)}(\mathbf{a}) = \Phi_Q^{(\lambda,\mu)}(\mathbf{b}) = \mathbf{c}, \tag{2.3.1}$$

we obtain finitely many solutions (λ, μ) and so, as we vary $P, Q \in \mathbb{F}_q[T] \setminus \{0\}$, it is expected that the ensuing set of solutions is Zariski dense in $\mathbb{A}^2$.

As an aside, we note that one could formulate a variant of Conjecture 2.3 involving *three* orbits colliding at the same point; once again, having 3 orbits and only 2 parameters would make this event unlikely and thus, a similar conclusion as in Conjecture 2.3 would be expected.

Remark 2.5. The following example shows the subtlety of the problem raised by Conjecture 2.3. So, consider the case when $r = sk$ for some integer $k > 1$, $\mathbf{c}_1 = 0$, while $\mathbf{c}_2 = u \cdot \mathbf{a}$ for some $u \in \mathbb{F}_q^*$, and moreover, $\mathbf{b} = v \cdot \mathbf{a}$ for some $v \in \mathbb{F}_{q^s} \setminus \mathbb{F}_q$. In this setting, actually conditions (i)-(ii) are *not* met. However, we will see that there exist *infinitely many* pairs $(\lambda, \mu) \in \overline{K} \times \overline{K}$ such that both $\mathbf{c}_1$ and $\mathbf{c}_2$ lie in both the orbits of $\mathbf{a}$ and of $\mathbf{b}$ under the action of $\Phi^{(\lambda,\mu)}$; but *all* these infinitely many found pairs (λ, μ) would lie on one single line in $\mathbb{A}^2$.

We consider the line $\lambda = 0$ in our parameter space, which means that we are considering Drinfeld modules $\Phi^{(0,\mu)} : \mathbb{F}_q[T] \longrightarrow \mathrm{End}(\mathbb{G}_a)$ given by

$$\Phi_T^{(0,\mu)}(x) = tx + \mu x^{q^s} + x^{q^{sk}}. \tag{2.3.2}$$

In particular, $\Phi_P^{(0,\mu)}(v\cdot x) = v\cdot\Phi_P^{(0,\mu)}(x)$ for any $P\in\mathbb{F}_q[T]$, thus showing that 0 is in the orbit of $\mathbf{a}$ under $\Phi^{(0,\mu)}$ if and only if 0 is in the orbit of $\mathbf{b}$ under $\Phi^{(0,\mu)}$. Furthermore, we can extend naturally the action of $\Phi^{(0,\mu)}$ to a Drinfeld module $\tilde{\Phi}^{(\mu)}:\mathbb{F}_{q^s}[T]\longrightarrow \mathrm{End}(\mathbb{G}_a)$ given by

$$\tilde{\Phi}_P^{(\mu)} = \Phi_P^{(0,\mu)} \text{ whenever } P\in\mathbb{F}_q[T]. \tag{2.3.3}$$

We let $w := u/v$; then $w\in\mathbb{F}_{q^s}^*$ and $\mathbf{c}_2 = w\cdot\mathbf{b}$. An argument almost identical with the one from Corollary 4.5 would yield that there exists an infinite set S of elements $\mu\in\overline{K}$ with the property that there exists some nonzero polynomial $P_\mu\in\mathbb{F}_q[T]$ such that

$$\tilde{\Phi}^{(\mu)}_{P_\mu(T)-w}(\mathbf{b}) = 0. \tag{2.3.4}$$

Equations (2.3.3) and (2.3.4) yield that $w\cdot\mathbf{b} = \mathbf{c}_2$ lies in the orbit of $\mathbf{b}$ under the action of $\Phi^{(0,\mu)}$. Moreover, equation (2.3.4) yields that there exists a nonzero polynomial $Q_\mu\in\mathbb{F}_q[T]$ such that $\tilde{\Phi}^{(\mu)}_{Q_\mu}(\mathbf{b}) = 0$ and so, using equation (2.3.3), we get that 0 is in the orbit of $\mathbf{b}$ under the action of $\Phi^{(0,\mu)}$ (and thus, 0 is also in the orbit of $\mathbf{a}$ under the action of $\Phi^{(0,\mu)}$).

Finally, our choice of $\mathbf{c}_2$ shows that for each $\mu\in\overline{K}$, we have that c_2 lies in the orbit of $\mathbf{a}$ under the action of $\Phi^{(0,\mu)}$. So, indeed, we have that for each μ in the infinite set S, both $\mathbf{c}_1$ and $\mathbf{c}_2$ lie in the orbits of both $\mathbf{a}$ and $\mathbf{b}$ under the action of $\Phi^{(0,\mu)}$.

In order to attack Conjecture 2.3, one would need to prove similar results as in [GH13], but this time for points of small height with respect to a 2-parameter family of Drinfeld modules. This is expected to be *difficult* since the similar problem in characteristic 0 (dealing with algebraic families of self-maps parameterized by higher dimensional varieties) is already quite challenging and only few results are known (see [GHT15, Theorem 1.4] and [GHT16]).

Finally, we note that Conjecture 1.3 with $k=\ell=1$ could be formulated in the context of unikely intersections of families of polynomials (or even more generally, rational functions) defined over number fields; the strategy to attack such a question would be similar to the one we employ in this paper. Also, both Conjectures 1.3 and 2.3 have natural formulations in the context of families of elliptic curves (or even more generally, abelian varieties) in which we work with finitely generated groups instead of orbits. However, as we previously stated, our focus is to introduce and study the questions of unlikely intersections in the context of Drinfeld modules both for their intrinsic interest, but also because often in the past, conjectures rising from Drinfeld modules led to deep arithmetic dynamics problems over number fields, as it was the case of the Dynamical Mordell-Lang Conjecture (see [GT08] which was the precursor of [BGT16]).

3. Heights for Drinfeld modules

3.1. **Our family of Drinfeld modules.** Throughout our paper, unless otherwise noted, we have $K := \overline{\mathbb{F}_q}(t)$ and we work with the 1-parameter family of Drinfeld modules $\Phi^{(\lambda)}$: $\mathbb{F}_q[T]\longrightarrow \mathrm{End}_{\overline{K}}(\mathbb{G}_a)$ for which

$$\Phi_T^{(\lambda)}(x) = tx + \lambda x^q + x^{q^r}. \tag{3.1.1}$$

For more details on Drinfeld modules, we refer the reader to [Gos96].

3.2. **The Weil height.** We consider the Weil height $h(\cdot)$ on K which is simply given by

$$h(A(t)/B(t)) = \max\{\deg_t(A), \deg_t(B)\}, \tag{3.2.1}$$

for any two coprime polynomials $A, B \in \overline{\mathbb{F}_q}[t]$ (with $B \neq 0$).

We let Ω_K be the set of inequivalent absolute values on K corresponding to the points of $\mathbb{P}^1(\overline{\mathbb{F}_q})$. More precisely, for each $\alpha \in \overline{\mathbb{F}_q}$, we consider the absolute value $|\cdot|_{v_\alpha}$ defined as follows:

$$\left|\frac{A(t)}{B(t)}\right|_{v_\alpha} = e^{\mathrm{ord}_\alpha(B)-\mathrm{ord}_\alpha(A)},$$

for any nonzero polynomials $A, B \in \overline{\mathbb{F}_q}[t]$, where $\mathrm{ord}_\alpha(C)$ is the order of vanishing at the point α of the nonzero polynomial $C \in \overline{\mathbb{F}_q}[t]$. Also, for the point at infinity of $\mathbb{P}^1(\overline{\mathbb{F}_q})$, we have the absolute value $|\cdot|_{v_\infty}$ for which

$$\left|\frac{A(t)}{B(t)}\right|_{v_\infty} = e^{\deg(A)-\deg(B)},$$

for any two nonzero polynomials $A, B \in \overline{\mathbb{F}_q}[t]$. We see that with this normalization for the absolute values from Ω_K, we have that for each $\mathbf{a} \in K$, we have

$$h(\mathbf{a}) = \sum_{v \in \Omega_K} \log^+ |\mathbf{a}|_v,$$

where for any nonnegative real number x, we have $\log^+(x) := \max\{0, \log(x)\}$.

Then we extend coherently the Weil height for all elements of $\overline{K}$; so, for any finite field extension L/K and for each place v of K, there exist finitely many places w of L lying over v (denoted $w|v$) with the corresponding absolute values $|\cdot|_w$ normalized such that for each nonzero $\mathbf{a} \in K$, we have

$$\log |\mathbf{a}|_v = \sum_{w|v} \log |\mathbf{a}|_w \tag{3.2.2}$$

In particular, each finite extension L of K is a *product formula field* with respect to the set Ω_L of normalized absolute values on L, i.e., for any nonzero $\alpha \in L$, we have that

$$\prod_{w \in \Omega_L} |\alpha|_w = 1. \tag{3.2.3}$$

Now, given a finite extension L/K, the Weil height of a point $\alpha \in L$ is given by

$$h(\alpha) = \sum_{w \in \Omega_L} \log^+ |\alpha|_w. \tag{3.2.4}$$

Due to the coherence of the normalization for the absolute values (see (3.2.2)), the formula (3.2.4) for the Weil height of a point $\alpha \in \overline{K}$ does not depend on the particular choice of the finite extension L/K containing α.

We also note that in our formulas (3.2.2) and (3.2.4), we do not divide the right-hand side by $[L : K]$; this is part of our normalization of the absolute values $w \in \Omega_L$, which is possible since for any nonarchimedean absolute value $|\cdot|$ and for any positive real number d, also $|\cdot|^d$ is a nonarchimedean absolute value (see also [GH13, Equation (2.2)]).

3.3. **The canonical height for Drinfeld modules.** For each $\lambda \in \overline{K}$, we define the canonical height $\widehat{h}_\lambda : \overline{K} \longrightarrow \mathbb{Q}_{\ge 0}$ corresponding to the Drinfeld module $\Phi^{(\lambda)}$ (see (3.1.1)) as follows:

$$\widehat{h}_\lambda(\alpha) := \lim_{n\to\infty} \frac{h\left(\Phi^{(\lambda)}_{T^n}(\alpha)\right)}{q^{rn}}. \tag{3.3.1}$$

The formula from (3.3.1) yields (see [Den92]) that for any nonzero $R \in \mathbb{F}_q[T]$, we have

$$\widehat{h}_\lambda\left(\Phi^{(\lambda)}_R(\alpha)\right) = q^{r\cdot \deg_T(R)} \cdot \widehat{h}_\lambda(\alpha). \tag{3.3.2}$$

In particular, a point $\alpha \in \overline{K}$ is torsion for Φ_λ (i.e., there exists some nonzero $R \in \mathbb{F}_q[T]$ such that $\Phi^{(\lambda)}_R(\alpha) = 0$) if and only if $\widehat{h}_\lambda(\alpha) = 0$; for more properties regarding the canonical height for arbitrary Drinfeld modules, see [Den92, Ghi07a, Ghi07b].

3.4. **Local canonical heights for Drinfeld modules.** The canonical height (from (3.3.1)) can also be expressed as a sum of local canonical heights. More precisely, given a finite field extension L/K and given $\alpha, \lambda \in L$, then for any $w \in \Omega_L$, we define the local canonical height of α with respect to the place w as follows:

$$\widehat{h}_{\lambda,w}(\alpha) := \lim_{n\to\infty} \frac{\log^+\left|\Phi^{(\lambda)}_{T^n}(\alpha)\right|_w}{q^{rn}}. \tag{3.4.1}$$

Then one can prove (see [Den92, Ghi07a, Ghi07b]) that

$$\widehat{h}_\lambda(\alpha) = \sum_{w\in\Omega_L} \widehat{h}_{\lambda,w}(\alpha), \tag{3.4.2}$$

where only finitely many terms from the right-hand side of equation (3.4.2) are nonzero.

3.5. **A technical result.** In our proof of Theorem 1.1, we will employ the following result regarding the variation of the canonical heights in the Drinfeld modules family $\left\{\Phi^{(\lambda)}\right\}_{\lambda\in\overline{K}}$.

Proposition 3.1. *Let $\mathbf{a} \in \overline{K}$. Then there exist positive constants M_1 and M_2 (depending only on $h(\mathbf{a})$) with the property that for each $\lambda \in \overline{K}$, we have*

$$-M_1 + \frac{h(\lambda)}{q^r} < \widehat{h}_\lambda(\mathbf{a}) < \frac{h(\lambda)}{q^r} + M_2. \tag{3.5.1}$$

Proposition 3.1 fits in a long series of results regarding the variation of the canonical height in an algebraic family of maps. Variants of Proposition 3.1 were proven over number fields in the case of elliptic curves by Tate [Tat83], and then extended to all abelian varieties by Silverman [Sil83] (see also [CS93, Ing13, GM13] for similar results for polynomial families and also certain families of rational functions).

Proof of Proposition 3.1. First, we note that if $\mathbf{a} = 0$, then we could simply take $M_1 = M_2 = 1$ and inequalities (3.5.1) would be verified since $\widehat{h}_\lambda(0) = 0$ for each $\lambda \in \overline{K}$. So, from now on, we assume that $\mathbf{a} \neq 0$.

Let $\lambda \in \overline{K}$ and let L be a finite extension of K containing both $\mathbf{a}$ and λ. The right-hand side inequality from (3.5.1) is easier. Indeed, for each $w \in \Omega_L$, we have that

$$\log^+\left|\Phi^{(\lambda)}_T(\mathbf{a})\right|_w \le \log^+|\lambda|_w + \log^+|t|_w + q^r \cdot \log^+|\mathbf{a}|_w \tag{3.5.2}$$

and then letting $M_w := \log^+|\lambda|_w + \log^+|t|_w + q^r\cdot\log^+|\mathbf{a}|_w$, a simple induction yields that for each $n\ge 1$, we have

$$\log^+\left|\Phi_{T^n}^{(\lambda)}(\mathbf{a})\right|_w \le q^{r(n-1)}\cdot M_w. \tag{3.5.3}$$

Therefore, using (3.4.1), we get

$$\widehat{h}_{\lambda,w}(\mathbf{a}) \le \frac{M_w}{q^r} = \frac{\log^+|\lambda|_w}{q^r} + \frac{\log^+|t|_w + q^r\log^+|\mathbf{a}|_w}{q^r}. \tag{3.5.4}$$

Using inequality (3.5.4) for each $w\in\Omega_L$ and also combining it with (3.4.2) along with the formula (3.2.4), we obtain that

$$\widehat{h}_{\lambda}(\mathbf{a}) \le \frac{h(\lambda)}{q^r} + \frac{h(t)+q^r\cdot h(\mathbf{a})}{q^r} \tag{3.5.5}$$

and thus, we obtain the right-hand side of our desired inequality (3.5.1) with $M_2 := 1 + h(\mathbf{a})$, for example (note that $h(t)=1$ according to (3.2.1)).

In order to establish the left-hand side of the inequality (3.5.1), we pick any $w\in\Omega_L$; then there are two cases depending on $|\lambda|_w$.

Case 1. Assume first that

$$\log^+|\lambda|_w \le q^r\cdot\left(\log^+|\mathbf{a}|_w + \log^+|1/\mathbf{a}|_w + \log^+|t|_w\right). \tag{3.5.6}$$

We let

$$N_w := \log^+|\mathbf{a}|_w + \log^+|1/\mathbf{a}|_w + \log^+|t|_w. \tag{3.5.7}$$

Inequality (3.5.6) yields simply that

$$\widehat{h}_{\lambda,w}(\mathbf{a}) \ge 0 \ge \frac{\log^+|\lambda|_w}{q^r} - N_w. \tag{3.5.8}$$

Case 2. Assume now that

$$\log^+|\lambda|_w > q^r\cdot N_w \tag{3.5.9}$$

(see (3.5.7) for the definition of N_w). In particular, we would have that $|\lambda|_w > 1$ since $N_w \ge 0$. The inequality (3.5.9) yields that

$$\left|\Phi_{T}^{(\lambda)}(\mathbf{a})\right|_w = |\lambda\cdot\mathbf{a}^q|_w \tag{3.5.10}$$

because $|\lambda\mathbf{a}^q|_w > \max\left\{|t\mathbf{a}|_w, |\mathbf{a}|_w^{q^r}\right\}$. Furthermore, inequality (3.5.9) combined with formula (3.5.10) yields that

$$\left|\Phi_{T}^{(\lambda)}(\mathbf{a})\right|_w = |\lambda|_w\cdot|\mathbf{a}|_w^q > |\lambda|_w^{\frac{q^r-q}{q^r}} \ge \max\left\{1, |t|_w^{\frac{1}{q^r-1}}, |\lambda|_w^{\frac{1}{q^r-q}}\right\} \tag{3.5.11}$$

where in the last inequality from (3.5.11), we also use the fact that $q,r\ge 2$. Inequality (3.5.11) yields that

$$\left|\Phi_{T^2}^{(\lambda)}(\mathbf{a})\right|_w = \left|\Phi_{T}^{(\lambda)}(\mathbf{a})^{q^r}\right|_w > \left|\Phi_{T}^{(\lambda)}(\mathbf{a})\right|_w. \tag{3.5.12}$$

Inequality (3.5.12) coupled with an easy induction yields that for each $n\in\mathbb{N}$, we have

$$\left|\Phi_{T^n}^{(\lambda)}(\mathbf{a})\right|_w = \left|\Phi_{T}^{(\lambda)}(\mathbf{a})\right|_w^{q^{r(n-1)}} = |\lambda\cdot\mathbf{a}|_w^{q^{r(n-1)}}. \tag{3.5.13}$$

Equation (3.5.13) along with formula (3.4.1) yields that

$$\widehat{h}_{\lambda,w}(\mathbf{a}) = \frac{\log|\lambda\cdot\mathbf{a}|_w}{q^r} \geq \frac{\log^+|\lambda|}{q^r} - \frac{\log^+|1/\mathbf{a}|_w}{q^r} \geq \frac{\log^+|\lambda|_w}{q^r} - N_w, \tag{3.5.14}$$

where in the inequality (3.5.14) we also employed the fact that $|\lambda|_w > 1$, along with the fact that $N_w \geq \log^+|1/\mathbf{a}|_w$ (see (3.5.7)).

Now, a simple computation using formula (3.2.4) gives:

$$\sum_{w\in\Omega_L} N_w \tag{3.5.15}$$

$$= \sum_{w\in\Omega_L}\log^+|\mathbf{a}| + \sum_{w\in\Omega_L}\log^+|1/\mathbf{a}|_w + \sum_{w\in\Omega_L}\log^+|t|_w \tag{3.5.16}$$

$$= h(\mathbf{a}) + h(1/\mathbf{a}) + h(t) \tag{3.5.17}$$

$$= 2h(\mathbf{a}) + 1, \tag{3.5.18}$$

where for the last equality, we employed both the fact that $h(t) = 1$ (see (3.2.1)) and also the fact that $h(1/\mathbf{a}) = h(\mathbf{a})$ (since $\mathbf{a}\neq 0$ and (L,Ω_L) is a product formula field according to the formula (3.2.3)). Then employing inequalities (3.5.14) and (3.5.8) for each $w\in\Omega_L$ coupled with formulas (3.4.2), (3.2.4) and (3.5.18), we get

$$\widehat{h}_\lambda(\mathbf{a}) \geq \frac{h(\lambda)}{q^r} - 2h(\mathbf{a}) - 1; \tag{3.5.19}$$

hence we may take $M_1 := 2h(\mathbf{a}) + 2$ for the left-hand side of our desired inequality (3.5.1). This concludes our proof of Proposition 3.1. □

4. Proof of our main result

We continue with our previous setup for the 1-parameter family of Drinfeld modules $\Phi^{(\lambda)}$ given by formula (3.1.1). We also employ the setup for heights associated to Drinfeld modules from Section 3 (see formula for $\widehat{h}_\lambda(\cdot)$ from equation (3.3.1)).

4.1. **Technical preliminaries.** We start with an easy result.

Lemma 4.1. *Let $\mathbf{a}\in\overline{K}^*$ and let $P\in\mathbb{F}_q[T]$ be a polynomial of degree $m\geq 1$. Then $\lambda\mapsto\Phi_P^{(\lambda)}(\mathbf{a})$ is a polynomial of degree $q^{r(m-1)}$.*

Proof. Using the definition of a Drinfeld module, it suffices to prove that $\lambda\mapsto\Phi_{T^m}^{(\lambda)}(\mathbf{a})$ is a polynomial of degree $q^{r(m-1)}$. This follows immediately by induction on m (using the formula (3.1.1) for $\Phi_T^{(\lambda)}$); a more general result is proven in [GH13, Lemma 4.2]. □

Lemma 4.1 has the following corollary.

Corollary 4.2. *Let $\mathbf{c}\in\overline{K}$ and let $\mathbf{a}\in\overline{K}^*$ such that $\mathbf{c}/\mathbf{a}\notin\mathbb{F}_q^*$. Then for each nonzero $P\in\mathbb{F}_q[T]$, there exist finitely many $\lambda\in\overline{K}$ such that $\Phi_P^{(\lambda)}(\mathbf{a}) = \mathbf{c}$.*

Proof. First of all, we see that if $P\in\mathbb{F}_q[T]\setminus\{0\}$ is a constant polynomial (equal to some $u\in\mathbb{F}_q^*$), then the equation $\Phi_P^{(\lambda)}(\mathbf{a}) = \mathbf{c}$ has no solutions $\lambda\in\overline{K}$ since this equation is equivalent with asking that $u\cdot\mathbf{a} = \mathbf{c}$.

Now, assuming that $\deg_T(P) = n\geq 1$, then Lemma 4.1 yields that $\deg_\lambda(\Phi_P^{(\lambda)}(\mathbf{a})) = q^{r(n-1)}$. Hence there exist finitely many $\lambda\in\overline{K}$ such that $\Phi_P^{(\lambda)}(\mathbf{a}) = \mathbf{c}$, as claimed. □

We will also employ in the proof of Theorem 1.1 the following technical consequence of [GH13, Theorem 1.5].

Proposition 4.3. *Let $r \ge 2$ be an integer, let $K = \overline{\mathbb{F}_q}(t)$ and let $\Phi^{(\lambda)} : \mathbb{F}_q[T] \longrightarrow \mathrm{End}_{K(\lambda)}(\mathbb{G}_a)$ be the Drinfeld module family given by*

$$\Phi_T^{(\lambda)}(x) = tx + \lambda x^q + x^{q^r}.$$

Let $\mathbf{a}, \mathbf{b} \in K^*$ with the property that there exist infinitely many $\lambda_n \in \overline{K}$ such that $\widehat{h}_{\lambda_n}(\mathbf{a}) \to 0$ and $\widehat{h}_{\lambda_n}(\mathbf{b}) \to 0$ as $n \to \infty$. Then $\mathbf{a}/\mathbf{b} \in \mathbb{F}_q^*$.

Proof. This is essentially proven in [GH13, Theorem 1.5]; there are two differences in the statement of [GH13, Theorem 1.5] from our Proposition 4.3 which we will explain below.

First, in [GH13, Theorem 1.5], there is the assumption that the λ_n belong to the *separable* closure of K; this is the underlying assumption throughout all the proofs of [GH13]. However, as explained also in [GS22, Remark 1.1], this is not necessary since throughout the proofs of [GH13], one could have worked with the perfect closure of a function field (i.e., work with K^{per} instead of K), which is still a product formula field and then the main equidistribution theorem of Baker-Rumely [BR10, Theorem 7.52] would still apply because their results only require an arbitrary product formula field (see [BR10, Definition 7.51, p. 185]); then the arguments from [GH13] follow verbatim.

Second, in [GH13, Theorem 1.5], we have the assumption that the parameters λ_n correspond to torsion points, i.e., $\widehat{h}_{\lambda_n}(\mathbf{a}) = \widehat{h}_{\lambda_n}(\mathbf{b}) = 0$. However, in order to apply the equidistribution theorem of Baker-Rumely [BR10, Theorem 7.52] for proving [GH13, Theorem 2.6] (which is the main technical result of [GH13] from which [GH13, Theorem 1.5] is derived), it suffices to assume that $\widehat{h}_{\lambda_n}(\mathbf{a}), \widehat{h}_{\lambda_n}(\mathbf{b}) \to 0$ as $n \to \infty$. ☐

4.2. **A key ingredient for the proof of our main result.** The next result is key to proving Theorem 1.1

Proposition 4.4. *Let $\mathbf{a} \in \overline{K}^*$ and $\mathbf{c} \in \overline{K}$. Let $\{P_n\}_{n\ge1}$ be a sequence of elements in $\mathbb{F}_q[T]$ and let $\{\lambda_n\}_{n\ge1}$ be a sequence of elements in $\overline{K}$ such that*

$$\Phi_{P_n}^{(\lambda_n)}(\mathbf{a}) = \mathbf{c}. \tag{4.2.1}$$

If $\deg_T(P_n) \to \infty$ as $n \to \infty$, then $\widehat{h}_{\lambda_n}(\mathbf{a}) \to 0$ as $n \to \infty$.

Proof. First we prove that $h(\lambda_n)$ is bounded above. Indeed, assuming otherwise, then (at the expense of replacing $\{\lambda_n\}$ with a subsequence) we would have that $h(\lambda_n) \to \infty$ as $n \to \infty$. Proposition 3.1 yields that there exists a positive constant M (depending on $h(\mathbf{a})$ and $h(\mathbf{c})$) such that

$$-M + \frac{h(\lambda_n)}{q^r} < \widehat{h}_{\lambda_n}(\mathbf{a}) < \frac{h(\lambda_n)}{q^r} + M \tag{4.2.2}$$

and also,

$$-M + \frac{h(\lambda_n)}{q^r} < \widehat{h}_{\lambda_n}(\mathbf{c}) < \frac{h(\lambda_n)}{q^r} + M. \tag{4.2.3}$$

Pick $N \in \mathbb{N}$ with the property that $h(\lambda_n) > 2q^r M$ whenever $n \ge N$ (since we asumed that $h(\lambda_n) \to \infty$ as $n \to \infty$). Then the first inequality in equation (4.2.2) yields

$$\frac{h(\lambda_n)}{2q^r} < \widehat{h}_{\lambda_n}(\mathbf{a}). \tag{4.2.4}$$

On the other hand, the second inequality from equation (4.2.3) yields (note also that $q^r > 2$) that

$$\widehat{h}_{\lambda_n}(\mathbf{c}) < h(\lambda_n). \tag{4.2.5}$$

Since $\deg_T(P_n) \to \infty$ and $\Phi^{(\lambda_n)}_{P_n}(\mathbf{a}) = \mathbf{c}$, while

$$\widehat{h}_{\lambda_n}\left(\Phi^{(\lambda_n)}_{P_n}(\mathbf{a})\right) = q^{r\cdot\deg_T(P_n)} \cdot \widehat{h}_{\lambda_n}(\mathbf{a}), \tag{4.2.6}$$

then we see that equations (4.2.4) and (4.2.5) are contradictory. So, indeed, we may assume that the sequence $\{h(\lambda_n)\}_n$ is bounded above. Then Proposition 3.1 yields that also the sequence $\{\widehat{h}_{\lambda_n}(\mathbf{c})\}_n$ is bounded above. Since $\mathbf{c} = \Phi^{(\lambda_n)}_{P_n}(\mathbf{a})$, then equation (4.2.6) along with the fact that $\deg_T(P_n) \to \infty$ allows us to obtain that $\widehat{h}_{\lambda_n}(\mathbf{a}) \to 0$, as $n \to \infty$.

This concludes our proof of Proposition 4.4. □

The following result is a consequence of our Proposition 4.4 and it will be used in proving the converse implication from Theorem 1.1.

Corollary 4.5. *Let $\mathbf{a} \in \overline{K}^*$ and let $\mathbf{c} \in \overline{K}$. There exist infinitely many $\lambda \in \overline{K}$ for which there exists some nonzero $P_\lambda \in \mathbb{F}_q[T]$ such that $\Phi^{(\lambda)}_{P_\lambda}(\mathbf{a}) = \mathbf{c}$.*

Proof. First, assume $\mathbf{c} = 0$. We consider an infinite sequence $\{P_n\}_{n\ge 1}$ of monic irreducible polynomials in $\mathbb{F}_q[T]$ with $\deg_T(P_n) = n$; in particular, the polynomials from our sequence are pairwise coprime. For each n, we consider the equation

$$\Phi^{(\lambda)}_{P_n}(\mathbf{a}) = 0; \tag{4.2.7}$$

then Lemma 4.1 yields that this is a polynomial equation of degree $q^{r(n-1)}$ in λ and so, we pick a solution $\lambda_n \in \overline{K}$ for equation (4.2.7). We claim that the sequence $\{\lambda_n\}_{n\ge 1}$ consists of distinct elements. Indeed, if $\lambda_m = \lambda_n$ for some $m < n$, then letting $\mu := \lambda_m = \lambda_n$, using (4.2.7) we get

$$\Phi^{(\mu)}_{P_m}(\mathbf{a}) = \Phi^{(\mu)}_{P_n}(\mathbf{a}) = 0. \tag{4.2.8}$$

Since P_m and P_n are coprime, then equation (4.2.8) yields that $\mathbf{a} = 0$, contradiction. So, indeed, the conclusion of Corollary 4.5 holds when $\mathbf{c} = 0$.

Now, assume $\mathbf{c} \ne 0$. For each $n \ge 1$, we let

$$R_n(T) := \prod_{\substack{Q\in\mathbb{F}_q[T]\\ 1\le\deg_T(Q)\le n}} Q(T).$$

Then for each positive integer n, we pick some $\lambda_n \in \overline{K}$ such that $\Phi^{(\lambda_n)}_{R_n}(\mathbf{a}) = \mathbf{c}$ (note that Lemma 4.1 yields that $\Phi^{(\lambda)}_{R_n}(\mathbf{a})$ is a polynomial of degree $q^{r(\deg_T(R_n)-1)}$ in λ). But then Proposition 4.4 yields that

$$\widehat{h}_{\lambda_n}(\mathbf{a}) \to 0. \tag{4.2.9}$$

If there are only finitely many distinct λ_n in our sequence, then equation (4.2.9) yields that actually for some $\tilde{\lambda}$ appearing infinitely often in our sequence $\{\lambda_n\}$, we would have that $\widehat{h}_{\tilde{\lambda}}(\mathbf{a}) = 0$; hence, $\mathbf{a}$ is a torsion point for $\Phi^{(\tilde{\lambda})}$, i.e., there exists some nonzero polynomial $Q \in \mathbb{F}_q[T]$ such that $\Phi^{(\tilde{\lambda})}_Q(\mathbf{a}) = 0$. Furthermore, Q must be a non-constant polynomial since $\mathbf{a} \ne 0$. However, since there exist infinitely many λ_n in our sequence which equal $\tilde{\lambda}$, then for

$n \geq \deg_T(Q)$, we see that $\Phi^{(\tilde{\lambda})}_{R_n}(\mathbf{a}) = 0$ because R_n is divisible by all polynomials (in T) of degree at most n (and so, Q divides R_n). But then

$$\mathbf{c} = \Phi^{(\lambda_n)}_{R_n}(\mathbf{a}) = \Phi^{(\tilde{\lambda})}_{R_n}(\mathbf{a}) = 0,$$

contradicting our assumption that $\mathbf{c} \neq 0$. So, indeed, there are infinitely many distinct elements λ_n in our sequence for which $\Phi^{(\lambda_n)}_{R_n}(\mathbf{a}) = \mathbf{c}$.

This concludes our proof of Corollary 4.5. □

4.3. **Proof of Theorem 1.1.** We work with the notation and hypotheses from Theorem 1.1.

We first note that it is immediate to get the *"moreover"* part in our desired conclusion (ii) since for any $\lambda, x \in \overline{K}$, for any $R \in \mathbb{F}_q[T]$ and for any $u \in \mathbb{F}_q$, we have that

$$\Phi^{(\lambda)}_R(u \cdot x) = u \cdot \Phi^{(\lambda)}_R(x). \tag{4.3.1}$$

Also, equation (4.3.1) along with Corollary 4.5 immediately delivers the reverse implication from our desired conclusion in Theorem 1.1. Indeed, if conclusion (ii) in Theorem 1.1 is met, i.e., if $\mathbf{b} = u \cdot \mathbf{a}$ for some $u \in \mathbb{F}_q^*$ then whenever $\Phi^{(\lambda)}_{P_\lambda}(\mathbf{b}) = \mathbf{c}$ for some $P_\lambda \in \mathbb{F}_q[T]$ yields that also $\Phi^{(\lambda)}_{u \cdot P_\lambda}(\mathbf{a}) = \mathbf{c}$. Furthermore, note that Corollary 4.5 yields the existence of infinitely many $\lambda \in \overline{K}$ such that for some $P_\lambda \in \mathbb{F}_q[T]$ we have $\Phi^{(\lambda)}_{P_\lambda}(\mathbf{b}) = \mathbf{c}$. On the other hand, if conclusion (i) is met, i.e., $v \cdot \mathbf{a} = \mathbf{c}$ for some $v \in \mathbb{F}_q^*$, then for *each* $\lambda \in \overline{K}$, we have that $\Phi^{(\lambda)}_v(\mathbf{a}) = \mathbf{c}$ and so, once again invoking Corollary 4.5, we get that there exist infinitely many $\lambda \in \overline{K}$ such that there exists some nonzero $Q_\lambda \in \mathbb{F}_q[T]$ for which $\Phi^{(\lambda)}_{Q_\lambda}(\mathbf{b}) = \mathbf{c}$, as desired.

So, we are left to prove the main implication in Theorem 1.1, i.e., we assume there exists an infinite sequence $\{\lambda_n\}_{n\geq 1}$ of elements in $\overline{K}$ and there exist two sequences $\{P_n\}_{n\geq 1}$ and $\{Q_n\}_{n\geq 1}$ of elements in $\mathbb{F}_q[T]$ such that

$$\Phi^{(\lambda_n)}_{P_n}(\mathbf{a}) = \mathbf{c} \text{ and } \Phi^{(\lambda_n)}_{Q_n}(\mathbf{b}) = \mathbf{c}. \tag{4.3.2}$$

Furthermore, we may assume that $\mathbf{c}/\mathbf{a} \notin \mathbb{F}_q^*$ and also $\mathbf{c}/\mathbf{b} \notin \mathbb{F}_q^*$ since otherwise condition (i) is met.

Using Corollary 4.2, we obtain that the degrees (in T) of the polynomials P_n and Q_n must grow to infinity. By Proposition 4.4, we obtain that $\widehat{h}_{\lambda_n}(\mathbf{a}) \to 0$ and $\widehat{h}_{\lambda_n}(\mathbf{b}) \to 0$ as $n \to \infty$. Then Proposition 4.3 yields that $\mathbf{b}/\mathbf{a} \in \mathbb{F}_q^*$, as desired.

This concludes our proof of Theorem 1.1.

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Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Canada V6T 1Z2

Email address: `dghioca@math.ubc.ca`