

FINITE INDEX THEOREMS FOR ITERATED GALOIS GROUPS OF UNICRITICAL POLYNOMIALS

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ABSTRACT. Let K be the function field of a smooth irreducible curve defined over $\overline{\mathbb{Q}}$. Let $f \in K[x]$ be of the form $f(x) = x^q + c$ where $q = p^r, r \geq 1$, is a power of the prime number p , and let $\beta \in \overline{K}$. For all $n \in \mathbb{N} \cup \{\infty\}$, the Galois groups $G_n(\beta) = \text{Gal}(K(f^{-n}(\beta))/K(\beta))$ embed into $[C_q]^n$, the n -fold wreath product of the cyclic group C_q . We show that if f is not isotrivial, then $[[C_q]^\infty : G_\infty(\beta)] < \infty$ unless β is postcritical or periodic. We are also able to prove that if $f_1(x) = x^q + c_1$ and $f_2(x) = x^q + c_2$ are two such distinct polynomials, then the fields $\bigcup_{n=1}^\infty K(f_1^{-n}(\beta))$ and $\bigcup_{n=1}^\infty K(f_2^{-n}(\beta))$ are disjoint over a finite extension of K .

1. INTRODUCTION AND STATEMENT OF RESULTS

Let K be a field. Let $f \in K(x)$ with $d = \deg f \geq 2$ and let $\beta \in \mathbb{P}^1(\overline{K})$. For $n \in \mathbb{N}$, let $K_n(f, \beta) = K(f^{-n}(\beta))$ be the field obtained by adjoining the n th preimages of β under f to $K(\beta)$. (We declare that $K(\infty) = K$.) Set $K_\infty(f, \beta) = \bigcup_{n=1}^\infty K_n(f, \beta)$. For $n \in \mathbb{N} \cup \{\infty\}$, define $G_n(f, \beta) = \text{Gal}(K_n(f, \beta)/K(\beta))$. In most of the paper, we will write $G_n(\beta)$ and $K_n(\beta)$, suppressing the dependence on f if there is no ambiguity.

The group $G_\infty(\beta)$ embeds into $\text{Aut}(T_\infty^d)$, the automorphism group of an infinite d -ary rooted tree T_∞^d . Recently there has been much work on the problem of determining when the index $[\text{Aut}(T_\infty^d) : G_\infty(\beta)]$ is finite. The group $G_\infty(\beta)$ is the image of an arboreal Galois representation, so this finite index problem is a natural analog in arithmetic dynamics of the finite index problem for the ℓ -adic Galois representations associated to elliptic curves, resolved by Serre's celebrated Open Image Theorem [Ser72]. By work of Odoni [Odo85], one expects that a generically chosen rational function has a surjective arboreal representation, i.e., that $[\text{Aut}(T_\infty^d) : G_\infty(\beta)] = 1$.

In this paper we study the family of polynomials $f(x) = x^d + c$ for $c \in K$, which up to change of variables over a finite extension of K represents all polynomials with precisely one (finite) critical point (note that the finite index property studied in our paper is unchanged when we replace K by

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a finite extension). If the field K contains a primitive d th root of unity, then it is easy to show that for f in this family, $G_\infty(\beta)$ sits in $[C_d]^\infty$, the infinite iterated wreath product of the cyclic group C_d (with d elements). As $\text{Aut}(T_n^d) \cong [S_d]^n$, this means that if $d \geq 3$, then $[\text{Aut}(T_\infty^d) : [C_d]^\infty] = \infty$. Thus it is impossible for $G_\infty(\beta)$ to have finite index within this family (except when $d = 2$). However, this simply means that, given the constraint on the size of $G_\infty(\beta)$, we should ask a different finite index question. We turn to the problem of when $G_\infty(\beta)$ has finite index in $[C_d]^\infty$; note that for a generic starting point β , then the image of $G_\infty(\beta)$ is indeed $[C_d]^\infty$ (see Proposition 2.3).

Before stating our main results, we set some notation. Throughout this paper, unless otherwise indicated, K will refer to a function field of transcendence degree 1 over its field of constants $\overline{\mathbb{Q}}$. In other words, K is the function field of a smooth projective irreducible curve C over $\overline{\mathbb{Q}}$. We say that $f \in K[x]$ is isotrivial if f is defined over $\overline{\mathbb{Q}}$ up to a change of variables, that is, if $\varphi^{-1} \circ f \circ \varphi \in \overline{\mathbb{Q}}[x]$ for some $\varphi \in \overline{K}[x]$ of degree 1. In the special case of a unicritical polynomial $f(x) = x^d + c \in K[x]$, we have that f is isotrivial if and only if $c \in \overline{\mathbb{Q}}$. We say $\beta \in \overline{K}$ is periodic for f if $f^n(\beta) = \beta$ for some $n \geq 1$, and we say β is preperiodic for f if $f^m(\beta)$ is periodic for some $m \geq 0$. Finally, we say that β is postcritical for f if $f^n(\alpha) = \beta$ for some $n \geq 1$ and some critical point α of f ; since in this paper we deal with the polynomial $f(z) := z^d + c$, then β is postcritical if it is in the orbit of 0 under f .

With this notation, our first main theorem is as follows.

Theorem 1.1. *Let $q = p^r$ ($r \geq 1$) be a power of the prime number p , let $c \in K \setminus \overline{\mathbb{Q}}$, let $f(x) = x^q + c \in K[x]$ and let $\beta \in K$. Then the following are equivalent:*

- (1) *The point β is neither periodic nor postcritical for f .*
- (2) *The group $G_\infty(\beta)$ has finite index in $[C_q]^\infty$.*

All the methods used in the proof of Theorem 1.1 work for unicritical polynomials of any degree d , except that we need the degree to be a prime power for proving the eventual stability of (f, β) (see Theorem 1.3 below and Section 6). More precisely, we require the degree d of our polynomial be a prime power p^r so that the map $x \mapsto x^d$ is injective on $\overline{\mathbb{F}_p}$; this fact is crucial in our proof of Lemma 6.2 (which follows from [JL17]). In the case where $q = 2$, this means that $G_\infty(\beta)$ has finite index in $\text{Aut}(T_\infty^2)$. For larger q this index is infinite, as mentioned previously. The case of isotrivial polynomials (i.e., when $c \in \overline{\mathbb{Q}}$ in Theorem 1.1) is very different and will be dealt with in Section 10.

It is fairly easy to see that the conditions on β in Theorem 1.1 are necessary. If β is periodic or postcritical, then $[[C_q]^\infty : G_\infty(\beta)] = \infty$ by a straightforward argument (see Proposition 3.2). Most of the paper is devoted to the showing that these conditions are sufficient.

Remark 1.2. In general one needs to rule out postcritically finite (PCF) maps in order to obtain a finite index result, as in the main result of [BT18b]. The reason we do not need to do this in Theorem 1.1 is that a PCF polynomial of the form $f(x) = x^q + c$ is automatically isotrivial. This is because c satisfies an equation of the form $f^n(c) = f^m(c)$ for some $n > m \geq 0$, and so $c \in \overline{\mathbb{Q}}$. For isotrivial polynomials the PCF distinction regains its importance; see Section 10.

One of the key steps in the proof of Theorem 1.1 is an eventual stability result. As is usual in arithmetic dynamics, we say that the pair (f, β) is eventually stable over the field K if the number of irreducible K -factors of $f^n(x) - \beta$ is uniformly bounded for all n .

Theorem 1.3. *Let $q = p^r$ ($r \geq 1$) be a power of the prime number p . Let $f \in K[x]$ be a polynomial of the form $x^q + t$ where $t \notin \overline{\mathbb{Q}}$. Then for any non-periodic $\beta \in K$, the pair (f, β) is eventually stable over K .*

We also prove the following disjointness theorem for fields generated by inverse images of different points under different maps.

Theorem 1.4. *For $i = 1, \dots, s$ let $f_i(x) = x^q + c_i \in K[x]$, where $c_i \notin \overline{\mathbb{Q}}$, and let $\alpha_i \in K$. Suppose that there are no distinct i, j with the property that (α_i, α_j) lies on a curve in \mathbb{A}^2 that is periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$. For each i , let M_i denote $K_\infty(f_i, \alpha_i)$. Then for each $i = 1, \dots, s$, we have that*

$$\left[M_i \cap \left(\prod_{j \neq i} M_j \right) : K \right] < \infty.$$

Remark 1.5. In Theorem 1.4, since each c_i is not in $\overline{\mathbb{Q}}$, then [GNV19, Theorems 1.4 and 1.5] yield the following:

- if there is no $(d - 1)$ -st root of unity ζ such that $c_i = \zeta c_j$, then (α_i, α_j) lies on a curve in \mathbb{A}^2 periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$ if and only if either α_i is periodic for f_i or α_j is periodic for f_j .
- if there is some $(d - 1)$ -st root of unity ζ such that $c_i = \zeta c_j$, then (α_i, α_j) lies on a periodic curve under the action of $(x, y) \mapsto (f_i(x), f_j(y))$ if and only if either α_i is periodic for f_i , or α_j is periodic for f_j , or $x = \zeta f_j^m(y)$ for some $m \geq 0$, or $y = f_j^m(\zeta^{-1}x)$ for some $m \geq 1$.

Theorem 1.4 also has a natural interpretation as a finite index result across pre-image trees of several points (see Section 9).

Remark 1.6. In light of Odoni's work, unicritical polynomials with degree $d \geq 3$ cannot be considered generic from the point of view of arboreal Galois theory (indeed, they are not a generic family in the moduli space of degree d polynomials in any reasonable sense). There are other families of

polynomials and rational functions (such as postcritically finite maps) that arise as obstructions to any potential classification of finite index arboreal representations – see [Jon13, Section 3] and [BT18b, Prop 3.3] for examples. One might hope that in these “exceptional” families, something similar to Theorem 1.1 could hold, in that a broad finite index result could be established for a natural overgroup other than $\text{Aut}(T_\infty^d)$. The authors will explore this in future work.

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2. WREATH PRODUCTS

In this section we give a brief introduction to wreath products, which arise naturally from the Galois theory of the preimage fields $K_n(\beta) = K(f^{-n}(\beta))$.

Let G be a permutation group acting on a set X , and let H be any group. Let H^X be the group of functions from X to H with multiplication defined pointwise, or equivalently the direct product of $|X|$ copies of H . The wreath product of G by H is the semidirect product $H^X \rtimes G$, where G acts on H^X by permuting coordinates: for $f \in H^X$ and $g \in G$ we have

$$f^g(x) = f(g^{-1}x)$$

for each $x \in X$. We will use the notation $G[H]$ for the wreath product, suppressing the set X in the notation. Another common convention is $H \wr G$ or $H \wr_X G$ if we wish to call attention to X .

Let $d \geq 2$ be a fixed integer. Let T_n^d be the complete d -ary rooted tree of level n , where each non-leaf vertex has precisely d child vertices. Let T_∞^d be the complete infinite d -ary rooted tree, which may be obtained as the direct limit of all T_n^d as $n \rightarrow \infty$.

Suppose $f \in K[x]$ with $\deg f = d$ and $\beta \in \mathbb{P}^1(\overline{K})$. The morphism $f : \mathbb{P}^1(\overline{K}) \rightarrow \mathbb{P}^1(\overline{K})$ is d -to-1 away from its critical points, so the set $f^{-n}(\beta)$ generically consists of d^n points. Let

$$T_n(\beta) = \bigsqcup_{i=0}^n f^{-i}(\beta).$$

For a generic choice of β , we see that $T_n(\beta)$ and T_n^d are isomorphic as rooted trees. The root is β , the vertices at the i th level are labeled by the elements of $f^{-i}(\beta)$, and the edges connect vertices labeled by z and $f(z)$ for all $z \in f^{-i}(\beta)$ with $1 \leq i \leq n$. However, if $\gamma \in f^{-m}(\beta)$ is a critical point of f , then $f(\gamma)$ has fewer than d preimages and $|f^{-n}(\beta)| < d^n$ for $n \geq m$. In this case, we can repair the construction as follows: we take $f^{-n}(\beta)$ to be the multiset of solutions to $f^n(x) = \beta$, each counted with multiplicity.

Then $|f^{-n}(\beta)| = d^n$ for all n , and once again we have $T_n(\beta) \cong T_n^d$. Let

$$T_\infty(\beta) = \bigsqcup_{i=0}^{\infty} f^{-i}(\beta)$$

be the infinite d -ary rooted tree formed by taking the direct limit of all the $T_n(\beta)$. With our convention on counting with multiplicity, $T_\infty(\beta)$ and T_∞^d are isomorphic as rooted trees for every $\beta \in \mathbb{P}^1(\overline{K})$.

It is easy to see that $\text{Aut}(T_1^d) \cong S_d$, and to show inductively that $\text{Aut}(T_n^d)$ satisfies the recursive formula

$$\text{Aut}(T_n^d) \cong \text{Aut}(T_{n-1}^d)[S_d],$$

or to be completely precise,

$$\text{Aut}(T_n^d) \cong S_d \wr_{T_{n-1}^d \setminus T_{n-2}^d} \text{Aut}(T_{n-1}^d),$$

with the obvious action of $\text{Aut}(T_{n-1}^d)$ on the $(n-1)$ th level of the tree. We may therefore think of $\text{Aut}(T_n^d)$ as the “ n th iterated wreath product” of S_d , which we will denote as $[S_d]^n$ (knowing that the implicit action in each wreath product is on a different level of the tree). From the recursive formulation of $[S_d]^n$ we deduce the order formula

$$|[S_d]^n| = (d!)^{d^{n-1}} |[S_d]^{n-1}| = (d!)^{d^{n-1} + d^{n-2} + \dots + 1} = (d!)^{\frac{d^n - 1}{d - 1}}.$$

The Galois group $G_n(\beta) = \text{Gal}(K_n(\beta)/K)$ embeds into $[S_d]^n$ via the faithful action of $G_n(\beta)$ on $T_n(\beta)$ (see [Odo85] or [BT18b, Section 2] for details). Taking inverse limits, there is an injection $G_\infty(\beta) \hookrightarrow \text{Aut}(T_\infty^d)$, and we may think of $\text{Aut}(T_\infty^d)$ as the infinitely iterated wreath product $[S_d]^\infty$.

Now consider the action of C_d on $\text{Aut}(T_1^d)$ by cyclically permuting the vertices. We may define an iterated wreath product of cyclic groups $[C_d]^n$ as subgroups of $\text{Aut}(T_n^d)$ by the recursive formula

$$[C_d]^n \cong [C_d]^{n-1}[C_d],$$

where again, the implied action of $[C_d]^{n-1}$ is on the $(n-1)$ th level of the tree. From this recursive formula, we deduce the order formula

$$|[C_d]^n| = d^{d^{n-1}} |[C_d]^{n-1}| = d^{d^{n-1} + d^{n-2} + \dots + 1} = d^{\frac{d^n - 1}{d - 1}}.$$

Taking an inverse limit, there is a group $[C_d]^\infty$ that arises as a subgroup of $\text{Aut}(T_\infty)$, and projects to $[C_d]^n$ when considering its action on the n th level of the tree and below.

Assume now that $f(x) := x^d + c \in K[x]$, where K is a field of characteristic 0 that contains the d th roots of unity. Let $\beta \in K$ be such that $\beta - c$ is not an m th power in K for any $m > 1$ dividing d . Then we have $K_1(\beta) = K((\beta - c)^{1/d})$ and $G_1(\beta) \cong C_d$. For any $n \geq 2$, the extension $K_n(\beta)$ is a Kummer extension attained by adjoining to $K_{n-1}(\beta)$ the d th roots of $z - c$

where z ranges over the roots of $f^{n-1}(x) = \beta$. Thus we have

$$\mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \leq \prod_{z \in f^{-(n-1)}(\beta)} \mathrm{Gal}(K_{n-1}(\beta)((z-c)^{1/d})/K_{n-1}(\beta)) \leq C_d^{d^{n-1}},$$

where we continue to consider the elements of $f^{-(n-1)}(\beta)$ with multiplicity.

Considering the Galois tower

$$K_n(\beta) \supseteq K_{n-1}(\beta) \supseteq K$$

we see that

$$G_n(\beta) \leq \mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \rtimes G_{n-1}(\beta) \leq G_{n-1}(\beta)[C_d],$$

where the implied action of $G_{n-1}(\beta)$ is on $f^{-(n-1)}(\beta)$. By induction, $G_n(\beta)$ embeds into $[C_d]^n$. Taking inverse limits, $G_\infty(\beta)$ embeds into $[C_d]^\infty$.

We summarize our basic strategy for proving that $G_\infty(\beta)$ has finite or infinite index in $[C_d]^\infty$ as Proposition 2.2. First we prove a preliminary lemma.

Lemma 2.1. *Let $f = x^d + c \in K[x]$.*

(1) $[K_n(\beta) : K_{n-1}(\beta)]$ divides $d^{d^{n-1}}$ and

$$[[C_d]^n : G_n(\beta)] = [[C_d]^{n-1} : G_{n-1}(\beta)] \frac{d^{d^{n-1}}}{[K_n(\beta) : K_{n-1}(\beta)]}$$

(2) $[[C_d]^\infty : G_\infty(\beta)] = \lim_{n \rightarrow \infty} [[C_d]^n : G_n(\beta)]$

Proof. Part (1) follows from the fact that $\mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \leq C_d^{d^{n-1}}$ and the recursive wreath product structure of $G_n(\beta)$. For part (2), consider the projection map $\pi_n : [C_d]^\infty \rightarrow [C_d]^n$. The restriction of π_n maps $G_\infty(\beta)$ to $G_n(\beta)$. By basic group theory,

$$[[C_d]^\infty : G_\infty(\beta)] \geq [[C_d]^n : G_n(\beta)].$$

for all n . Conversely, suppose the RHS is some finite number M . Then, appealing to the profinite structure of $[C_d]^\infty$, we see that if there are more than M cosets of $G_\infty(\beta)$ in $[C_d]^\infty$, these would project to more than M cosets of $G_n(\beta)$ in $[C_d]^n$ under π_n for some n . Therefore

$$[[C_d]^\infty : G_\infty(\beta)] \leq [[C_d]^n : G_n(\beta)]$$

and we are done. \square

Proposition 2.2. *Let $f = x^d + c \in K[x]$. Then $[[C_d]^\infty : G_\infty(\beta)] < \infty$ if and only if $\mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \cong C_d^{d^{n-1}}$ for all sufficiently large n .*

Proof. This immediately follows from Lemma 2.1. \square

We conclude this Section by discussing the *generic* case for the arboreal Galois representation associated to a unicritical polynomial.

Proposition 2.3. *Let \mathcal{F} be a field containing $\overline{\mathbb{Q}}$, let $c, \beta \in \mathcal{F}$ be algebraically independent elements over $\overline{\mathbb{Q}}$ and let $K = \overline{\mathbb{Q}}(c, \beta)$. We let $d \geq 2$ be an integer and let $f(x) := x^d + c$. Then the image of $G_\infty(\beta)$ is the entire $[C_d]^\infty$.*

Proof. This is a consequence of [JKMT16, Theorem 3.1]. Indeed, since β is transcendental over $\overline{\mathbb{Q}}(c)$, then the groups G_N from [JKMT16, Theorem 3.1] are precisely our groups $G_n(\beta)$; furthermore, as explained in [JKMT16, Remark 3.2], condition (2) of [JKMT16, Theorem 3.1] is automatically satisfied for unicritical polynomials. Finally, condition (1) of [JKMT16, Theorem 3.1] is easily satisfied when c is transcendental since then the orbit of 0 must be infinite. \square

3. NECESSARY CONDITIONS

We prove that the conditions in Theorem 1.1 are necessary for finite index. For this part of the theorem, we do not need to assume that f has prime power degree, or that f is not isotrivial. The argument relies on a basic fact of algebra known as Capelli's Lemma, which we will use many times throughout the paper. We state it below without proof.

Lemma 3.1 (Capelli's Lemma). *Let K be any field and let $f, g \in K[x]$. Suppose $\alpha \in \overline{K}$ is any root of f . Then $f(g(x))$ is irreducible over K if and only if both $f(x)$ is irreducible over K and $g(x) - \alpha$ is irreducible over $K(\alpha)$.*

Proposition 3.2. *Suppose $f(x) = x^d + c \in K[x]$ with $d \geq 2$, and let $\beta \in K$. If β is either periodic or postcritical for f , then $[[C_d]^\infty : G_\infty(\beta)] = \infty$.*

Proof. First assume that β is postcritical for f , i.e., that there is some critical point α of f with $f^m(\alpha) = \beta$ for some $m \geq 1$. This means that the tree of preimages of β is degenerate at the m th level: as $f^m(x) - \beta$ has at least one repeated root, we have

$$|f^{-n}(\beta)| \leq (d^m - 1)d^{n-m}$$

for every $n \geq m$. As in Section 2, the Galois group $\text{Gal}(K_n(\beta)/K_{n-1}(\beta))$ embeds into the direct product of $|f^{-(n-1)}(\beta)|$ copies of C_d . In particular,

$$|\text{Gal}(K_n(\beta)/K_{n-1}(\beta))| \leq d^{(d^m - 1)d^{n-1-m}} < d^{d^{n-1}}$$

for all sufficiently large n . By Proposition 2.2, we conclude that $G_\infty(\beta)$ has infinite index in $[C_d]^\infty$.

Now assume that β is periodic for f and *not* postcritical, so that the tree of preimages of β can be identified with the complete d -ary tree T_∞^d . The pair (f, β) cannot be eventually stable by a straightforward argument using Capelli's Lemma [BT18b, Prop 4.2]. This implies that the number of Galois orbits in $f^{-n}(\beta)$ is unbounded as $n \rightarrow \infty$, and thus that there are an infinite number of orbits in the action of $G_\infty(\beta)$ on ∂T_∞^d , where ∂T_∞^d is the boundary (or the "ends") of the tree T_∞^d , which can be identified with the set of infinite paths starting from the root of the tree [JL17, Prop 2.2]. But as $[C_d]^n$ acts transitively on the n th level of the tree for every n , we see that $[C_d]^\infty$ acts transitively on ∂T_∞^d . A simple argument in group theory then implies that $[[C_d]^\infty : G_\infty(\beta)] = \infty$ [JL17, Prop 3.3]. \square

4. HEIGHT ESTIMATES

In this section we present two lemmas that give key height inequalities, which will be used in the proofs of the main theorems. For background on heights, see [HS00, GNT13, BT18a].

First we set some notation. We continue with the assumption that K is a function field of transcendence degree 1 over $\overline{\mathbb{Q}}$, i.e., K is the function field of a smooth projective irreducible curve C defined over $\overline{\mathbb{Q}}$. Let \mathfrak{p} be a place of K , i.e., a point of $C(\overline{\mathbb{Q}})$. We let $k_{\mathfrak{p}}$ be the residue field of the associated discrete valuation $v_{\mathfrak{p}}$; note that $k_{\mathfrak{p}}$ is naturally isomorphic to $\overline{\mathbb{Q}}$. Then for each point $z \in K$, we have its Weil height

$$h(z) := \sum_{\mathfrak{p} \text{ place of } K} -\min\{0, v_{\mathfrak{p}}(z)\}.$$

For $f \in K[x]$ with $\deg f = d \geq 2$, let $h_f(z)$ be the Call-Silverman canonical height of z relative to f [CS93], defined by

$$h_f(z) = \lim_{n \rightarrow \infty} \frac{h(f^n(z))}{d^n}.$$

We will often write sums indexed by primes of K that satisfy some condition. As an example of our indexing convention, observe that

$$\sum_{v_{\mathfrak{p}}(z) > 0} v_{\mathfrak{p}}(z) \leq h(z)$$

for all $z \in K^{\times}$ by the product formula for K . Also define the forward orbit of $\gamma \in K$ under $f \in K[x]$ to be

$$\mathcal{O}_f(\gamma) = \bigcup_{n \geq 1} \{f^n(\gamma)\} = \{f(\gamma), f^2(\gamma), f^3(\gamma), \dots\}.$$

With this notation we have the following two lemmas.

Lemma 4.1. *Let $f \in K[x]$ with $d = \deg(f) \geq 2$. Let $\gamma \in K$ with $h_f(\gamma) > 0$. Let $\beta_1, \beta_2 \in K$ such that $\beta_2 \notin \mathcal{O}_f(\beta_1)$. For $n > 0$, let $\mathcal{X}(n)$ denote the set of places \mathfrak{p} of K such that*

$$\min(v_{\mathfrak{p}}(f^m(\gamma) - \beta_1), v_{\mathfrak{p}}(f^n(\gamma) - \beta_2)) > 0$$

for some $0 < m < n$. Then for any $\delta > 0$, we have

$$\#\mathcal{X}(n) \leq \delta d^n h_f(\gamma) + O_{\delta}(1).$$

for all n . (Note that, with the notation as in [BT18b], we have that $N_{\mathfrak{p}} = 1$ for each place \mathfrak{p} since $k_{\mathfrak{p}}$ is isomorphic to the field of constants.)

Proof. See [BT18b, Section 5]. Note that β_1 and β_2 need not be distinct. \square

Lemma 4.2. *Let $f \in K[x]$ with $d = \deg(f) \geq 2$, and assume that f is not isotrivial. Let $\gamma, \beta \in K$ be such that $\beta \notin \mathcal{O}_f(\gamma)$ and that β is also not postcritical. For every $\epsilon > 0$, there exists a constant C_{ϵ} such that*

$$\#\{\mathfrak{p}: v_{\mathfrak{p}}(f^n(\gamma) - \beta) = 1\} \geq (d - \epsilon)d^{n-1}h_f(\gamma) + C_{\epsilon}$$

for all $n \geq 1$.

Proof. This follows immediately from [BT18a, Lemma 4.2]. \square

Lemma 4.2 is sometimes called the ‘‘Roth- abc ’’ estimate because of its similarity to Roth’s theorem; for our case of function fields, this is a consequence of Yamanoi’s proof of Vojta’s $(1 + \epsilon)$ -conjecture [Yam04].

5. FINITENESS OF GCD

We derive the following theorem by combining the results from [GKNY17] and [CS93].

Proposition 5.1. *Let d be an integer larger than 1. Let K be the function field of a smooth projective irreducible curve C defined over $\overline{\mathbb{Q}}$. For $i = 1, 2$, let $f_i(x) = x^d + t_i \in K[x]$ be non-isotrivial with $t_1 \neq \xi t_2$ for any ξ such that $\xi^{d-1} = 1$. Let $c_1, c_2 \in K$ be such that $f_i^\ell(0) \neq c_i$ for $i = 1, 2$ and for all non-negative integers ℓ . Then there are at most finitely many places v of K such that there are positive integers m, n with the property that*

$$(5.1.1) \quad \min(v(f_1^m(0) - c_1), v(f_2^n(0) - c_2)) > 0.$$

In order to prove Proposition 5.1, we need a Bogomolov-type version of the main theorem of [GKNY17].

Theorem 5.2. *Let d be an integer larger than 1, let F be a number field, let K be the function field of a smooth projective geometrically irreducible curve C defined over F , and let $t_1, t_2 \in K \setminus F$ such that t_1/t_2 is not a $(d-1)$ -st root of unity. We let $f_i(x) := x^d + t_i \in K[x]$ for $i = 1, 2$, and for each point $\lambda \in C(\overline{\mathbb{Q}})$ which is not a pole for either t_1 or t_2 , we consider the specialization of the polynomials f_i at λ , denoted as $f_{i,\lambda}(x) := x^d + t_i(\lambda) \in \overline{\mathbb{Q}}[x]$; for each such $\lambda \in C(\overline{\mathbb{Q}})$, we denote by $h_{f_{i,\lambda}} : \overline{\mathbb{Q}} \rightarrow \mathbb{R}_{\geq 0}$ the corresponding canonical heights. Then there exists $\epsilon > 0$ such that there are finitely many points $\lambda \in C(\overline{\mathbb{Q}})$ for which $\max\{h_{f_{1,\lambda}}(0), h_{f_{2,\lambda}}(0)\} \leq \epsilon$.*

Proof. We argue by contradiction and therefore assume there exists an infinite sequence of points $\{\lambda_i\}_{i \geq 1} \subseteq C(\overline{\mathbb{Q}})$ such that

$$(5.2.1) \quad \lim_{i \rightarrow \infty} h_{f_{1,\lambda_i}}(0) = \lim_{i \rightarrow \infty} h_{f_{2,\lambda_i}}(0) = 0.$$

We proceed as in [GKNY17] and for each $j = 1, 2$, we construct adelic metrized line bundles $\overline{\mathcal{L}}_j$ on the curve C corresponding to the families of dynamical systems $z \mapsto z^d + t_1(\lambda)$, respectively $z \mapsto z^d + t_2(\lambda)$ (parametrized by the $\overline{\mathbb{Q}}$ -points $\lambda \in C$). In general, given a rational function $\Psi : C \rightarrow \mathbb{P}^1$ (defined over $\overline{\mathbb{Q}}$), there exists an adelic metrized line bundle $\overline{\mathcal{L}}_\Psi$ associated to the family of dynamical systems $g_\lambda^\Psi(z) := z^d + \Psi(\lambda)$ (as we vary $\lambda \in C(\overline{\mathbb{Q}})$), where \mathcal{L}_Ψ is the line bundle on C obtained by pulling-back $\mathcal{O}(1)$ through the morphism $\Psi : C \rightarrow \mathbb{P}^1$; for more details, see [GKNY17, Sections 3.2 and 4.1]. In particular, this gives rise to height functions

$h_{\overline{\mathcal{L}}_j} : C(\overline{\mathbb{Q}}) \rightarrow \mathbb{R}_{\geq 0}$ (associated to the metrized line bundles $\overline{\mathcal{L}}_j$, for $j = 1, 2$) for which we have:

$$(5.2.2) \quad h_{\overline{\mathcal{L}}_j}(\lambda) = \frac{d}{\deg(t_j)} \cdot h_{f_{j,\lambda}}(0),$$

for each $\lambda \in C(\overline{\mathbb{Q}})$, where $\deg(t_j)$ is the degree of the rational function $t_j : C \rightarrow \mathbb{P}^1$; see [GKNY17, Proposition 3.5]. Our hypothesis (see (5.2.1)), coupled with (5.2.2), yields that for the infinite sequence $\{\lambda_i\}_{i \geq 1} \subseteq C(\overline{\mathbb{Q}})$, we have

$$(5.2.3) \quad \lim_{i \rightarrow \infty} h_{\overline{\mathcal{L}}_1}(\lambda_i) = \lim_{i \rightarrow \infty} h_{\overline{\mathcal{L}}_2}(\lambda_i) = 0.$$

Using (5.2.3) coupled with [CL11, Proposition 3.4.2] (which uses crucially the inequalities established by Zhang [Zha95] regarding the successive minima associated to a metrized line bundle), we derive that there exist positive integers ℓ_j (for $j = 1, 2$) such that the two line bundles $\mathcal{L}_1^{\ell_1}$ and $\mathcal{L}_2^{\ell_2}$ are linearly equivalent and, moreover, the two heights $h_{\overline{\mathcal{L}}_j}$ are proportional. In particular, this means that for each $\lambda \in C(\overline{\mathbb{Q}})$, we have that $h_{\overline{\mathcal{L}}_1}(\lambda) = 0$ if and only if $h_{\overline{\mathcal{L}}_2}(\lambda) = 0$. Using this last equivalence along with equation (5.2.2), we obtain that for each point $\lambda \in C(\overline{\mathbb{Q}})$,

$$(5.2.4) \quad h_{f_{1,\lambda}}(0) = 0 \text{ if and only if } h_{f_{2,\lambda}}(0) = 0.$$

Using (5.2.4) and the fact that only preperiodic points have canonical height equal to 0 (for a rational function defined over $\overline{\mathbb{Q}}$), we get that 0 is preperiodic for the dynamical system $z \mapsto z^d + t_1(\lambda)$ if and only if 0 is preperiodic for the dynamical system $z \mapsto z^d + t_2(\lambda)$. In other words, $\gamma_1 := t_1(\lambda) \in \overline{\mathbb{Q}}$ is a PCF (postcritically-finite) parameter for the family of unicritical polynomials $z \mapsto z^d + \gamma$ (parametrized by $\gamma \in \overline{\mathbb{Q}}$) if and only if $\gamma_2 := t_2(\lambda)$ is a PCF parameter for the same dynamical system $z \mapsto z^d + \gamma$. Because there exist infinitely many PCF parameters $\gamma \in \overline{\mathbb{Q}}$ for the family of polynomials $z \mapsto z^d + \gamma$, we conclude that there exist infinitely many $\lambda \in C(\overline{\mathbb{Q}})$ such that

$$(5.2.5) \quad t_1(\lambda) \text{ and } t_2(\lambda) \text{ are PCF parameters.}$$

Now, we let Y be the Zariski closure in the plane of the image of C under the rational map $C \dashrightarrow \mathbb{A}^2$ given by $x \mapsto (t_1(x), t_2(x))$; then (5.2.5) yields that there exist infinitely many points $(\gamma_1, \gamma_2) \in Y(\overline{\mathbb{Q}})$ with both coordinates PCF parameters for the unicritical dynamical system $z \mapsto z^d + \gamma$. But then [GKNY17, Theorem 1.1] yields that $Y \subset \mathbb{A}^2$ is given by an equation of the form $y = \zeta \cdot x$ for some $(d-1)$ -st root of unity ζ , where (x, y) are the coordinates of \mathbb{A}^2 (note that Y is neither a horizontal line, nor a vertical line because both t_1 and t_2 are non-constant rational functions, and so possibilities (1)-(2) in [GKNY17, Theorem 1.1] cannot occur). However, our hypothesis regarding t_1/t_2 not being a $(d-1)$ -st root of unity prevents Y from satisfying such an equation and this contradiction proves that there exists no infinite sequence $\{\lambda_i\}_{i \geq 1} \subseteq C(\overline{\mathbb{Q}})$ as in (5.2.1). This concludes our proof of Theorem 5.2. \square

We now state a simple lemma that follows from work of Call and Silverman [CS93]. We recall the following lemma from [BT18b, Lemma 8.3].

Lemma 5.3. *Let K be the function field of a smooth irreducible curve C defined over $\overline{\mathbb{Q}}$. For each $t \in K$ and each $\lambda \in C(\overline{\mathbb{Q}})$ which is not a pole of t , we denote by $t_\lambda := t(\lambda)$ the specialization of t at λ . Let $\varphi \in K(x)$ have degree $d \geq 2$. Let $y, z \in K$ with $h_\varphi(y) > 0$ (where h_φ is the canonical height associated to the rational function φ). Let $(\lambda_i)_{i=1}^\infty$ be a sequence of points of $C(\overline{\mathbb{Q}})$ satisfying $\varphi_{\lambda_i}^{n_i}(y_{\lambda_i}) = z_{\lambda_i}$ for a sequence $(n_i)_{i=1}^\infty$ of positive integers with $\lim_{i \rightarrow \infty} n_i = \infty$. Then*

$$\lim_{i \rightarrow \infty} h_{\varphi_{\lambda_i}}(y_{\lambda_i}) = 0.$$

Proof of Proposition 5.1. Let F be a number field such that C is a geometrically irreducible curve defined over F . If there were infinitely many places v of the function field K such that (5.1.1) holds, then this means there exists an infinite sequence of points $\{\lambda_i\}_{i \geq 1} \subseteq C(\overline{\mathbb{Q}})$ such that there exist some nonnegative integers m_i and n_i , for which we have

$$(5.3.1) \quad f_{1, \lambda_i}^{m_i}(0) = c_1(\lambda_i) \text{ and } f_{2, \lambda_i}^{n_i}(0) = c_2(\lambda_i).$$

Also, since $f_1^m(0) \neq c_1$ for all integers $m \geq 0$ and $f_2^n(0) \neq c_2$ for all integers $n \geq 0$, we derive that the integers m_i and n_i appearing in (5.3.1) must tend to infinity. But then Lemma 5.3 yields that

$$\lim_{i \rightarrow \infty} h_{f_{1, \lambda_i}}(0) = \lim_{i \rightarrow \infty} h_{f_{2, \lambda_i}}(0) = 0,$$

contradicting thus Theorem 5.2. \square

6. EVENTUAL STABILITY

The results of this section are valid (with only a few changes) in the more general setting of any function field of a curve defined over a finitely generated field of characteristic 0. However, we will restrict to the case relevant for our results. So, let L be a number field, let k be the function field of a smooth projective geometrically irreducible curve C defined over L , let $q = p^r$ be a power of a prime number p and let $f(x) := x^q + t \in k[x]$ for some $t \in k \setminus L$. (Note that our hypothesis yields that L is algebraically closed in k .) Let $\beta \in k$ be a point which is not periodic under f ; then we will prove that the pair (f, β) is eventually stable over k .

We note that the places of k correspond to points of $C(\overline{\mathbb{Q}})$. For any element $c \in k$ and any point λ of $C(\overline{\mathbb{Q}})$ such that c does not have a pole at λ , we let c_λ denote the specialization of c to $\overline{\mathbb{Q}}$ at λ (see [CS93] for more details); in other words, seeing c as a rational function $C \rightarrow \mathbb{P}^1$ defined over L , then $c_\lambda := c(\lambda)$. Likewise for a rational function $\varphi \in k(x)$, we let φ_λ denote the specialization of φ to $\overline{\mathbb{Q}}(x)$ at λ for any λ such that the coefficients of φ do not have poles at λ ; so, for our polynomial $f(x) = x^q + t$, we simply have that $f_\lambda(x) := x^q + t_\lambda$ for any point $\lambda \in C(\overline{\mathbb{Q}})$ which is not a pole of $t \in k$. Finally, for any $\lambda \in C(\overline{\mathbb{Q}})$, we let $L(\lambda)$ be the field of definition for

the point λ ; in particular, this means that $[L(\lambda) : L] < \infty$ and furthermore, for each $z \in k$, we have that $z_\lambda \in L(\lambda)$.

With notation as above, the following lemma follows from the work of [CS93].

Lemma 6.1. *Let $\varphi \in k(x)$ be a non-isotrivial rational function of degree greater than one and let $\beta \in k$. Then we have the following:*

- (a) *if $h_\varphi(\beta) > 0$, then the set of specializations λ from k to $\overline{\mathbb{Q}}$ such that $h_{\varphi_\lambda}(\beta_\lambda) = 0$ has bounded height (with respect to some degree-1 divisor on C); and*
- (b) *if β is not periodic under φ , then the set of specializations λ from k to $\overline{\mathbb{Q}}$ such that β_λ is periodic under φ_λ has bounded height.*

Proof. The statement of (a) follows directly from [CS93, Theorem 4.1]. We now prove (b). If β is not preperiodic under φ , then $h_\varphi(\beta) > 0$, by [Bak09]. Then, from (a), it follows that the set of λ such that $h_{\varphi_\lambda}(\beta_\lambda) = 0$ has bounded height. Now, if β is strictly preperiodic, then there are at most finitely many λ such that β_λ is periodic under φ_λ , so the set of such λ clearly has bounded height. \square

The following lemma is a simple consequence of the main theorem of [JL17].

Lemma 6.2. *Let $g(x) = x^q + c$ where c is an element of a number field L with the property that $|c|_v \leq 1$ for some non-archimedean place v of L such that $v|p$. Then for any $\beta \in L$ that is not periodic under g , the pair (g, β) is eventually stable over L .*

Proof. Let k_v be the residue field at v . Then reducing g at v induces a map $g_v : \mathbb{P}^1(k_v) \rightarrow \mathbb{P}^1(k_v)$ such that every point of $\mathbb{P}^1(k_v)$ has exactly one inverse image under g_v . Let $\beta_v \in \mathbb{P}^1(k_v)$ denote the reduction of β at v . Then there is an n such that $g_v^{-n}(\beta_v) = \{\beta_v\}$. Theorem 1.7 of [JL17] states that (g, β) must therefore be eventually stable over L . \square

Now we can state the main result of this section, which is instrumental in proving Theorem 1.3.

Proposition 6.3. *Let k be the function field of a smooth projective geometrically irreducible curve C defined over a number field L . Let $f(x) = x^q + t \in k[x]$, where $t \notin L$. Then for any $\beta \in k$ that is not periodic under f , the pair (f, β) is eventually stable over k .*

Proof. We may choose a specialization λ of k to $\overline{\mathbb{Q}}$ such that:

- (i) t_λ is an algebraic integer; and
- (ii) β_λ is not periodic under f_λ .

Indeed, for all but finitely many $z \in \overline{\mathbb{Q}}$, there is a $\lambda \in C(\overline{\mathbb{Q}})$ such that $t_\lambda = z$; furthermore, condition (ii) is achieved for all points $\lambda \in C(\overline{\mathbb{Q}})$ of sufficiently large height (by Lemma 5.3), while on the other hand, if the algebraic integer

t_λ has large height, then the point λ must have large height (on C) as well. Then, by Lemma 6.2, the pair $(f_\lambda, \beta_\lambda)$ is eventually stable over $L(\lambda)$, which implies that (f, β) is eventually stable over k . \square

7. RAMIFICATION AND GALOIS THEORY

We note that from now on we follow the general strategy from [BT18b].

Let $f(x) = x^q + t$ with $t \in K$. In this section we define *Condition R* and *Condition U* in terms of primes dividing certain elements of K related to the forward orbits of 0. In Proposition 7.4 and 7.5 we show that these conditions control ramification in the extensions $K(\beta) \subseteq K_n(\beta)$, with consequences for the Galois theory of these extensions. We begin with the following standard lemma from Galois theory (see also [BT18b, Lemma 6.1]).

Lemma 7.1. *Let L_1, \dots, L_n and M be fields all contained in some larger field. Assume that L_1, \dots, L_n are finite extensions of M .*

- (i) *If L_1, L_2 are Galois over M with $L_1 \cap L_2 = M$, then $L_1 L_2$ is Galois over L_2 and $\text{Gal}(L_1 L_2 / L_2) \cong \text{Gal}(L_1 / M)$.*
- (ii) *If L_1, \dots, L_n are Galois over M with $L_i \cap \prod_{j \neq i} L_j = M$ for each i , then $\text{Gal}(\prod_{i=1}^n L_i / M) \cong \prod_{i=1}^n \text{Gal}(L_i / M)$.*

Conditions R and U make use of the notion of good reduction of a map $f \in K(x)$ at a prime \mathfrak{p} . A polynomial

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \dots + a_1 x + a_0,$$

has good reduction at \mathfrak{p} if $v_{\mathfrak{p}}(a_d) = 0$ and $v_{\mathfrak{p}}(a_i) \geq 0$ for $0 \leq i \leq d-1$. See [MS94] or [Sil07, Theorem 2.15] for a more careful definition that also applies to rational functions. Clearly any f has good reduction at all but finitely many \mathfrak{p} . The idea behind the definition is that if f has good reduction at \mathfrak{p} , then f commutes with the reduction mod \mathfrak{p} map $\bar{\cdot} : \mathbb{P}^1(\overline{K}) \rightarrow \mathbb{P}^1(\overline{k}_{\mathfrak{p}})$. This is clear for polynomials (see [Sil07, Theorem 2.18] for a proof for rational functions). We say that f has good separable reduction at \mathfrak{p} if the reduced map $\bar{f} : \mathbb{P}^1(\overline{k}_{\mathfrak{p}}) \rightarrow \mathbb{P}^1(\overline{k}_{\mathfrak{p}})$ is separable.

Definition 7.2. *Let $\beta \in \overline{K}$. We say that a prime \mathfrak{p} of $K(\beta)$ satisfies **Condition R** at β for f and n if the following hold:*

- (a) *f has good separable reduction at \mathfrak{p} ;*
- (b) *$v_{\mathfrak{p}}(f^i(0) - \beta) = 0$ for all $0 \leq i < n$;*
- (c) *$v_{\mathfrak{p}}(f^n(0) - \beta) = 1$;*
- (d) *$v_{\mathfrak{p}}(\beta) = 0$.*

Definition 7.3. *Let $\beta \in \overline{K}$. We say that a prime \mathfrak{p} of $K(\beta)$ satisfies **Condition U** at β for f and n if the following hold:*

- (a) *f has good separable reduction at \mathfrak{p} ;*
- (b) *$v_{\mathfrak{p}}(f^i(0) - \beta) = 0$ for all $0 \leq i \leq n$;*
- (c) *$v_{\mathfrak{p}}(\beta) = 0$.*

Proposition 7.4. *Let $\beta \in \overline{K}$. Let \mathfrak{p} be a prime of $K(\beta)$ that satisfies Condition U at β for f and n . Then \mathfrak{p} is unramified in $K_n(\beta)$.*

Proof. This is the content of [BT18a, Proposition 3.1]. The proof in [BT18a] is stated for $\beta \in K$, but works exactly the same if we allow $\beta \in \overline{K}$ and replace K with $K(\beta)$. \square

The following result is similar to [BT18b, Proposition 6.5].

Proposition 7.5. *Let $\beta \in \overline{K}$. Suppose that \mathfrak{p} is a prime of $K(\beta)$ that satisfies Condition R at β for n and that $f^n(x) - \beta$ is irreducible over $K(\beta)$. Then*

$$\mathrm{Gal}(K_n(\beta)/K_{n-1}(\beta)) \cong C_q^{q^{n-1}}.$$

Furthermore, \mathfrak{p} does not ramify in $K_{n-1}(\beta)$ and does ramify in any field E such that $K_{n-1}(\beta) \subsetneq E \subset K_n(\beta)$.

Proof. Observe that Condition R at β for n implies Condition U at β for $n - 1$. By Proposition 7.4, \mathfrak{p} does not ramify in $K_{n-1}(\beta)$.

Let \bar{z} denote the image of $z \in \mathbb{P}^1(\overline{K})$ under the reduction mod \mathfrak{p} map (which is always well defined once we fix an extension of $v_{\mathfrak{p}}$ to \overline{K}). Consider the map $\bar{f} : \mathbb{P}^1(\overline{k}_{\mathfrak{p}}) \rightarrow \mathbb{P}^1(\overline{k}_{\mathfrak{p}})$ that comes from reducing f at \mathfrak{p} , and recall that Condition R assumes that f has good reduction at \mathfrak{p} . The unique critical point of \bar{f} is $0 \in k_{\mathfrak{p}}$. By (b) of Condition R, we see that $\bar{f}^{n-1}(x) - \bar{\beta}$ has no repeated roots. By (c) of Condition R, we see that $0 \in \bar{f}^{-n}(\beta)$, and 0 is totally ramified over $\bar{f}(0) = \bar{t}$ (in the sense of \bar{f} as a morphism of $\mathbb{P}^1(\overline{k}_{\mathfrak{p}})$). So $\bar{f}^n(x) - \bar{\beta}$ has 0 as a root of multiplicity q , and has no other repeated roots. So we may write

$$(7.5.1) \quad \bar{f}^n(x) - \bar{\beta} = x^q h(x)$$

where $h(0) \neq 0$ and h has distinct roots.

Let $z_1, \dots, z_{q^{n-1}}$ be the roots of $f^{n-1}(x) - \beta$. We have the factorization

$$(7.5.2) \quad f^n(x) - \beta = \prod_{i=1}^{q^{n-1}} (f(x) - z_i)$$

in $K_{n-1}(\beta)[x]$. For each i , let L_i denote the splitting field of $f(x) - z_i$ over $K(z_i)$, and let M_i denote the splitting field of $f(x) - z_i$ over $K_{n-1}(\beta)$. Note that $M_i = K_{n-1}(\beta) \cdot L_i$. By Capelli's Lemma, for each i , $f(x) - z_i$ is irreducible over $K(z_i)$. Also note that the z_i are all distinct mod \mathfrak{p} , because $\bar{f}^{n-1}(x) - \bar{\beta}$ has no repeated roots.

Let \mathfrak{q} be a prime of $K_{n-1}(\beta)$ that lies over \mathfrak{p} . By (7.5.1) and (7.5.2), there is precisely one $i \in \{1, \dots, q^{n-1}\}$ for which $f(x) - z_i$ has a repeated root mod \mathfrak{q} , and this root is repeated with multiplicity q . For this i , let \mathfrak{m} be a prime of M_i that lies over \mathfrak{q} . Then we have

$$(7.5.3) \quad f(x) - z_i \equiv x^q \pmod{\mathfrak{m}}.$$

By (c) of Condition R we have $v_{\mathfrak{q}}(f(0) - z_i) = v_{\mathfrak{p}}(f(0) - z_i) = 1$, as \mathfrak{q} is unramified over \mathfrak{p} . By (7.5.3) we compute $v_{\mathfrak{m}}(f(0) - z_i) = q$. We conclude

that \mathfrak{q} is totally ramified in M_i with $e(\mathfrak{m}/\mathfrak{q}) = q$, and for all $j \neq i$, \mathfrak{q} does not ramify in M_j .

Now, \mathfrak{p} is unramified in $K(z_i)$ (which is a subextension of $K_{n-1}(\beta)$), so $\mathfrak{m} \cap L_i$ ramifies over $\mathfrak{m} \cap K(z_i)$ with ramification index q . Using that $f(x) - z_i$ is irreducible over $K(z_i)$, $\text{Gal}(L_i/K(z_i))$ contains the order q inertia subgroup $I(\mathfrak{m} \cap L_i/\mathfrak{m} \cap K(z_i))$ and is therefore cyclic of order q (as we know it is a subgroup of C_q). Taking the base change by $K_{n-1}(\beta)$, $\text{Gal}(M_i/K_{n-1}(\beta))$ is a normal subgroup of C_q that contains the order q inertia subgroup $I(\mathfrak{m}/\mathfrak{q})$, so $\text{Gal}(M_i/K_{n-1}(\beta)) \cong C_q$ as well.

As $f^n(x) - \beta$ is irreducible over $K(\beta)$, it follows easily that $f^{n-1}(x) - \beta$ is irreducible over $K(\beta)$ as well. Therefore all of the z_i are Galois-conjugate. That is, for any $z_j \neq z_i$, there exists $\sigma \in G_{n-1}(\beta)$ such that $\sigma(z_i) = z_j$. Applying σ to \mathfrak{q} , we obtain a prime $\sigma(\mathfrak{q})$ of $K_{n-1}(\beta)$ that ramifies in M_j with ramification index q and does not ramify in M_k for any $k \neq j$. Repeating the same argument as above, it follows that $\text{Gal}(M_j/K_{n-1}(\beta)) \cong C_q$. The disjointness of ramification shows that for each j we have

$$(7.5.4) \quad M_j \cap \left(\prod_{k \neq j} M_k \right) = K_{n-1}(\beta).$$

By Lemma 7.1, we conclude that $\text{Gal}(K_n(\beta)/K_{n-1}(\beta)) \cong C_q^{q^{n-1}}$. To see the statement about intermediate fields, let \mathfrak{m}_j be a prime of M_j lying over $\sigma(\mathfrak{q})$. Then the inertia group $I(\mathfrak{m}_j/\sigma(\mathfrak{q}))$ extends to an inertia group $I(\mathfrak{m}'_j/\sigma(\mathfrak{q}))$ for a prime \mathfrak{m}'_j of $K_n(\beta)$ lying over \mathfrak{m}_j . The group $I(\mathfrak{m}'_j/\sigma(\mathfrak{q}))$ restricts to the identity on M_k for $k \neq j$. Therefore $\text{Gal}(K_n(\beta)/K_{n-1}(\beta))$ is generated by inertia groups of the form $I(\mathfrak{m}'_j/\sigma(\mathfrak{q}))$ for various $\sigma \in G_{n-1}(\beta)$. It follows that any intermediate field E lying strictly between $K_{n-1}(\beta)$ and $K_n(\beta)$ is ramified over some $\sigma(\mathfrak{q})$, and thus ramified over \mathfrak{p} . \square

Before stating the last result from Galois theory we need, we need a little notation.

Definition 7.6. *Let s be a positive integer, let $\alpha = (\alpha_1, \dots, \alpha_s) \in \overline{K}^s$, and let $\mathbf{f} = (f_1, \dots, f_s)$, where for each i , $f_i = x^q + t_i$ for some $t_i \in K \setminus \overline{\mathbb{Q}}$. We define $K_n(\mathbf{f}, \alpha)$ to be $\prod_{i=1}^s K_n(f_i, \alpha_i)$.*

With this notation we have the following result, which is similar to [BT18b, Proposition 6.7].

Proposition 7.7. *Let $\alpha = (\alpha_1, \dots, \alpha_s) \in L^s$ for L a finite extension of K . Suppose there exist primes $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ of L such that*

- (a) $\mathfrak{p}_i \cap K(\alpha_i)$ satisfies Condition R at α_i for f_i and n ;
- (b) $\mathfrak{p}_i \cap K(\alpha_j)$ satisfies Condition U at α_j for f_j and n for all $j \neq i$;
- (c) $\mathfrak{p}_i \cap K(\alpha_i)$ does not ramify in L ; and
- (d) $f_i^n(x) - \alpha_i$ is irreducible over $K(\alpha_i)$ for $i = 1, \dots, s$.

Then $\text{Gal}(K_n(\mathbf{f}, \boldsymbol{\alpha})/K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})) \cong C_q^{sq^{n-1}}$. Furthermore, there is no field E with $K_{n-1}(\mathbf{f}, \boldsymbol{\alpha}) \subsetneq E \subset K_n(\mathbf{f}, \boldsymbol{\alpha})$ that is unramified over $K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})$.

Proof. Choose any i with $1 \leq i \leq s$. By Proposition 7.5, we have

$$\text{Gal}(K_n(f_i, \alpha_i)/K_{n-1}(f_i, \alpha_i)) \cong C_q^{q^{n-1}}.$$

Further, \mathfrak{p}_i does not ramify in $K_{n-1}(f_i, \alpha_i)$ and does ramify in every field E with $K_{n-1}(f_i, \alpha_i) \subsetneq E \subseteq K_n(f_i, \alpha_i)$. Let $L_i = K_n(f_i, \alpha_i) \cdot K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})$. By (b), (c), and Proposition 7.4, \mathfrak{p}_i does not ramify in $K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})$ or in L_j for $j \neq i$. Therefore

$$K_n(f_i, \alpha_i) \cap \prod_{j \neq i} L_j = K_{n-1}(f_i, \alpha_i)$$

and

$$K_n(f_i, \alpha_i) \cap K_{n-1}(\mathbf{f}, \boldsymbol{\alpha}) = K_{n-1}(f_i, \alpha_i).$$

By Lemma 7.1 and the fact that $L_i \cdot \prod_{j \neq i} L_j = K_n(f_i, \alpha_i) \cdot \prod_{j \neq i} L_j$,

$$\begin{aligned} \text{Gal} \left(L_i \cdot \prod_{j \neq i} L_j / \prod_{j \neq i} L_j \right) &\cong \text{Gal}(L_i/K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})) \\ &\cong \text{Gal}(K_n(f_i, \alpha_i)/K_{n-1}(f_i, \alpha_i)) \\ &\cong C_q^{q^{n-1}}. \end{aligned}$$

Therefore we have

$$q^{q^{n-1}} = [L_i : K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})] \geq \left[L_i : L_i \cap \prod_{j \neq i} L_j \right] \geq \left[L_i \cdot \prod_{j \neq i} L_j : \prod_{j \neq i} L_j \right] = q^{q^{n-1}}.$$

We conclude that $L_i \cap \prod_{j \neq i} L_j = K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})$. Applying Lemma 7.1 again and using that $L_i \cdot \prod_{j \neq i} L_j = K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})$, we are done. \square

8. PROOF OF MAIN THEOREMS

The proofs of the main theorems combine the preliminary arguments from throughout the paper with the following proposition, which uses height arguments to produce primes with certain ramification behavior in $K_n(\beta)$. Recall the definitions of Condition R and Condition U from Section 7.

Proposition 8.1. *Let s be a positive integer, let K be the function field of a smooth projective curve defined over $\overline{\mathbb{Q}}$. For $i = 1, \dots, s$, let $f_i(x) = x^q + t_i \in K[x]$, where t_i is not in the constant field of K . Let L be a finite extension of K . Let $\alpha_1, \dots, \alpha_s$ be distinct elements of L such that for all ℓ , the point α_ℓ is not postcritical for f_ℓ , and for any $i \neq j$, the point $(\alpha_i, \alpha_j) \in \mathbb{A}^2$ is not on a curve that is periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$. Then, for all sufficiently large n , there exist primes $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ of L such that*

- (a) for each i , we have that $\mathfrak{p}_i \cap K(\alpha_i)$ satisfies Condition R at α_i and f_i for n ;

- (b) for each $i \neq j$, we have that $\mathfrak{p}_i \cap K(\alpha_j)$ satisfies Condition U at α_j and f_j for n ;
(c) $\mathfrak{p}_i \cap K(\alpha_i)$ does not ramify in L .

Proof. For any $1 \leq i \leq s$, let $\mathcal{A}_i(n)$ be the set of primes \mathfrak{p} of L such that (a), (b), and (c) hold. If a prime \mathfrak{p} of L satisfies condition R or condition U at α_i for n , then it is easy to see that the prime $\mathfrak{p} \cap K(\alpha_i)$ of $K(\alpha_i)$ also satisfies condition R or condition U at α_i for n . Therefore we will establish Conditions R and U for primes of L rather than primes of the various $K(\alpha_i)$, which will make the argument less cumbersome to state. Thus all sums below are indexed by primes of \mathfrak{o}_L as in Section 4.

There are only finitely many primes \mathfrak{p} of \mathfrak{o}_L for which some f_i does not have good separable reduction at \mathfrak{p} , $v_{\mathfrak{p}}(\alpha_i) \neq 0$ for some i , or $\mathfrak{p} \cap K(\alpha_i)$ ramifies in L for some α_i . The contributions of these primes to our estimates will be absorbed into the constant term C'_δ at the end of the proof.

Now, note that after changing variables, we may assume that for any i, j , we have $(t_i/t_j)^{q-1} = 1$ if and only if $t_i = t_j$. Choose any i, j with $1 \leq i, j \leq s$ such that $t_i \neq t_j$. By Proposition 5.1, there are at most finitely many primes \mathfrak{p} such that there are m, n such that

$$(8.1.1) \quad \min(v(f_i^m(0) - \alpha_i), v(f_j^n(0) - \alpha_j)) > 0.$$

As above, the contributions of these primes to our estimates will be absorbed into the constant term C'_δ at the end of the proof.

Now, take any i and j with $1 \leq i, j \leq s$ (and possibly $i = j$) such that $t_i = t_j$. Note that when $t_i = t_j$, the curves $(z, f_j^m(z))$ and $(f_i^m(z), z)$ are periodic under (f_i, f_j) so we may assume that there is no m such that $\alpha_i = f_j^m(\alpha_j)$ or $\alpha_j = f_i^m(\alpha_i)$ for any $m \geq 0$, so we may use Lemma 4.1. Let $\mathcal{X}(n)$ be the set of primes \mathfrak{p} with $\min(v_{\mathfrak{p}}(f_i^n(0) - \alpha_j), v_{\mathfrak{p}}(f_i^m(0) - \alpha_i)) > 0$ for some $1 \leq m \leq n-1$. By Lemma 4.1 with $\gamma = 0$, $\beta_1 = \alpha_i$, and $\beta_2 = \alpha_j$, for any $\delta > 0$ we have

$$(8.1.2) \quad \#\mathcal{X}(n) \leq \delta q^n h_{f_i}(0) + O_\delta(1),$$

Furthermore, if $j \neq i$, then the set of primes \mathfrak{p} such that $\min(v_{\mathfrak{p}}(f_i^n(0) - \alpha_i), v_{\mathfrak{p}}(f_i^n(0) - \alpha_j)) > 0$ is a finite set depending only on α_i and α_j (and not on n), because $\alpha_i \equiv \alpha_j \pmod{\mathfrak{p}}$ for such \mathfrak{p} , so

$$(8.1.3) \quad \#\{\mathfrak{p}: \min(v_{\mathfrak{p}}(f_i^n(0) - \alpha_i), v_{\mathfrak{p}}(f_i^n(0) - \alpha_j)) > 0\} = O_\delta(1).$$

Now, fix an i . By Lemma 4.2, for any $\delta > 0$ there is a constant C_δ such that

$$(8.1.4) \quad \#\{\mathfrak{p}: v_{\mathfrak{p}}(f_i^n(0) - \alpha_i) = 1\} \geq (q - \delta)q^{n-1}h_{f_i}(0) + C_\delta.$$

We subtract out (8.1.2); for all such j such that $j \neq i$ we subtract (8.1.3) as well. This sieves out all $\mathfrak{p} \notin \mathcal{A}_i(n)$ from the sum in (8.1.4), and we have

$$(8.1.5) \quad \#\mathcal{A}_i(n) \geq q^n h_{f_i}(0)(1 - sq\delta - \delta/q) + C'_\delta$$

where C'_δ is a constant obtained by combining all the $O_\delta(1)$ terms. For sufficiently large n , we can choose some δ to make the right hand side of (8.1.5) positive. Repeating this process for each i , we are done. \square

Proof of Theorem 1.3. We know that K is the function field of a projective smooth irreducible curve C defined over $\overline{\mathbb{Q}}$; so, we let L be a number field such that C is a geometrically irreducible curve defined over L . Then we let k be the function field $L(C)$; at the expense of replacing k by a finite extension, we may assume that $t, \beta \in k$. By Proposition 6.3, the pair (f, β) is eventually stable over k . It will suffice to show that

$$\overline{\mathbb{Q}} \cap \left(\bigcup_{n=1}^{\infty} k(f^{-n}(\beta)) \right) \text{ is a finite extension of } L.$$

Since (f, β) is eventually stable over k , by Capelli's Lemma there exists an m such that $f^n(x) - \alpha_i$ is irreducible over $k(\alpha_i)$ for all α_i such that $f^m(\alpha_i) = \beta$ and all $n \geq m$ (see [BT18b, Prop 4.2] for a proof of this fact). Applying Propositions 7.7 and 8.1, we see then that there is an integer n_1 such that for all $n > n_1$, the field $k_n(\beta) := k(f^{-n}(\beta))$ contains no nontrivial extensions of $k_{n-1}(\beta)$ that are unramified over $k_{n-1}(\beta)$. Let γ be any element of $\overline{\mathbb{Q}} \cap k_\ell(\beta)$ for some ℓ . Let N be minimal among all integers such that $\gamma \in k_N(\beta)$. Since $k_{N-1}(\beta)(\gamma)$ is unramified over $k_{N-1}(\beta)$, it follows that $N \leq n_1$. Hence $\overline{\mathbb{Q}} \cap \bigcup_{n=1}^{\infty} k(f^{-n}(\beta))$ is a finite extension of L , as desired. \square

Proof of Theorem 1.1. In Proposition 3.2 we have already proved the conditions are necessary. Therefore assume that β is not postcritical nor periodic for f . By Theorem 1.3, the pair (f, β) is eventually stable. Again using Capelli's Lemma, there is some m such that for all $\alpha \in f^{-m}(\beta)$ and for all $n \geq 1$, $f^n(x) - \alpha$ is irreducible over $K_n(\beta)$. By Propositions 7.7 and 8.1, there exists n_0 such that for all $n \geq n_0$, we have

$$\text{Gal}(K_n(\beta)/K_{n-1}(\beta)) \cong C_q^{q^{n-1}}.$$

By Proposition 2.2, we are done. \square

Proof of Theorem 1.4. By Theorem 1.3, each pair (f_i, α_i) is eventually stable. Arguing as in the proof of Theorem 1.3, we let γ in $M_i \cap \prod_{j \neq i} M_j$. Let N be minimal among all positive integers n such that $\gamma \in K_n(f_i, \alpha_i) \cap \prod_{j \neq i} K_n(f_j, \alpha_j)$. Applying Propositions 7.7 and 8.1, we see that there is an integer n_2 such for all $n > n_2$, we have

$$\text{Gal}(K_n(\mathbf{f}, \boldsymbol{\alpha})/K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})) \cong C_q^{sq^{n-1}}$$

(in the notation of Proposition 7.7). Since

$$[K_n(f_\ell, \alpha_\ell) : K_{n-1}(f_\ell, \alpha_\ell)] = [K_n(f_\ell, \alpha_\ell) \cdot K_{n-1}(\mathbf{f}, \boldsymbol{\alpha}) : K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})] = q^{q^{n-1}}$$

for each ℓ , it follows that any element of $K_n(f_i, \alpha_i) \cap \prod_{j \neq i} K_n(f_j, \alpha_j)$ is in $K_{n-1}(\mathbf{f}, \boldsymbol{\alpha})$, and thus in both $K_n(f_i, \alpha_i) \cap K_{n-1}(\mathbf{f}, \boldsymbol{\alpha}) = K_{n-1}(f_i, \alpha_i)$ and

$$\left(\prod_{j \neq i} K_n(f_j, \alpha_j) \right) \cap K_{n-1}(\mathbf{f}, \boldsymbol{\alpha}) = \prod_{j \neq i} K_{n-1}(f_j, \alpha_j)$$

Thus, we must have $N \leq n_2$ by minimality of N . This shows that $M_i \cap \prod_{j \neq i} M_j$ is a finite extension of K , as desired. \square

9. THE MULTITREE

In this section we introduce a generalization of trees, which we call multitrees, in order to give a pleasant interpretation of Theorem 1.4 in terms of a finite index statement. For our purposes, we can simplify the presentation of multitrees in [BT18b, Section 11] by avoiding the use of stunted trees.

Let $f \in K(x)$ with $\deg f \geq 2$ and set $\boldsymbol{\alpha} = \{\alpha_1, \dots, \alpha_s\} \subseteq K$. Define

$$\mathcal{M}_n(\boldsymbol{\alpha}) = \bigcup_{i=0}^n \bigcup_{j=1}^s f^{-i}(\alpha_j)$$

and

$$G_n(\boldsymbol{\alpha}) = \text{Gal}(K(\mathcal{M}_n(\boldsymbol{\alpha}))/K(\boldsymbol{\alpha})).$$

We refer to $\mathcal{M}_n(\boldsymbol{\alpha})$ as a *multitree*. It can be pictured as the union of s distinct trees of level n , rooted at the α_i .

As $n \rightarrow \infty$, define the direct limit

$$\mathcal{M}_\infty(\boldsymbol{\alpha}) = \varinjlim \mathcal{M}_n(\boldsymbol{\alpha})$$

and the inverse limit

$$G_\infty(\boldsymbol{\alpha}) = \varprojlim G_n(\boldsymbol{\alpha})$$

just as in the single tree case. For each n , $G_n(\boldsymbol{\alpha})$ acts faithfully on $\mathcal{M}_n(\boldsymbol{\alpha})$ in the usual way. So there are injections $G_n(\boldsymbol{\alpha}) \hookrightarrow \text{Aut}(\mathcal{M}_n(\boldsymbol{\alpha}))$, and thus an injection $G_\infty(\boldsymbol{\alpha}) \hookrightarrow \text{Aut}(\mathcal{M}_\infty(\boldsymbol{\alpha}))$, where an automorphism of the multitree must fix each root α_i .

Suppose that the individual trees rooted at α_i are disjoint, and that each α_i is neither periodic nor postcritical for f . Then the automorphism group of the infinite multitree has the simple description

$$\text{Aut}(\mathcal{M}_\infty(\boldsymbol{\alpha})) \cong \text{Aut}(T_\infty^q)^s,$$

that is, the direct product of s copies of $\text{Aut}(T_\infty^q)$. This group has a subgroup $([C_q]^\infty)^s$, which is the direct product of s copies of the permutation group given by the infinite iterated wreath product action of C_q on T_∞^q . If there are s different polynomial maps $f_i(x) = x^q + c_i$ that satisfy the hypotheses of Theorem 1.1, then it is easy to see that $G_\infty(\boldsymbol{\alpha})$ embeds into $([C_q]^\infty)^s$. Thus we may rephrase Theorem 1.4 as a finite index statement.

Theorem 9.1. *Let K be the function field of a smooth projective irreducible curve defined over $\overline{\mathbb{Q}}$. For $i = 1, \dots, s$ let $f_i(x) = x^q + t_i \in K[x]$, where $t_i \notin \overline{\mathbb{Q}}$, and suppose that $\alpha_i \in K$ are neither periodic nor postcritical for f . Suppose that there are no i, j with the property that (α_i, α_j) lies on a curve in \mathbb{A}^2 that is periodic under the action of $(x, y) \mapsto (f_i(x), f_j(y))$. Then*

$$[[C_q]^\infty]^s : G_\infty(\boldsymbol{\alpha}) < \infty.$$

Proof. The group $G_\infty(\boldsymbol{\alpha})$ equals $\text{Gal}(\prod_{i=1}^s K_\infty(f_i, \alpha_i)/K)$, which has finite index in the direct product $G_\infty(f_1, \alpha_1) \times \cdots \times G_\infty(f_s, \alpha_s)$ by Theorem 1.4 and basic Galois theory. This group in turn has finite index in $([C_q]^\infty)^s$ by applying Theorem 1.1 to each G_∞ separately. \square

10. THE ISOTRIVIAL CASE

In this section we treat the case when the polynomial $f(x) := x^q + c$ is isotrivial, i.e. $c \in \overline{\mathbb{Q}}$; also, we may assume the starting point $\beta \notin \overline{\mathbb{Q}}$ since otherwise all the Galois groups corresponding to preimages of β under iterates of f would be trivial. So, throughout this section, we assume K is the function field of a smooth projective irreducible curve defined over $\overline{\mathbb{Q}}$. First we deal with the case when the polynomial f is not PCF (see Proposition 10.1); note the similarity in the statements of Proposition 10.1 and [BT18b, Prop 12.1]. In the special case f is PCF, we will see in Proposition 10.2 that we can answer even the case when the degree of the unicritical polynomial f is not a prime power.

Proposition 10.1. *Let $f(x) := x^q + c \in \overline{\mathbb{Q}}[x]$ with $q = p^r$ a power of the prime number p and let $\beta \in K \setminus \overline{\mathbb{Q}}$. If f is not PCF, then $[[C_q]^\infty : G_\infty(\beta)] = 1$.*

Proof. The unique critical point of f is 0, which means that there are no integers $n > m > 0$ such that $f^n(0) = f^m(0)$. In particular, $\mathcal{O}_f(0)$ is infinite.

Put $L = \overline{\mathbb{Q}}(\beta)$. Examining the Newton polygon for $f^n(x) - \beta$ with respect to the place of L at infinity, we see that $f^n(x) - \beta$ is irreducible over L for every n . Using the fact that the orbit $\mathcal{O}_f(0)$ is infinite and non-repeating, it is easy to see that, for every n , the prime of L generated by $\beta - f^n(0)$ satisfies Condition R for β at n . We are done by Propositions 7.5 and 2.2. \square

If f is PCF of degree $d \geq 2$ (not necessarily a prime power), then we have a refined variation of the well-known fact that $G_\infty(\beta)$ has infinite index in $\text{Aut}(T_\infty^d)$. The proof of Proposition 10.2 does not explicitly use the fact that f is isotrivial, but clearly, $c \in \overline{\mathbb{Q}}$ since (as mentioned in Section 1) f cannot be PCF otherwise.

Proposition 10.2. *Let $f(x) := x^d + c \in K[x]$ with $d \geq 2$ be any integer, and let $\beta \in K$. If f is PCF, then $[[C_d]^\infty : G_\infty(\beta)] = \infty$.*

Proof. We argue as in [Jon13, Thm 3.1], which shows that the image of the arboreal Galois representation attached to any PCF rational map of

degree $d \geq 2$ has infinite index in $\text{Aut}(T_\infty^d) \cong [S_d]^\infty$. Using the fact that the abelianization of $G[H]$ is $G^{\text{ab}} \times H^{\text{ab}}$, we see that the abelianization of $[C_d]^n$ is C_d^n . Thus there are group homomorphisms from $[C_d]^n \rightarrow C_d^n$ for every $n \geq 1$, which induce a homomorphism $\tau : [C_d]^\infty \rightarrow C_d^\mathbb{N}$.

As f is PCF, by the main result of [CH12] (see also [BIJ⁺17]), the extension $K_\infty(\beta)/K$ is unramified outside a finite set of primes S . By work of Ihara [NSW00, Cor 10.1.6], the Galois group $G_{K,S}$ of the maximal extension of K unramified outside of S is topologically generated by the conjugacy classes of finitely many elements. (Here it is important that K has characteristic 0 in order to ensure that all ramification is tame.) Of course, $G_\infty(\beta)$ is a quotient of $G_{K,S}$ and therefore shares this property. So $\tau(G_\infty(\beta))$ is topologically finitely generated in $C_d^\mathbb{N} \cong \tau([C_d]^\infty)$. We conclude that $[\tau([C_d]^\infty) : \tau(G_\infty(\beta))] = \infty$, and therefore $[[C_d]^\infty : G_\infty(\beta)] = \infty$. \square

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