A REFORMULATION OF THE DYNAMICAL MANIN-MUMFORD CONJECTURE

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Abstract. We advance a new conjecture in the spirit of the Dynamical Manin-Mumford Conjecture. We show that our conjecture holds for all polarizable endomorphisms of abelian varieties and for all polarizable endomorphisms of $(\mathbb{P}^1)^N$. Furthermore, we show various examples which highlight the restrictions one would need to consider in formulating any general conclusion in the Dynamical Manin-Mumford Conjecture.

1. Introduction

As usual in algebraic dynamics, given a self-map $\Phi\colon X \longrightarrow X$ of a quasi-projective variety X, we denote by Φ^n the n-th iterate of Φ. Given a point $x \in X$, we let $\mathcal{O}_\Phi(x) = \{\Phi^n(x)\colon n \in \mathbb{N}\}$ be the orbit of x. Recall that a point x is *periodic* if there exists some $n \in \mathbb{N}$ such that $\Phi^n(x) = x$; a point y is *preperiodic* if there exists $m \in \mathbb{N}$ such that $\Phi^m(y)$ is periodic.

One of the most fundamental questions in algebraic dynamics is the Dynamical Manin-Mumford Conjecture, originally formulated by Zhang in early 1990's. Essentially, one expects the following be true: given an endomorphism Φ of a quasiprojective variety X defined over a field K of characteristic 0, then for any subvariety $Z \subseteq X$, if Z contains a Zariski dense set of preperiodic points under the action of Φ, then Z is itself preperiodic, i.e., there exist integers $0 \le m < n$ such that $\Phi^m(Z) = \Phi^n(Z)$ (or, at least, the Zariski closure of $\Phi^m(Z)$ is the Zariski closure of $\Phi^n(Z)$). However, as noted by Zhang in [Zha06], one would need to impose an additional hypothesis on Φ in order to rule out simple counterexamples coming from the coordinatewise action of different iterates of the same rational function acting on the coordinate axes of $(\mathbb{P}^1)^N$. So, Zhang [Zha06] advanced the above conjecture in the case of *polarizable* endomorphisms Φ of projective varieties X (i.e., there exists an ample line bundle $\mathcal{L}$ such that $\Phi^*\mathcal{L}$ is linearly equivalent with $\mathcal{L}^{\otimes d}$ for some $d > 1$).

However, there are counterexamples to the above conjecture using polarizable endomorphisms Φ of a power of a complex multiplication (CM) elliptic curve E (for more details, see Section 3). In light of such counterexamples, an extra hypothesis in the Dynamical Manin-Mumford Conjecture was added (see the paper [GTZ11]). More precisely, one expects that if Z contains a Zariski dense of preperiodic points x with the property that the tangent subspace of Z at x is preperiodic under the induced action of Φ on the Grassmanian $\mathrm{Gr}_{\dim(Z)}(T_{X,x})$, then Z should be itself preperiodic under the action of Φ.

However, the above extra hypothesis regarding the induced action of Φ on the tangent subspace seems too strong; we still expect that the *governing principle* in the conjecture formulated originally by Zhang should hold (see also the discussion of the Dynamical Manin-Mumford Conjecture from [YZ17] and also the results of [DF17], which hold beyond the case of polarizable endomorphisms). Therefore, we expect that if a subvariety Z of a projective variety X endowed with a polarizable endomorphism Φ defined over a field K of characteristic 0 contains a Zariski dense set of preperiodic points, then either Z is itself preperiodic, or there exists an additional rigidity for the dynamical system, which may be expressed in the existence of an endomorphism Ψ under which Z

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is preperiodic and moreover, Ψ commutes with a suitable iterate of Φ. Before properly stating our conjecture, we start with the definition of *dynamically special* subvarieties with respect to a given polarizable endomorphism.

Definition 1.1. Let Φ be a polarizable endomorphism of a smooth projective variety X defined over a field of characteristic 0. We say that $Z \subseteq X$ is a *dynamically special* subvariety with respect to Φ, if there exists a positive integer n, there exists a subvariety $Z \subseteq Y \subseteq X$ and there exists a polarizable endomorphism Ψ of X such that the following holds:

- Y is invariant under both Φ^n and Ψ;
- $\Phi^n \circ \Psi = \Psi \circ \Phi^n$ on Y; and
- Z is preperiodic under the action of Ψ, i.e., there exist distinct positive integers k and ℓ such that $\Psi^k(Z) = \Psi^\ell(Z)$.

The above Definition 1.1 is partially motivated by a similar phenomenon occurring in the arithmetic of Drinfeld modules regarding endomorphisms which admit suitable commuting iterates (see [Ghi05] and also, [Pin06] for the introduction of the so-called *brother structure* in the context of Drinfeld modules).

Conjecture 1.2. *Let Φ be a polarizable endomorphism of a projective smooth variety X defined over a field K of characteristic 0. Let $Z \subseteq X$ be a subvariety containing a Zariski dense set of preperiodic points. Then Z is dynamically special with respect to Φ.*

In Section 2, we will prove that Conjecture 1.2 holds for polarizable endomorphisms of abelian varieties and also holds for polarizable endomorphisms of $(\mathbb{P}^1)^N$. We note that the aforementioned two cases were also the most studied cases of the Dynamical Manin-Mumford Conjecture (see [BH05, GT10, GTZ11, GNY18, GNY19]) and therefore, we are able to infer in our proofs some of the most difficult results previously established for the Dynamical Manin-Mumford question (such as the result from [GNY18] for endomorphisms of $(\mathbb{P}^1)^N$ given by the coordinatewise action of non-Lattés maps). We will prove our results each time under the assumption that the varieties in question are defined over a number field; most of our proofs extend easily to the function field case but we prefer at this (still) early stage in dealing with the Dynamical Manin-Mumford Conjecture to deal with the number field case alone (just as it was the case of the conjecture advanced by Zhang in his groundbreaking article [Zha06]).

Our paper grew from our search for finding the most natural conclusion in the Dynamical Manin-Mumford Conjecture, which would survive the various counterexamples we knew existed for the earlier version of this conjecture. We were also encouraged in our efforts by Shouwu Zhang's question that he posed to us for formulating a characterization of the class of dynamically special subvarieties in the Dynamical Manin-Mumford Conjecture which would be closed under taking intersections. Unfortunately, we could not find such a characterization and we will present in Section 3 various counterexamples in this direction. In particular, our various examples from Section 3 show the difficulty one encounters in formulating a satisfying conclusion in the Dynamical Manin-Mumford Conjecture which would take into account all possible constructions.

2. Proofs

We start with an easy standard lemma.

Lemma 2.1. *Let Φ and Ψ be commuting polarizable endomorphisms of a smooth projective variety X defined over a number field K. Let $x \in X(K)$ be a preperiodic point of Φ. Then x is a preperiodic point for Ψ.*

Proof. So, we know there exist integers $0 \le m < n$ such that $\Phi^m(x) = \Phi^n(x)$. We let $\widehat{h}_\Phi : X(K) \longrightarrow \mathbb{R}_{\ge 0}$ be the canonical height constructed with respect to the polarizable endomorphism

Φ (for more details, see [CS93]). Then we know that $\widehat{h}_\Phi$ differs by a bounded amount from the naive Weil height $h : X(K) \longrightarrow \mathbb{R}_{\geq 0}$. Since each preperiodic point has canonical height equal to 0 (once again, see [CS93]), we have that the Weil height of each preperiodic point of Φ is uniformly bounded by some positive constant C.

Now, since $\Psi \circ \Phi = \Phi \circ \Psi$, we get that also $\Phi^m(\Psi(x)) = \Phi^n(\Psi(x))$, i.e., also $\Psi(x)$ is a preperiodic point for Φ; therefore, $h(\Psi(x)) < C$. Repeating this argument for any positive integer ℓ, we get that $h\left(\Psi^\ell(x)\right) < C$ for all $\ell \in \mathbb{N}$. However, there exist only finitely many points of bounded Weil height defined over a given number field (by Northcott's theorem) and therefore, we must have that x is preperiodic also under the action of Ψ, as desired. $\square$

Lemma 2.1 coupled with the conclusion from the classical Manin-Mumford Conjecture (which was proven by Raynaud [Ray83]) immediately solves Conjecture 1.2 for polarizable endomorphisms of abelian varieties.

Proof of Conjecture 1.2 for polarizable endomorphisms of abelian varieties defined over number fields. Let Φ be a polarizable endomorphism of the abelian variety X defined over a number field K. We assume $Z \subseteq X$ contains a Zariski dense set of preperiodic points under the action of Φ. Lemma 2.1 yields that each preperiodic point of Ψ is also preperiodic under the action of the multiplication-by-2 map [2] on X (since [2] and Φ are commuting endomorphisms of X). Then Raynaud's theorem [Ray83] yields that Z is a torsion translate of a an algebraic subgroup of X. Therefore X is preperiodic under the multiplication-by-2 map (which commutes with Φ) and so, Z is dynamically special. $\square$

Using [GNY18] (which actually builds on the work from [GNY19]) along with a more careful analysis of the case of polarizable endomorphisms of powers of CM elliptic curves, we can prove Conjecture 1.2 for all polarizable endomorphisms of $(\mathbb{P}^1)^N$.

Proof of Conjecture 1.2 for polarizable endomorphisms of $(\mathbb{P}^1)^N$ defined over number fields. So, we assume that $Z \subseteq (\mathbb{P}^1)^N$ contains a Zariski dense set of preperiodic points under the action of Φ. We know that the polarizable endomorphism Φ of $(\mathbb{P}^1)^N$ is given by the action of

$$(x_1, \dots, x_N) \mapsto (f_1(x_1), \dots, f_N(x_N)),$$

where each f_i is a rational function of degree $d > 1$. We let S be the subset of all indices in $\{1, \dots, N\}$ with the property that f_i is not a Lattés map; without loss of generality, we assume $S = \{1, \dots, m\}$. We let Φ_1 and Φ_2 be the induced action of Φ on the first m coordinate axes, respectively on the remaining $(N - m)$ coordinate axes of $(\mathbb{P}^1)^N$.

Then, as proven in [GNY18, Section 2], a subvariety containing a Zariski dense set of preperiodic points under the action of Φ is of the form $Z_1 \times Z_2$, where for each $j = 1, 2$, we have that Z_j contains a Zariski dense set of preperiodic points under the induced action Φ_j of Φ on the corresponding set of coordinate axes of $(\mathbb{P}^1)^N$. The proof of the Dynamical Manin-Mumford Conjecture for rational functions which are not Lattés maps (see [GNY18, Theorem 1.1]) yields that Z_1 must be preperiodic under the action of Φ_1.

Next we study Z_2, which we know it contains a Zariski dense set of periodic points under the coordinatewise action of $(N-m)$ Lattés maps. In this case, for each $i = m+1, \dots, N$, the Lattés map f_i corresponds to an elliptic curve E_i equipped with an endomorphism g_i along with a projection map $\pi_i : E_i \longrightarrow \mathbb{P}^1$ such that $\pi_i \circ g_i = f_i \circ \pi_i$. We let $A := \prod_{i=m+1}^N E_i$ and let g be the endomorphism of A given by the coordinatewise action of each $g_i : E_i \longrightarrow E_i$; also, we let $\pi : A \longrightarrow (\mathbb{P}^1)^{N-m}$ be the projection map induced by each coordinatewise projection $\pi_i : E_i \longrightarrow \mathbb{P}^1$. Then $W := \pi^{-1}(Z_2)$ is a subvariety of A containing a Zariski dense set of preperiodic points under the action of Φ and so, by Lemma 2.1, we obtain that each irreducible component of W is a torsion translate of an algebraic subgroup of A. In particular, a given irreducible component W_2 of W which projects dominantly onto Z_2 (through π) is of the form $\zeta + H$, where $\zeta \in A_{\mathrm{tor}}$ and H is a (connected) algebraic subgroup of A. So, W_2 is preperiodic under the action of the multiplication-by-$[d]$ map on A, which also commutes with g. Therefore, the Lattés maps $h_i : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ induced by the multiplication-by-d maps on each E_i commute with f_i for each $i = m+1, \dots, N$; furthermore, Z_2 is preperiodic under the induced action Ψ_2 on the last $(N-m)$ coordinate axes of $(\mathbb{P}^1)^N$ given by the coordinatewise action of the Lattés maps g_i. Also, note that $\deg(g_i) = d^2$ for each $i = m+1, \dots, N$.

Now, we let $\Psi_1 := \Phi_1^2$, i.e., the coordinatewise action given by f_i^2 on the first m coordinate axes of $(\mathbb{P}^1)^N$. Then $\Psi := \Psi_1 \times \Psi_2$ is a polarizable endomorphism of $(\mathbb{P}^1)^N$ (of degree d^2), commuting with Φ. Furthermore, Z is preperiodic under the action of Ψ, as desired. □

3. Examples

In [GTZ11], examples are given of polarizable morphisms $\Phi : X \longrightarrow X$ with subvarieties $Z \subset X$ such that Z contains a Zariski dense set of (pre)periodic points under the action of Φ but Z is not preperiodic under Φ. One such example is given by taking E to be an elliptic curve such that $\mathrm{End}(E)$ is $\mathbb{Z}[i]$ and letting $X = E \times E$ with $\Phi = ([3+4i, 3-4i])$ (that is, the map that acts as multiplication-by-$(3+4i)$ in the first coordinate and multiplication-by-$(3-4i)$ in the second coordinate) with Z the diagonal in $E \times E$. One can easily see then that not only does Z contain a Zariski dense set of preperiodic points under Φ but it also contains a Zarkisi dense set of periodic points under Φ. We note that of course Z is invariant under (f, f) for any $f \in \mathrm{End}(E)$.

Since in the classical case of the Manin-Mumford conjecture, the special subvarieties (i.e., the ones containing a Zariski dense set of torsion points of a given abelian variety X) are the torsion translates of X, then the intersection of two such special subvarieties is once again special. So, it was a very natural question posed by Shouwu Zhang whether a similar conclusion might be achieved in the dynamical setting, i.e., intersection of two dynamically special subvarieties (with a proper definition) would be again dynamically special. The following example shows that the intersection of preperiodic subvarieties (under the action of a polarizable endomorphism) does not have to be itself a preperiodic subvariety; furthermore, it does not even have to contain a Zariski dense set of preperiodic points.

Example 3.1. Let $\Phi : \mathbb{P}^2 \longrightarrow \mathbb{P}^2$ by $\Phi([x : y : z]) = [x^2 + z^2, y^2 + z^2 : z^2]$ and let Z_1 be the curve given by $x = y$. Then $\Phi^{-1}(Z_1)$ is the union of Z_1 with the curve $x = -y$, which we call Z_2. So, both Z_1 and Z_2 are preperiodic under the action of Φ. However, the intersection $Z_1 \cap Z_2$ is the point $[0 : 0 : 1]$, which is not preperiodic under Φ. ◇

Example 3.1 shows that one cannot formulate a characterization of the dynamically special subvarieties in the conclusion of the Dynamical Manin-Mumford Conjecture involving preperiodic points and still expect that the intersection of two dynamically special subvarieties would be dynamically special. Note that Lemma 2.1 would yield that a non-preperiodic point for a polarizable endomorphism Φ would remain non-preperiodic under any polarizable endomorphism Ψ commuting with Φ. Another reasonable problem would be to consider subvarieties containing a Zariski dense set of periodic points, especially in the light of the following conjecture we propose.

Conjecture 3.2. *Let Φ be a polarizable endomorphism of a projective smooth variety X defined over a field of characteristic 0. If $Z \subseteq X$ contains a Zariski dense set of preperiodic points, then there exists $\ell \in \mathbb{N}$ such that $\Phi^\ell(Z)$ contains a Zariski dense set of periodic points.*

We note that if the dynamical system (X, Φ) from Conjecture 3.2 were to satisfy the conclusion from the original Dynamical Manin-Mumford Conjecture formulated by Zhang, then we would know that there exists $\ell \in \mathbb{N}$ such that $\Phi^\ell(Z)$ is periodic under the induced action of Φ^ℓ. Then, using Fakhruddin's theorem [Fak03] regarding the existence of a Zariski closed set of periodic points for any polarizable dynamical system, we would obtain the desired conclusion in Conjecture 3.2.

In particular, the conclusion of Conjecture 3.2 is known for polarizable endomorphisms of $(\mathbb{P}^1)^N$ which are not given by coordinatewise action of Lattés maps (see [GNY18, Theorem 1.1]). Employing Raynaud's theorem [Ray83], one can easily deduce that Conjecture 3.2 holds for polarizable endomorphisms of abelian varieties. A similar analysis as in our proof of Conjecture 1.2 for the case of all polarizable endomorphisms of $(\mathbb{P}^1)^N$ shows that Conjecture 3.2 also holds for arbitrary polarizable endomorphisms of $(\mathbb{P}^1)^N$.

On the other hand, as shown in the following example, the polarizability condition is crucial in Conjecture 3.2.

Example 3.3. This is an example of a surjective endomorphism Φ of an affine variety X (defined over a field of characteristic 0) with the property that there exists a Zariski dense set of preperiodic points, but on the other hand, the periodic points of Φ are not Zariski dense in X; in particular, this justifies the polarizability assumption from Conjecture 3.2 (since of course we have $\Phi^n(X) = X$ for all X). We let Φ be the endomorphism of $X = \mathbb{A}^2$ given by $\Phi(x, y) = (x + y - 1, y^2)$. Then each point of the form (x, ζ_{2^n}) is a preperiodic point for Φ (for any x and any 2^n-th root of unity ζ_{2^n}). On the other hand, the only periodic points for Φ are the points fixed by Φ, which are of the form $(x, 1)$. ⋄

Now, we return to the question of finding a suitable property characterizing the dynamically special subvarieties so that intersections of such varieties would still satisfy the same property. The next example shows that even the intersection of two subvarieties that are periodic under a polarizable morphism Φ need not contain a Zariski dense set of periodic points under Φ (though it will contain a dense set of *preperiodic* points).

Example 3.4. Let E be an elliptic curve, let $X = E \times E$, let $\Phi(x, y) = ([3]x, [3]y)$, let Z_1 be the diagonal in $E \times E$, and let Z_2 be the set of all points $(x, [4]x)$ where $x \in E$. Then Z_1 and Z_2 are both invariant under Φ but $Z_1 \cap Z_2$ contains only points of form (z, z) where z is a 3-torsion point of E; all of these are strictly preperiodic under Φ except for $(0, 0)$ (where 0 is the identity element of E). ⋄

Note that in Example 3.4, all of the points in $Z_1 \cap Z_2$ are however periodic under Ψ where $\Psi(x, y) = ([m]x, [m]y)$ for any m that is not divisible by 3. In particular, this also motivated our Definition 1.1 for dynamically special subvarieties, which allow in their characterization another polarizable endomorphism Ψ commuting with a suitable iterate of Φ. Unfortunately, even this formulation is not enough for being preserved under taking intersections, as the following (more elaborate) example will show it.

Example 3.5. Let E be a (non-CM) elliptic curve and let $f(z)$ be a polynomial of degree 25, which is *disintegrated*, according to the definition from [MS14] (i.e., $f(z)$ is conjugated neither to z^{25} nor to C_{25}, where for each $n \in \mathbb{N}$, C_n is the n-th Chebyshev polynomial, which is the unique polynomial satisfying $C_n(x + 1/x) = x^n + 1/x^n$). Then, as shown in [GN16, Proposition 2.3], each polynomial commuting with an iterate of $f(z)$ must have degree a power of 5.

Now, let $X := E^2 \times (\mathbb{P}^1)^2$ and let Φ be the endomorphism of X given by

$$(w_1, w_2, z_1, z_2) \mapsto ([5]w_1, [5]w_2, f(z_1), f(z_2)) .$$

We let $\Gamma_f \subset (\mathbb{P}^1)^2$ be the graph of the polynomial f. We let $\Delta \subset E^2$ be the diagonal (i.e., the set of all points $(w, w) \in E^2$); also, we let $\Delta_6 \subset E^2$ be the set of all points of the form $(w, [6]w)$. Then we let $Z_1 := \Delta \times \Gamma_f$ and $Z_2 := \Delta_6 \times \Gamma_f$. Both Z_1 and Z_2 contain a Zariski dense set of periodic points under the action of Φ (they are actually both invariant under the action of Φ).

Now, $Z := Z_1 \cap Z_2$ is the set $\left(\Delta \cap E[5]^2\right) \times \Gamma_f$, i.e., it consists of all points of the form

$$(w_5, w_5, z, f(z)) ,$$

where $w_5 \in E[5]$ is a 5-torsion point of E, while z is an arbitrary point in $\mathbb{P}^1$.

Our choice for a disintegrated polynomial $f(z)$ of degree 25 yields that any polarizable endomorphism Ψ of X which commutes with some iterate of Φ needs to have degree equal to a power of 5. However, then the induced polarizable endomorphism Ψ_1 of E^2 also has its degree equal to a power of 5 and therefore, the points of the form (w_5, w_5) (for $w_5 \neq 0$) are strictly preperiodic under the action of Ψ_1. $\diamond$

The examples from this Section simply show the difficulty one encounters in trying to formulate the most satisfying conclusion in the Dynamical Manin-Mumford Conjecture. From all the examples we saw, it seems to us that Conjecture 1.2 is the most natural conclusion one could expect for this difficult problem.

References

[BH05] M. H. Baker and L.-C. Hsia, *Canonical heights, transfinite diameters, and polynomial dynamics*, J. reine angew. Math. **585** (2005), 61–92.

[CS93] G. S. Call and J. H. Silverman, *Canonical heights on varieties with morphisms*, Compos. Math. **89** (1993), 163–205.

[DF17] R. Dujardin and C. Favre, *The dynamical Manin-Mumford problem for plane polynomial automorphisms*, J. Eur. Math. Soc. (JEMS) **19** (2017), no. 11, 3421–3465.

[Fak03] N. Fakhruddin, *Questions on self maps of algebraic varieties*, J. Ramanujan Math. Soc. **18** (2003), no. 2, 109-122.

[Ghi05] D. Ghioca, *The Mordell-Lang theorem for Drinfeld modules*, Int. Math. Res. Not. **2005**, no. 53, 3273–3307.

[GN16] D. Ghioca and K. D. Nguyen, *Dynamical anomalous subvarieties: structure and bounded height theorems*, Adv. Math. **288** (2016), 1433–1462.

[GNY18] D. Ghioca, K. D. Nguyen, and H. Ye, *The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for endomorphisms of* $(\mathbb{P}^1)^n$, Compos. Math. **154** (2018), no. 7, 1441–1472.

[GNY19] D. Ghioca, K. D. Nguyen, and H. Ye, *The dynamical Manin-Mumford conjecture and the dynamical Bogomolov conjecture for split rational maps*, J. Eur. Math. Soc. (JEMS) **21** (2019), no. 5, 157–1594.

[GT10] D. Ghioca and T. J. Tucker, *Proof of a dynamical Bogomolov conjecture for lines under polynomial actions*, Proc. Amer. Math. Soc. **138** (2010), no. 3, 937–942.

[GTZ11] D. Ghioca, T. J. Tucker and S. Zhang, *Towards a Dynamical Manin-Mumford Conjecture*, Int. Math. Res. Not. **IMRN 2011** (2011), 5109–5122.

[Lau84] M. Laurent, *Equations diophantiennes exponentielles*, Invent. Math. **78** (1984), 299–327.

[MS14] A. Medvedev and T. Scanlon, *Invariant varieties for polynomial dynamical systems*, Ann. of Math. (2) **179** (2014), 81–177.

[Pin06] R. Pink, *The Galois representations associated to a Drinfeld module in special characteristic. II. Openness*, J. Number Theory **116** (2006), no. 2, 348–372.

[Ray83] M. Raynaud, *Sous-variétés d'une variété abélienne et points de torsion*, in Arithmetic and geometry, Vol. I, Progress in Mathematics, vol. 35 (Birkhauser, Boston, MA, 1983), 327–352.

[YZ17] X. Yuan and S. Zhang, *The arithmetic Hodge index theorem for adelic line bundles*, Math. Ann. **367** (2017), no. 3-4, 1123–1171.

[Zha06] S. Zhang, *Distributions in algebraic dynamics, Survey in Differential Geometry*, vol. **10**, International Press, 2006, pp. 381–430.

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