INTERSECTIONS OF ORBITS OF SELF-MAPS WITH SUBGROUPS IN SEMIABELIAN VARIETIES

JASON BELL AND DRAGOS GHIOCA

ABSTRACT. Let G be a semiabelian variety defined over an algebraically closed field K, endowed with a rational self-map Φ. Let $\alpha \in G(K)$ and let $\Gamma \subseteq G(K)$ be a finitely generated subgroup. We show that the set $\{n \in \mathbb{N} \colon \Phi^n(\alpha) \in \Gamma\}$ is a union of finitely many arithmetic progressions along with a set S of Banach density equal to 0. In addition, assuming Φ is regular, we prove that the set S must be finite.

1. INTRODUCTION

The Mordell-Lang conjecture, now a theorem due to Faltings [Fal83], asserts that a subvariety of an abelian variety A defined over a field of characteristic zero intersects a finitely generated subgroup Γ of A in a finite union of cosets of subgroups Γ. This was later extended by Vojta [Voj96, Voj99] (see also [McQ95]), who showed the analogous result holds for semiabelian varieties; that is, (commutative) algebraic groups G that lie in a short exact sequence of algebraic groups

$$1 \to \mathbb{G}_m^N \to G \to A \to 1$$

with A an abelian variety and N a nonnegative integer. This result is noteworthy in that it shows that the interaction between the geometric structure with the underlying group theoretic structure of a semiabelian variety is well-behaved. In recent years, the Mordell-Lang conjecture has inspired the so-called Dynamical Mordell-Lang conjecture, in which one now has an algebraic variety with a rational self-map and one now seeks to show that the interaction between the geometric structure and the dynamical structure is again well-behaved. More precisely, the Dynamical Mordell-Lang conjecture asserts that if X is a quasi-projective variety defined over a field of characteristic zero, and $\Phi : X \dashrightarrow X$ is a rational self-map and c and Y are respectively a closed point of X whose forward orbit under Φ is well-defined and a Zariski closed subset of X, then the set $\{n \in \mathbb{N}_0 \colon \Phi^n(c) \in Y\}$ is a finite (possibly empty) union of infinite arithmetic progressions along with a (possibly empty) finite set. (We recall that for a rational self-map Φ on a variety X and for $n \in \mathbb{N}_0$, Φ^n denotes the n-th iterate of Φ, where Φ^0 is taken to be the identity map by convention.).

For more details regarding the Dynamical Mordell-Lang conjecture, we refer the reader to the book [BGT16].

If, however, one returns to the setting of semiabelian varieties, which gave impetus to the Dynamical Mordell-Lang conjecture, it is very natural to ask whether the interaction between the dynamical structure and the group theoretic structure is similarly well-behaved when one has a rational self-map Φ of a semiabelian variety G. In this paper we study this question and show that in the case that Φ is a morphism one in fact obtains the analogous conclusion as in the statement of the Dynamical Mordell-Lang conjecture. In the case when Φ is a rational self-map the conclusion does not hold in general, but a weaker version holds in which the finite set is replaced by a set of zero Banach density. Intuitively, a set of zero Banach density is a *very sparse* set. Precisely, given a subset $I \subseteq \mathbb{N}_0$, we define the (upper) Banach density, $\bar{\delta}(I)$, of I using the formula

$$\bar{\delta}(I) := \limsup_{|J| \to \infty} \frac{|I \cap J|}{|J|}, \tag{1}$$

where the above lim sup is computed with respect to finite intervals J in the natural numbers. We note that due to definition (1), one could have potentially a set of natural density 0, but of Banach density equal to 1, and so the condition that a set have Banach density zero is a much stronger constraint than merely being of zero natural density.

Our main result is given by the following theorem.

Theorem 1.1. *Let G be a semiabelian variety defined over an algebraically closed field K endowed with a rational self-map Φ, let $\alpha \in G(K)$ be a point for which its orbit under the action of Φ is well-defined, and let $\Gamma \subset G(K)$ be a finitely generated subgroup. Then the following hold:*

  (i) *the set $\{n \colon \Phi^n(\alpha) \in \Gamma\}$ is a finite union of arithmetic progressions along with a set of zero Banach density.*
  (ii) *if, in addition, Φ is regular then $\{n \colon \Phi^n(\alpha) \in \Gamma\}$ is a finite union of arithmetic progressions along with a finite set.*

Remark 1.2. The conclusions to the statement of Theorem 1.1 (i) and (ii) both hold if we replace Γ by a coset of a finite of a finitely generated group, which can be seen by conjugating Φ by a suitable translation map and suitably modifying the point α.

Theorem 1.1 (ii), while not obviously connected to the Dynamical Mordell-Lang conjecture, in fact quickly implies that the Dynamical Mordell-Lang conjecture holds for dynamical systems (G, Φ) when G is a semiabelian variety and Φ is a regular self-map, if one also uses the famous theorem of Vojta [Voj96, Voj99] proving the classical Modell-Lang Conjecture (see Corollary 3.3). The Dynamical Mordell-Lang conjecture for semiabelian varieties was proven in [GT09] for regular self-maps on semiabelian varieties, but the method employed in [GT09] does not use the above route and instead employs a p-adic approach which is (along with the main result of [Bel06]) the precursor of the so-called *p-adic arc lemma* from [BGT10].

Our Theorem 1.1 is also connected to the main result of [BGT15] in which it was shown that for any algebraic dynamical system (X, Φ), where X is a variety and Φ is even a rational self-map, then given $x \in X$, the set of n for which $\Phi^n(x)$ lies in a fixed subvariety Y of X is a union of finitely many arithmetic progressions along with a set of Banach density 0. However, the method of proof from [BGT15] employs Noetherian induction and it cannot be modified to prove the conclusion from Theorem 1.1, which has a more algebraic flavour in the spirit of the classical Mordell-Lang conjecture.

Our Theorem 1.1 is connected to a result of the authors regarding a fusion variant of the classical and dynamical Mordell-Lang conjectures, which itself built on the paper [BCH21]. More precisely, in [BCH21, BG], it is shown that for a dominant rational self-map Φ on a variety X defined over a field K, endowed with a rational function $f : X \dashrightarrow \mathbb{P}^1$, then for a starting point $\alpha \in X(K)$ with a well-defined orbit under Φ and for a finitely generated subgroup $\Gamma \subset \mathbb{G}_m(K)$, the set of all $n \in \mathbb{N}_0$ for which $f(\Phi^n(\alpha)) \in \Gamma$ is a union of finitely many arithmetic progressions along with a set of Banach density equal to 0. This result plays a crucial role in our proof of Theorem 1.1 (i).

Finally, we note that in the conclusion to the statement of Theorem 1.1 (i), one cannot in general replace the set of zero Banach density with a finite set, as the following examples show.

Example 1.3. For the rational self-map Φ on $\mathbb{G}_m$ given by $x \mapsto x + 1$, along with the starting point $\alpha := 1$ and the subgroup $\Gamma \subset \mathbb{G}_m(\mathbb{Q})$ generated by 2, we see

$$\{n \colon \Phi^n(\alpha) \in \Gamma\} = \{0\} \cup \{2^n \colon n \geq 0\}.$$

Similar examples arise in positive characteristic, such as the following one.

Example 1.4. For the map $\Phi : \mathbb{G}_m \dashrightarrow \mathbb{G}_m$ defined over $\mathbb{F}_p(t)$ by the formula: $\Phi(x) = tx - t + 1$, we see that $\Phi^n(t + 1) = t^{n+1} + 1$ for each $n \geq 0$. So for the cyclic subgroup $\Gamma \subset \mathbb{G}_m(\mathbb{F}_p(t))$ generated by $t + 1$ and $\alpha = t + 1$, we have

$$\{n \colon \Phi^n(\alpha) \in \Gamma\} = \{p^n - 1 \colon n \in \mathbb{N}_0\}.$$

It is tempting to conjecture that the sparse sets $R(G, \Phi, \alpha, \Gamma)$ from the conclusion of Theorem 1.1 are always the image of some exponential function such as in Examples 1.3 and 1.4. However, as shown in [CGSZ21, Section 4], it is very hard to predict the exact shape of return sets associated with questions having the flavour of the Dynamical Mordell-Lang problem in characteristic p.

The outline of this paper is as follows. We begin by presenting various technical results in Section 2 regarding linear recurrence sequences, semiabelian varieties and also basic theory of finitely generated groups, which will later be employed in our proof of Theorem 1.1. Then we prove Theorem 1.1 (ii) in Section 3. We conclude with our proof of Theorem 1.1 (i) in Section 4.

2. Technical background

We collect here the various technical results later employed in our proofs.

2.1. **Linear recurrence sequences.** In this section we state the Skolem-Mahler-Lech theorem which will be used in our proof. First we need to introduce the basic set-up for *linear recurrence sequences* (see [Sch03] for more details on linear recurrent sequences).

Definition 2.1. Let $(H, +)$ be an abelian group. The sequence $\{u_n\}_{n \in \mathbb{N}_0} \subseteq H$ is a linear recurrence sequence (defined over the integers), if there exists a positive integer k and there exist constants $c_1, \ldots, c_k \in \mathbb{Z}$ such that

$$u_{n+k} = \sum_{i=1}^{k} c_i u_{n+k-i}, \text{ for each } n \in \mathbb{N}_0. \tag{2}$$

The following result is the well-known Skolem-Mahler-Lech theorem which applies to general linear recurrence sequences of complex numbers (not necessarily defined over the integers); for more details, see [Sko34, Mah35, Lec53].

Proposition 2.2. *Let $\{u_k\}_{k \in \mathbb{N}_0} \subset \mathbb{C}$ be a linear recurrence sequence and let $C \in \mathbb{C}$. Then the set $\{n \in \mathbb{N}_0 : u_n = C\}$ is a union of finitely many arithmetic progressions along with a finite set.*

2.2. **Semiabelian varieties.** For a semiabelian variety G defined over an algebraically closed field K there exists a short exact sequence of algebraic groups defined over K:

$$1 \longrightarrow \mathbb{G}_m^N \longrightarrow G \longrightarrow A \longrightarrow 1, \tag{3}$$

where $N \in \mathbb{N}_0$ and A is an abelian variety. Furthermore, each regular self-map Φ of G is a composition of a translation with a group endomorphism. Finally, for each group endomorphism Ψ of G, there exist integers $a_0, a_1, \ldots, a_{g-1}$ (where $g \leq 2 \dim(G)$) such that

$$\Psi^g = \sum_{i=1}^{g} a_i \Psi^{g-i}. \tag{4}$$

For more details regarding semiabelian varieties, we refer the reader to [Iit76, NW14].

2.3. **Finitely generated subgroups.** We conclude by deriving a useful result regarding finitely generated subgroups which will be employed in our proof of Theorem 1.1. First we need a definition.

Definition 2.3. Let Γ be a finitely generated subgroup of an abelian group $(G, +)$. We say that the elements $z_1, \ldots, z_n \in G$ are linearly independent with respect to Γ if for each integers $k_1, \ldots, k_n$, we have that

$$\sum_{i=1}^{n} k_i z_i \in \Gamma$$

if and only if $k_1 = \cdots = k_n = 0$.

Proposition 2.4. *Let Γ be a finitely generated subgroup of the abelian group $(G, +)$. Let $m \in \mathbb{N}$ and $y_1, \ldots, y_n, z_1, \ldots, z_n \in G$ with the property that $z_1, \ldots, z_n$ are linearly independent with respect to the subgroup Γ_1 spanned by Γ along with $y_1, \ldots, y_n$. Let $y_i' := y_i + z_i$ for each $i = 1, \ldots, n$, and we let the subgroup $\Gamma_1' \subseteq G$ spanned by Γ along with $y_1', \ldots, y_n'$. Then $\Gamma_1 \cap \Gamma_1' = \Gamma$.*

Proof. Clearly, Γ is contained in $\Gamma_1 \cap \Gamma_1'$, so it suffices to show that $\Gamma_1 \cap \Gamma_1' \subseteq \Gamma$. Let $\gamma \in \Gamma_1 \cap \Gamma_1'$. Then there exist integers $k_1, \ldots, k_n$ and $k_1', \ldots, k_n'$ such that $\gamma \in k_1 y_1 + \cdots + k_n y_n + \Gamma$ and $\gamma \in k_1' y_1' + \cdots + k_n' y_n' + \Gamma$. Hence

$$k_1 y_1 + \cdots + k_n y_n - k_1' y_1' - \cdots - k_n' y_n' \in \Gamma.$$

But this gives that $-k_1' z_1 - \cdots - k_n' z_n \in \Gamma_1$ and since $z_1, \ldots, z_n$ are linearly independent with respect to the subgroup Γ_1, we must have $k_1' = \cdots = k_n' = 0$ and so $\gamma \in \Gamma$ as desired. The result follows. $\square$

3. PROOF OF THEOREM 1.1 (II)

For this case, the strategy is inspired by the proofs from [Ghi08, CGSZ21], even though our arguments are simpler.

We begin with a useful result about orbits of points in semiabelian varieties.

Lemma 3.1. *Let G be a semiabelian variety and let $\Phi : G \to G$ be a regular self-map of G. If $\alpha \in G$ then $\{\Phi^n(\alpha) : n \geq 0\}$ is contained in a finitely generated subgroup of G.*

Proof. The regular self-map $\Phi : G \longrightarrow G$ can be written as $T_\beta \circ \Psi$, where T_β is the translation-by-β map on G and Ψ is a group endomorphism. Since every endomorphism of a semiabelian variety is integral over $\mathbb{Z}$ (see (4)), we may let $X^g - \sum_{i=1}^g e_i X^{g-i}$ (for some $g \in \mathbb{N}$) be the minimal polynomial of Ψ over $\mathbb{Z}$. Then, for each $k \geq 0$, we have that for a point $\gamma \in G$:

$$\Psi^{n+g}(\gamma) = \sum_{i=1}^g e_i \Psi^{n+g-i}(\gamma), \tag{5}$$

for each $n \in \mathbb{N}_0$. Furthermore, given $\alpha \in G(K)$, we have the general formula:

$$\Phi^n(\alpha) = \Psi^n(\alpha) + \sum_{j=0}^{n-1} \Psi^j(\beta). \tag{6}$$

Employing equation (6) and the recurrence relation (5) applied to the points α and β, we see that

$$\Phi^{n+g+1}(\alpha) = (e_1 + 1)\Phi^{n+g}(\alpha) + \sum_{i=2}^g (e_i - e_{i-1})\Phi^{n+1+g-i}(\alpha) - e_g \Phi^n(\alpha). \tag{7}$$

So, because $\{\Phi^n(\alpha)\}_{n \in \mathbb{N}_0}$ is a linear recurrence sequence of order $g + 1$, then the orbit of α under Φ is contained in Γ_1, where Γ_1 is the finitely generated subgroup of G spanned by the generators of Γ, along with $\Phi^i(\alpha)$ for $i = 0, \ldots, g$. $\square$

Hence, it suffices to show the following general statement.

Proposition 3.2. *Let Γ_1 be a finitely generated group, and let Γ be a subgroup of Γ_1. Then for every linear recurrence sequence $\{x_n\}_{n \in \mathbb{N}_0} \subset \Gamma_1$ defined over the integers, the set of all $n \in \mathbb{N}_0$ such that $x_n \in \Gamma$ is a union of finitely many arithmetic progressions along with a finite set.*

Proof. We first note that it suffices to prove the proposition in the case that $\Gamma = \{0\}$. Indeed, we let $\pi : \Gamma_1 \longrightarrow \Gamma_1/\Gamma$ be the canonical projection. Then $y_n := \pi(x_n)$, for each $n \in \mathbb{N}$, forms another linear recurrence sequence, also defined over the integers. Hence, each $n \in \mathbb{N}$ satisfies $x_n \in \Gamma$ if and only if $y_n = 0$.

Thus we may assume that $\Gamma = \{0\}$ for the remainder of the proof and so we only need to show that the set of all integers n such that $x_n = 0$ consists of at most finitely many arithmetic progressions in $\mathbb{N}$ along with a finite set.

Using the fact that the intersection of two arithmetic progressions is another arithmetic progression (or the empty set), we see that if we can write $\Gamma_1 = H_1 \oplus H_2$, and let $\pi_1 : \Gamma_1 \longrightarrow H_1$ and $\pi_2 : \Gamma_1 \longrightarrow H_2$ be the canonical projections, then for $n \in \mathbb{N}_0$, let $y_n := \pi_1(x_n)$ and $z_n := \pi_2(x_n)$. It suffices to prove that the set of n for which $y_n \in H_1$ and the set of n for which $z_n \in H_2$ are both expressible as a finite union of arithmetic progressions along with a finite set.

Then since Γ_1 can be written as a finite direct sum of cyclic groups, we see it suffices to prove Proposition 3.2 when Γ_1 is cyclic and $\Gamma = \{0\}$. Now if Γ_1 is an infinite cyclic group, then Γ_1 is isomorphic to $\mathbb{Z}$ (as a group) and the desired conclusion follows from Proposition 2.2. Thus we may assume that Γ_1 is finite and so $\Gamma_1 \cong \mathbb{Z}/N\mathbb{Z}$, for some positive integer N. Because $\{x_n\}_{n \in \mathbb{N}_0}$ is a linear recurrence sequence (over $\mathbb{Z}$) contained in a finite group, we conclude that $\{x_n\}_{n \in \mathbb{N}_0}$ is preperiodic (i.e., there exist integers $k \geq 0$ and $\ell \geq 1$ such that $x_{n+\ell} = x_n$ for each $n \geq k$). Thus the set of all $n \in \mathbb{N}$ such that $x_n = 0$ consists of at most finitely many arithmetic progressions along with a finite set. The result follows. $\square$

Proof of Theorem 1.1 (ii). By Lemma 3.1, we have $\{\Phi^n(\alpha) : n \geq 0\}$ is contained in finitely generated group. Hence there exists a finitely generated subgroup Γ_1 of G that contains both Γ and $\Phi^j(\alpha)$ for $j \geq 0$. Then $x_n := \Phi^n(\alpha)$ lies in Γ_1 for each $n \geq 0$ and the sequence $\{x_n\}$ satisfies a linear recurrence as the arguments in Lemma 3.1 and Equation (7) show. Then applying Proposition 3.2 gives the desired result. $\square$

As an immediate consequence, we obtain the Dynamical Mordell-Lang conjecture or semiabelian varieties, which was proved by different methods in [GT09].

Corollary 3.3. *Let Φ be a regular self-map of a semiabelian variety G. Then the Dynamical Mordell-Lang conjecture holds for the dynamical system (G, Φ).*

Proof. By Lemma 3.1 we have that the orbit of c under Φ is contained in a finitely generated subgroup Γ of G. Then by Vojta's theorem [Voj96, Voj99], $\Gamma \cap Y$ is a finite union of cosets of subgroups of Γ, say $\{\gamma_i + \Gamma_i : i = 1, \ldots, m\}$;

by construction $\Phi^n(c) \in Y$ if and only if $\Phi^n(c) \in \gamma_i + \Gamma_i$ for some i. Then by Theorem 1.1 (ii) and Remark 1.2 we have that the set of n for which $\Phi^n(c) \in \gamma_i + \Gamma_i$ is a finite union of arithmetic progressions along with a finite set for $i = 1, \ldots, m$ and since such sets are closed under the process of taking finite unions, we obtain the desired result. $\qquad\square$

4. Proof of Theorem 1.1 (i)

Throughout this section we let G be a semiabelian variety and let Γ be a finitely generated subgroup of G. Given a rational self-map Ψ of G and $\beta \in G$ whose forward orbit under Ψ is defined, we define the *return set*

$$R(G, \Psi, \beta, \Gamma) := \{n \in \mathbb{N}_0 \colon \Psi^n(\beta) \in \Gamma\}. \tag{8}$$

We proceed first with a useful reduction for our proof.

Remark 4.1. Throughout our proof of Theorem 1.1 we will often find useful to replace (α, Φ) by $(\Phi^\ell(\alpha), \Phi^k)$. Indeed, given $k \in \mathbb{N}$, once we prove that for each $\ell = 0, \ldots, k - 1$, the sets

$$R(G, \Phi^k, \Phi^\ell(\alpha), \Gamma) := \{n \in \mathbb{N}_0 \colon \Phi^{kn}(\Phi^\ell(\alpha)) \in \Gamma\}$$

is a union of finitely many arithmetic progressions along with a set of Banach density equal to 0, then clearly also $R(G, \Phi, \alpha, \Gamma)$ is a union of finitely many arithmetic progressions along with a set of Banach density equal to 0.

4.1. **General setup.** We have a rational self-map Φ on a semiabelian variety G defined over an algebraically closed field K. In particular, there exists a short exact sequence of algebraic groups (see (3))

$$1 \longrightarrow \mathbb{G}_m^N \longrightarrow G \longrightarrow A \longrightarrow 1, \tag{9}$$

where A is an abelian variety; we denote by $\pi \colon G \longrightarrow A$ the projection from (9). Also, if π were an isomorphism (i.e., $N = 0$ in (9)), then the rational self-map Φ would actually be regular and so, we would be done by Theorem 1.1 (ii) proven in Section 3. Thus we may assume henceforth that N is a positive integer (and also identify the maximal algebraic torus inside G with $\mathbb{G}_m^N$).

Now, because the only rational maps $\mathbb{A}^1 \dashrightarrow A$ are constant (since A is an abelian variety), we see that Φ induces a rational self-map $\bar{\Phi}$ on A, i.e., $\pi \circ \Phi = \bar{\Phi} \circ \pi$. On the other hand, since any rational self-map on an abelian variety is regular, we get that $\bar{\Phi}$ is a regular self-map on A.

4.2. **A linear recurrence relation.** For the regular self-map $\bar{\Phi}$ of the abelian variety A, arguing identically as in Section 3 (see equation (7)), we obtain that there exists a positive integer $m \leq 2\dim(A) + 1$ with the property that there exist integers $c_1, \ldots, c_m$ such that

$$\bar{\Phi}^m = \sum_{j=1}^m c_j \bar{\Phi}^{m-j}. \tag{10}$$

We derive a relation such as (10) by noting that $\bar{\Phi}$ is the composition of a translation with a group endomorphism of A. For any group endomorphism of A there exists a recurrence relation of the form (10) with $m \leq 2\dim(A)$.

Then, arguing as in the transition from Equation (5) to Equation (7), we obtain a recurrence relation of the form (10) for $\bar{\Phi}$ with $m \leq 1 + 2\dim(A)$.

We define a group endomorphism $\Psi : G^m \longrightarrow G^m$ as follows

$$\Psi(y_1, \ldots, y_m) = \left(y_2, \ldots, y_m, \sum_{i=1}^{m} c_j y_{m+1-j} \right). \tag{11}$$

We also let $\tilde{\pi} : G^m \longrightarrow A^m$ be defined as

$$\tilde{\pi}(y_1, \ldots, y_m) = (\pi(y_1), \ldots, \pi(y_m)).$$

Then letting $\bar{x} := \pi(x)$ for each $x \in G(K)$, and also using Equations (11) and (10), we obtain that for each $n \in \mathbb{N}_0$, we have that

$$(\tilde{\pi} \circ \Psi^n)\left(\alpha, \Phi(\alpha), \ldots, \Phi^{m-1}(\alpha) \right) = \left(\bar{\Phi}^n(\bar{\alpha}), \bar{\Phi}^{n+1}(\bar{\alpha}), \ldots, \bar{\Phi}^{n+m-1}(\bar{\alpha}) \right). \tag{12}$$

For the sake of simplifying our notation later, we denote:

$$\bar{\alpha}_i := \bar{\Phi}^i(\bar{\alpha}) \text{ for } i = 0, \ldots, m-1.$$

4.3. **Another finitely generated subgroup.** We let $\beta_0, \ldots, \beta_{m-1} \in G(K)$ such that

$$\pi(\beta_i) = \bar{\alpha}_i \text{ for } i = 0, \ldots, m-1. \tag{13}$$

We let Γ_1 be the finitely generated subgroup of $G(K)$ spanned by Γ along with $\beta_0, \ldots, \beta_{m-1}$.

Remark 4.2. The subgroup Γ_1 depends on the choice for $\beta_0, \ldots, \beta_{m-1}$ verifying equation (13) (one obvious choice is $\beta_i = \Phi^i(\alpha)$ for $i = 0, \ldots, m-1$, but there are infinitely many possibilities for the β_i's since $N \geq 1$ and thus, $\pi : G \longrightarrow A$ has infinite kernel). We will prove that (regardless of the choice for the β_i's) the set

$$R_1 := R(G, \Phi, \alpha, \Gamma_1) := \{n \in \mathbb{N}_0 : \Phi^n(\alpha) \in \Gamma_1\} \tag{14}$$

is a union of finitely many arithmetic progressions along with a set of Banach density equal to 0. Then working with two different liftings $\beta_0, \ldots, \beta_{m-1}$ and $\beta'_0, \ldots, \beta'_{m-1}$ of $\bar{\alpha}_0, \ldots, \bar{\alpha}_{m-1}$ (which satisfy the hypotheses of Proposition 2.4) which generate two subgroups Γ_1 and Γ'_1 for which $\Gamma = \Gamma_1 \cap \Gamma'_1$ will allow us to derive the desired conclusion in Theorem 1.1 (i).

In Section 4.4 we reformulate the desired conclusion for the set R_1 from Equation (14) in terms of a new finitely generated subgroup of $\mathbb{G}_m^N$.

4.4. **Reformulating the desired conclusion for our new finitely generated subgroup.** We consider the following rational self-map $\tilde{\Phi} : G^{m+1} \dashrightarrow G^{m+1}$ given by

$$\tilde{\Phi}(x, y_1, \ldots, y_m) := (\Phi(x), \Psi(y_1, \ldots, y_m)). \tag{15}$$

Then using Equation (12) we see that for each $n \in \mathbb{N}_0$, we have that

$$\tilde{\Phi}(\alpha, \beta_0, \ldots, \beta_{m-1}) := (\Phi^n(\alpha), \beta_n, \ldots, \beta_{n+m-1}) \tag{16}$$

for some points $\beta_n \in G(K)$, where (very importantly) due to Equations (12) and (13), we have that for each $n \in \mathbb{N}_0$:

$$\pi\left(\Phi^n(\alpha)\right) = \pi(\beta_n). \tag{17}$$

Letting $\Theta: G^{m+1} \longrightarrow G^{m+1}$ be the group endomorphism defined by the rule

$$\Theta(x, y_1, \ldots, y_m) := (x - y_1, y_1, \ldots, y_m), \tag{18}$$

we have that for each $n \in \mathbb{N}_0$,

$$\left(\Theta \circ \tilde{\Phi}^n\right)(\alpha, \beta_0, \ldots, \beta_{m-1}) \in \mathbb{G}_m^N(K) \times G^m(K). \tag{19}$$

We let $Y \subset G^{m+1}$ be the Zariski closure of the orbit of $\tilde{\alpha} := (\alpha, \beta_0, \ldots, \beta_{m-1})$ under the action of $\tilde{\Phi}$; clearly, $\tilde{\Phi}(Y) \subseteq Y$. For the sake of not complicating the notation, we let $\tilde{\Phi}$ also denote the induced rational self-map on Y.

Using Equation (19) along with the fact that Y is the Zariski closure of the orbit of α under $\tilde{\Phi}$ on which the property (19) holds, we conclude that $\Theta(Y) \subseteq \mathbb{G}_m^N \times G^m$. Therefore, for each $i = 1, \ldots, N$, we have well-defined projection maps $\pi_i: \Theta(Y) \dashrightarrow \mathbb{G}_m$ onto each one of the N coordinates of $\mathbb{G}_m^N$.

We now consider the intersection

$$H := \Gamma_1 \cap \mathbb{G}_m^N(K), \tag{20}$$

and let $\tilde{\pi}: \mathbb{G}_m^N \times G^m \longrightarrow \mathbb{G}_m^N$ be the natural projection.

Lemma 4.3. *Adopt the notation above. Then $R_1 = R(G, \Phi, \alpha, \Gamma_1)$ is equal to the set*

$$\tilde{R}_1 := \left\{ n \in \mathbb{N}_0 \colon \left(\tilde{\pi} \circ \Theta \circ \tilde{\Phi}^n\right)(\alpha, \beta_0, \ldots, \beta_{m-1}) \in H \right\}. \tag{21}$$

Proof. We recall that $\Psi^n(\beta_0, \ldots, \beta_{m-1}) = (\beta_n, \ldots, \beta_{n+m-1})$ for each $n \geq 0$ and so from the definition of Ψ we have that β_n is contained in the linear span of $\beta_0, \ldots, \beta_{m-1}$. Hence $\beta_n \in \Gamma_1$ for all n. Therefore, we have that $\Phi^n(\alpha) \in \Gamma_1$ if and only if $(\Phi^n(\alpha) - \beta_n) \in \Gamma_1$. Furthermore, since $\Phi^n(\alpha) - \beta_n \in \mathbb{G}_m^N(K)$ (see equation (17)), then $(\Phi^n(\alpha) - \beta_n) \in \Gamma_1$ if and only if $\Phi^n(\alpha) - \beta_n \in H$, as desired. $\qquad\square$

In Section 4.5 we show that the set R_1 from Equation (14) (which is the same as the set $\tilde{R}_1$ from (21), as shown in Lemma 4.3) is indeed a union of finitely many arithmetic progressions along with a set of Banach density equal to 0.

4.5. Deriving the conclusion for the new finitely generated subgroup.
Throughout this section we let H be as in Equation (20). Then there exists a finitely generated subgroup $E \subset \mathbb{G}_m(K)$ such that $H \subseteq E^N$ (we can, for example, take E to be the subgroup of K^* spanned by all the entries of the generators of H).

Next, for each $i = 1, \ldots, N$, we let $f_i: Y \dashrightarrow \mathbb{G}_m$ be the rational map

$$f_i := \pi_i \circ \Theta. \tag{22}$$

Now, [BCH21, Theorem 1.1] and [BG, Theorem 1.1] yield that for each $i = 1, \ldots, N$, the set

$$U_i := \left\{ n \in \mathbb{N}_0 \colon f_i\left(\tilde{\Phi}^n\left(\tilde{\alpha}\right) \right) \in E \right\} \tag{23}$$

is a union of finitely many arithmetic progressions along with a set of Banach density equal to 0. Therefore, using that intersections of finite unions of arithmetic progressions is also a finite union of arithmetic progressions, along with the fact that $H \subseteq E^N$, then we see that

$$R_1 \subseteq \tilde{U} := \bigcap_{i=1}^{N} U_i \tag{24}$$

and moreover, $\tilde{U}$ is a union of finitely many arithmetic progressions along with a set of Banach density equal to 0.

Remark 4.4. The variety Y is not necessarily irreducible. However, it has finitely many irreducible components Y_j and an iterate $\tilde{\Phi}^\ell$ of $\tilde{\Phi}$ induces rational self-maps on each irreducible component of Y. Hence the result of [BG] can be applied to each $(Y_j, \tilde{\Phi}^\ell)$, which still allows us to derive the desired conclusion about each U_i (see also Remark 4.1). Also, note that the results of [BCH21] (which were then extended in [BG] for fields of arbitrary characteristic) are written for arbitrary varieties, not necessarily irreducible. Therefore, even when Y is not irreducible, we still obtain the desired description of each U_i from Equation (24) as a union of finitely many arithmetic progressions along with a set of Banach density equal to 0.

Next we fix such an (infinite) arithmetic progressions $\{nk+\ell\}_{n\in\mathbb{N}_0}$ appearing in $\tilde{U}$ (for some given $k \in \mathbb{N}$ and $\ell \in \mathbb{N}_0$). Recalling the definition of the β_n's as in Equation (16) (see also equation (17)), we know that

$$\Phi^{nk+\ell}(\alpha) - \beta_{nk+\ell} \in E^N \text{ for each } n \in \mathbb{N}_0. \tag{25}$$

Lemma 4.5. *Let k and ℓ be as in Equation (25). Then the set of integers of the form $nk + \ell$ with the property that $\Phi^{nk+\ell}(\alpha) \in \Gamma_1$ is a union of finitely many arithmetic progressions along with a set of Banach density equal to 0.*

Proof. First, we recall (as shown in Lemma 4.3) that $\Phi^{nk+\ell}(\alpha) \in \Gamma_1$ if and only if $\Phi^{nk+\ell}(\alpha) - \beta_{nk+\ell} \in H$.

On the other hand, using Equation (25) along with [BCH21, Corollary 1.3] and [BG, Corollary 1.3] yields that for some generators $v_1, \ldots, v_s$ of E (and also at the expense of replacing the arithmetic progressions $\{nk+\ell\}$ by finitely many arithmetic progressions, which is again admissible due to Remark 4.1), we can write

$$\Phi^{nk+\ell}(\alpha) - \beta_{nk+\ell} = \left(\prod_{i=1}^{s} v_i^{b_i^{(j)}(n)} \right)_{1 \leq j \leq N}, \tag{26}$$

where each sequence $\{b_i^{(j)}(n)\}$ is a linear recurrence sequence of integers. Applying Proposition 2.2 to the linear recurrence sequence

$$\left(\prod_{i=1}^{s} v_i^{b_i^{(j)}(n)}\right)_{1 \leq j \leq N}$$

with respect to the subgroup H of E^N yields that the set of $n \in \mathbb{N}_0$ for which $\Phi^{nk+\ell}(\alpha) - \beta_{nk+\ell} \in H$ is a union of finitely many arithmetic progressions along with a finite set. $\qquad\square$

Lemma 4.5 yields that the set $R_1 = \tilde{R}_1$ (see also Lemma 4.3) is a finite union of arithmetic progressions along with a set of Banach density equal to 0.

4.6. Conclusion of the proof of Theorem 1.1 (i).

We have shown that the set $R_1 = R(G, \Phi, \alpha, \Gamma_1)$ is a union of finitely many arithmetic progressions along with a set of Banach density equal to 0.

Next, we let $\epsilon_0, \ldots, \epsilon_{m-1} \in \mathbb{G}_m^N(K) \subseteq G(K)$ be elements that are linearly independent with respect to Γ_1; the existence of such points $\epsilon_0, \ldots, \epsilon_{m-1}$ is guaranteed by the fact that

$$\dim_{\mathbb{Q}} \mathbb{G}_m^N(K) \otimes_{\mathbb{Z}} \mathbb{Q} = \infty.$$

Then we let $\beta_i' := \beta_i + \epsilon_i$ for each $i = 0, \ldots, m-1$. We let Γ_1' be the finitely generated subgroup spanned by Γ along with $\beta_0', \ldots, \beta_{m-1}'$. Then Proposition 2.4 yields that

$$\Gamma_1 \cap \Gamma_1' = \Gamma. \qquad (27)$$

Applying the exact same argument as before to $R_1' := R(G, \Phi, \alpha, \Gamma_1')$, we conclude that R_1' is also a union of finitely many arithmetic progressions along with a set of Banach density equal to 0. But then using Equation (27), we get that $R = R(G, \Phi, \alpha, \Gamma)$ equals $R_1 \cap R_1'$ and since intersections of finitely many arithmetic progressions is a union of finitely many arithmetic progressions, then we obtain the desired conclusion in Theorem 1.1.

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University of Waterloo, Department of Pure Mathematics, Waterloo, On-
tario, N2L 3G1, Canada
    Email address: jpbell@uwaterloo.ca

University of British Columbia, Department of Mathematics, Vancouver,
BC, V6T 1Z2, Canada
    Email address: dghioca@math.ubc.ca