A BERTINI TYPE THEOREM FOR PENCILS OVER FINITE FIELDS

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Abstract. We study the question of finding smooth hyperplane sections to a pencil of hypersurfaces over finite fields.

1. Introduction

Given a smooth projective variety $X \subset \mathbb{P}^n$ over the complex numbers, the classical Bertini theorem asserts the existence of a hyperplane H such that $X \cap H$ is smooth. The statement remains valid over an arbitrary infinite field k. For example, every smooth $\mathbb{Q}$-variety admits a smooth $\mathbb{Q}$-hyperplane section. However, if $k = \mathbb{F}_q$ is a finite field, there are counter-examples to the statement. The following example is due to Nick Katz [Kat99]. Consider the surface $S \subset \mathbb{P}^3_{\mathbb{F}_q}$ defined by

$$X^qY - XY^q + Z^qW - ZW^q = 0$$

One can check that each $\mathbb{F}_q$-hyperplane $H \subset \mathbb{P}^3$ is tangent to the surface S, and so $S \cap H$ is singular for every choice of H in this case [ADL19, Example 3.4].

If the field $\mathbb{F}_q$ has sufficiently large cardinality with respect to the degree of X, then we still expect to find smooth hyperplane sections. A theorem of Ballico [Bal03] shows that for $q \geq d(d-1)^{n-1}$, any smooth hypersurface $X \subset \mathbb{P}^n$ of degree d admits an $\mathbb{F}_q$-hyperplane H such that $X \cap H$ is smooth. When X is a plane curve, a sharper bound of $q \geq d-1$ has been obtained under a stronger hypothesis of reflexivity [Asg19].

We restrict our attention to the case of hypersurfaces. If $X \subset \mathbb{P}^n$ is a hypersurface, we say that a given hyperplane H is *transverse* to X if $X \cap H$ is smooth.

In this paper, we study a pencil of hypersurfaces defined over $\mathbb{F}_q$ and ask for an $\mathbb{F}_q$-hyperplane which is simultaneously transverse to all the $\mathbb{F}_q$-members of the pencil. We take two different hypersurfaces $X_1 = \{F = 0\}$ and $X_2 = \{G = 0\}$ of the same degree, and consider the $\mathbb{F}_q$-members of the pencil generated by X_1 and X_2. In other words, we examine the $q+1$ hypersurfaces,

$$X_{[s:t]} = \{sF + tG = 0\}$$

where $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$. The main question can be phrased as follows:

Question 1.1. Suppose that each member of the pencil spanned by X_1 and X_2 admits a transverse hyperplane over $\overline{\mathbb{F}_q}$. Provided that q is sufficiently large with respect to d, can we find an $\mathbb{F}_q$-hyperplane H such that H is simultaneously transverse to $X_{[s:t]}$ for each $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$?

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The case $d = 1$ is clear, because we can simply pick H to be any hyperplane that is not in the pencil, and any two distinct hyperplanes intersect transversely. We assume $d > 1$ throughout the paper. In a similar vein with Question 1.1, one may be inclined to ask for the existence of an $\overline{\mathbb{F}_q}$-hyperplane H such that H is transverse to all the $\overline{\mathbb{F}_q}$-members of a given pencil. However, this cannot be attained because any hyperplane H must intersect some members of the pencil non-transversely. This is proved in Lemma 3.1.

Our main result asserts that the answer to Question 1.1 is positive if we allow a base extension. The result rests on the following natural assumption on the pencil:

Assumption on the pencil. Suppose that $X_1, X_2 \subset \mathbb{P}^n$ are two hypersurfaces of degree d defined over a finite field k. We will say that the pencil generated by X_1 and X_2 satisfies the condition **(T)** if the following hold:

(1) Each member of the pencil has a transverse hyperplane over $\overline{k}$.
(2) The pencil has a smooth member defined over $\overline{k}$.

Theorem 1.2. *Let $n \geq 2$ and $d \geq 2$ be positive integers with $p \nmid n(d-1)$. Suppose that $X_1, X_2 \subset \mathbb{P}^n$ are two hypersurfaces of degree d defined over a finite field k of characteristic p satisfying the assumption (**T**). Then there exists a finite field extension k'/k such that the following holds: for all finite fields $\mathbb{F}_q \supseteq k'$, there exists an $\mathbb{F}_q$-hyperplane H such that H is transverse to $X_{[s,t]}$ for each $[s:t] \in \mathbb{P}^1(\mathbb{F}_q)$.*

Remark 1.3. The finite field extension k'/k depends only on n and d, but not on the pencil itself. This assertion will be explicitly justified in the proof.

Remark 1.4. As it will be mentioned in the proof, the hypothesis $p \nmid n(d-1)$ is needed to ensure that a certain map is separable. The required separability condition would also follow if we had instead imposed the following geometric condition: there exists a hyperplane H defined over $\overline{k}$ such that H is tangent to $n(d-1)^{n-1}$ many *distinct* hypersurfaces in the pencil (see Lemma 3.1 for more context).

Remark 1.5. The hypothesis that a pencil has at least one smooth member defined over $\overline{k}$ is fairly mild. Indeed, a pencil can be viewed as a $\mathbb{P}^1$ inside the parameter space of all hypersurfaces of degree d in $\mathbb{P}^n$. The condition that the pencil admits a smooth member is equivalent to the statement that the corresponding $\mathbb{P}^1$ is not contained inside the discriminant hypersurface $\mathcal{D}_{d,n}$, which parametrizes singular hypersurfaces of degree d in $\mathbb{P}^n$. A generically chosen line is not contained inside $\mathcal{D}_{d,n}$, and so a generic pencil contains a smooth member.

Remark 1.6. According to our definition, a hyperplane H is said to be transverse to X if H provides a smooth hyperplane section of X. This condition automatically implies that $H \notin X^*$ where X^* is the dual hypersurface parametrizing tangent hyperplanes to X. More precisely, X^* is the closure of the image of the Gauss map of X. However, the converse implication is not true. For example a line L passing through the singularity of an irreducible nodal cubic C is not transverse according to our definition, but still satisfies $L \notin C^*$. Some authors, such as [Bal03], defines H to be transverse when the weaker condition $H \notin X^*$ is satisfied. Note that if X is smooth, then $H \notin X^*$ if and only if $X \cap H$ is smooth. Thus, for smooth hypersurfaces, these two definitions of "transverse hyperplane" coincide.

We sketch here the plan for our paper. In Section 2 we discuss our Question 1.1 in the context of plane curves. Then we prove Theorem 1.2 in Section 3. Finally, we conclude our paper by a brief discussion of whether we need to consider a base extension from k to k' as in the conclusion of Theorem 1.2; in particular, we prove in Proposition 3.3 that for a pencil of reduced plane conics (with at least one smooth conic in the $\overline{\mathbb{F}_q}$-pencil), there always exists a common transverse line to each element of the $\mathbb{F}_q$-pencil (as long as $q \geq 16$).

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2. Plane curves

In this Section, we discuss more broadly Question 1.1 in the context of plane curves. In particular, we show (see Proposition 2.3) that given any N reduced plane curves of degree d, there exists a common $\mathbb{F}_q$-line transverse to each one of these N curves, as long as $q \geq 2Nd(d-1)$. Therefore, it makes sense to consider our Question 1.1 in which we search for a common $\mathbb{F}_q$-line transverse to each curve in a given set of $q+1$ curves. On the other hand, we show in Example 2.6 that there exists a set of $q+1$ smooth plane curves with the property that no $\mathbb{F}_q$-line is simultaneously transverse to each curve in our set. Hence, this suggests even more the setup considered in Question 1.1 in which we consider a *pencil* of plane curves, or more generally of hypersurfaces in $\mathbb{P}^n$.

The setup for this Section is to have two plane curves $C_1 = \{F = 0\}$ and $C_2 = \{G = 0\}$ in $\mathbb{P}^2$ defined over $\mathbb{F}_q$. The polynomials $F, G \in \mathbb{F}_q[x, y, z]$ are homogenous of degree d, and we assume that $C_1 \cap C_2$ is finite, i.e. the curves C_1 and C_2 do not share any components. We consider the pencil of plane curves,

$$C_{[s:t]} = \{sF + tG = 0\}$$

We are interested in finding a line $L \subset \mathbb{P}^2$ defined over $\mathbb{F}_q$ such that L is simultaneously transverse to the $q+1$ members $C_{[s:t]}$ as $[s:t]$ varies in $\mathbb{P}^1(\mathbb{F}_q)$. Note that a line $L \subset \mathbb{P}^2$ is transverse to a curve $C \subset \mathbb{P}^2$ if and only if $L \cap C$ consists of $d = \deg(C)$ distinct points (over $\overline{\mathbb{F}_q}$).

We need the following result on the number of $\mathbb{F}_q$-points to an arbitrary plane curve which is used in the proof of Proposition 2.2.

Lemma 2.1. *Suppose $X \subset \mathbb{P}^2$ is a plane curve of degree d defined over $\mathbb{F}_q$. Then the number of $\mathbb{F}_q$-points of X can be bounded by:*

$$\#X(\mathbb{F}_q) \leq dq + 1$$

The equality occurs if X is a union of d lines, each defined over $\mathbb{F}_q$, passing through a common $\mathbb{F}_q$-point P_0.

We note that Lemma 2.1 is covered by a result of Serre [Ser91] who proved an upper bound on the number of $\mathbb{F}_q$-points for an arbitrary projective hypersurface in $\mathbb{P}^n$. Serre's result was generalized to all projective varieties by [Cou16].

Proposition 2.2. *Let $C \subset \mathbb{P}^2$ be a reduced plane curve of degree d defined over $\mathbb{F}_q$. If $q \geq 2d(d-1)$, then there exists a transverse $\mathbb{F}_q$-line to C.*

Proof. Given a line $L = \{ax + by + cz = 0\} \subset \mathbb{P}^2$, we will show that the condition that L is *not* transverse to $C = \{F = 0\}$ can be expressed in terms of vanishing of a certain discriminant. Indeed, we can solve for the intersection points $C \cap L$ by substituting $z = -(a/c)x - (b/c)y$ into the equation of $F(x, y, z) = 0$ to obtain $F(x, y, -(a/c)x - (b/c)y) = 0$. After homogenizing (which takes care of the possibility that c could be 0 in the above expression), the equation represents vanishing of a binary form $B_L(x, y)$ of degree d in variables x and y with coefficients that are homogenous in variables a, b, c with degree d. The line L is non-transverse to C if this binary form B_L has a repeated root on $\mathbb{P}^1$, i.e. the discriminant of B_L vanishes. Since $\mathrm{disc}(B_L)$ has degree $2d-2$ in the coefficients of the binary form, and the coefficients themselves are degree d in variables a, b, c, we can view

$$\mathrm{disc}(B_L) \in \mathbb{F}_q[a, b, c]$$

as a homogenous form H of degree $(2d-2)d = 2d(d-1)$ in variables a, b, c. By viewing a particular line L as a point $[p : q : r] \in (\mathbb{P}^2)^*$ in the dual space, we deduce that L is tangent to C if and only if the point $[p : q : r]$ lies on the plane curve $D = \{H = 0\}$. In particular,

$$\#\{L \in (\mathbb{P}^2)^*(\mathbb{F}_q) \mid L \text{ is a line not transverse to } C\} \leq \#D(\mathbb{F}_q)$$

Since D is a plane curve of degree $2d(d-1)$, the number of $\mathbb{F}_q$-points of D can be bounded by $2d(d-1)q + 1$ by Lemma 2.1. Since the total number of $\mathbb{F}_q$-lines in $\mathbb{P}^2$ is $q^2 + q + 1$, we will obtain a transverse $\mathbb{F}_q$-line to C provided that

$$q^2 + q + 1 > 2d(d-1)q + 1$$

This last inequality is equivalent to $q + 1 > 2d(d-1)$, that is, $q \geq 2d(d-1)$. ☐

Using the same idea as in the previous proposition, we obtain:

Proposition 2.3. *Let $C_1, C_2, ..., C_N$ be N reduced plane curves of degree $d > 1$ in $\mathbb{P}^2$ defined over $\mathbb{F}_q$. If $q \geq 2Nd(d-1)$, then there exists a common $\mathbb{F}_q$-line which is simultaneously transverse to C_i for each $1 \leq i \leq N$.*

Proof. As in the proof of the previous proposition, we obtain that the number of non-transverse $\mathbb{F}_q$-lines to C_i is at most $2d(d-1)q + 1$. Thus, the number of lines that are non-transverse to at least one of the curves $C_1, C_2, ..., C_N$ is at most $N \cdot (2d(d-1)q + 1)$. So, we will obtain a common transverse $\mathbb{F}_q$-line to all C_i if

$$q^2 + q + 1 > N \cdot (2d(d-1)q + 1)$$

This inequality will be satisfied for $q \geq 2Nd(d-1)$ according to the following computation.

$$\begin{aligned} q^2 + q + 1 &= q(q+1) + 1 \geq q(2Nd(d-1) + 1) + 1 \\ &= 2Nd(d-1)q + q + 1 > 2Nd(d-1)q + N = N \cdot (2d(d-1)q + 1) \end{aligned}$$

where in the last inequality we used the fact that $q + 1 > N$ which is valid under the assumption $q \geq 2d(d-1)N$. ☐

However, if the number of curves depend also on q, then the existence of a simultaneous transverse $\mathbb{F}_q$-line is not guaranteed.

Proposition 2.4. *For each $d \geq 2$, there exist $q + 1$ plane curves $C_1, C_2, ..., C_{q+1}$ of degree d such that there is no $\mathbb{F}_q$-line which is transverse to each C_i.*

Proof. Fix an $\mathbb{F}_q$-line L_0 in $\mathbb{P}^2$. After enumerating the $q+1$ $\mathbb{F}_q$-points $P_1, P_2, ..., P_{q+1}$ on $L_0 = \mathbb{P}^1$, construct the curve C_i such that C_i is *any* given degree d curve that is singular at the point P_i. The resulting collection of curves $C_1, ..., C_{q+1}$ satisfy the conclusion of the claim. Indeed, each $\mathbb{F}_q$-line L meets L_0 at a unique point $P_i \in L_0$ (depending on L), and so L passes through the singular point of C_i, implying that L is not transverse to C_i. Thus, no $\mathbb{F}_q$-line L can be simultaneously transverse to all the $q+1$ curves $C_1, C_2, ..., C_{q+1}$. □

It would be more satisfying to have examples of smooth curves satisfying the conclusion of Proposition 2.4. We conjecture that such a collection of $q+1$ curves exist.

Conjecture 2.5. For each $d \geq 2$, there exist $q+1$ *smooth* curves $C_1, C_2, ..., C_{q+1}$ in $\mathbb{P}^2$ of degree d such that there is no $\mathbb{F}_q$-line which is transverse to each C_i.

We can prove the conjecture in the special case when $d = 2$.

Example 2.6. Suppose that the characteristic of the field is $p > 2$. We want to construct $q+1$ smooth conics $C_1, ..., C_{q+1}$ such that each $\mathbb{F}_q$-line L in $\mathbb{P}^2$ is tangent to at least one of C_i. The set of tangent lines to a given smooth conic C is parametrized by the dual curve C^* which also has degree $d(d-1) = 2$. The condition that no $\mathbb{F}_q$-line is transverse to all of $C_1, ..., C_{q+1}$ can be translated into the statement that the $\mathbb{F}_q$-points of the corresponding dual curves $C_1^*, ..., C_{q+1}^*$ fill up all the $\mathbb{F}_q$-points of $(\mathbb{P}^2)^*$.

Motivated by the observation above, we proceed to construct $q+1$ smooth conics $D_1, D_2, ..., D_{q+1}$ such that

$$\bigcup_{i=1}^{q+1} D_i(\mathbb{F}_q) = \mathbb{P}^2(\mathbb{F}_q)$$

Consider the collection of 4 points $\{P_1, P_2, P_3, P_4\} \subset \mathbb{P}^2(\overline{\mathbb{F}_q})$ such that $\{P_1, P_2, P_3\}$ is a $\mathrm{Gal}(\mathbb{F}_{q^3} / \mathbb{F}_q)$-orbit of the point $P_1 \in \mathbb{P}^2(\mathbb{F}_{q^3})$, while $P_4 \in \mathbb{P}^2(\mathbb{F}_q)$. In other words, if we write $P_1 = [a : b : c] \in \mathbb{P}^2(\mathbb{F}_{q^3})$, then $P_2 = [a^q : b^q : c^q]$ and $P_3 = [a^{q^2} : b^{q^2} : c^{q^2}]$.

Furthermore, we can pick the collection $B := \{P_1, P_2, P_3, P_4\}$ in such a way that no three of P_i are collinear. The vector space of homogeneous quadratic polynomials in 3 variables passing through B has dimension $6 - 4 = 2$, and so we get a pencil of conics with base locus B. If $\{F_1, F_2\}$ is an $\mathbb{F}_q$-basis for this vector space, then we consider the $q+1$ members of the pencil,

$$D_{[s:t]} := \{sF_1 + tF_2 = 0\}$$

where $[s : t] \in \mathbb{P}^1(\mathbb{F}_q)$. We claim that each $D_{[s:t]}$ is smooth. Indeed, there are only three singular conics (geometrically) in this pencil, and they are union of two lines passing through $B = \{P_1, P_2, P_3, P_4\}$. Using the notation $\overline{PQ}$ for the line passing through P and Q, these 3 singular conics are:

$$S_1 := \overline{P_1P_2} \cup \overline{P_3P_4}$$
$$S_2 := \overline{P_2P_3} \cup \overline{P_1P_4}$$
$$S_3 := \overline{P_1P_3} \cup \overline{P_2P_4}$$

However, none of the S_i for $1 \leq i \leq 3$ is defined over $\mathbb{F}_q$. In fact, S_1 is strictly defined over the field $\mathbb{F}_{q^3}$, and Frobenius action sends $S_1 \to S_2 \to S_3 \to S_1$, and so

$\{S_1, S_2, S_3\}$ is a Galois orbit of the Frobenius. In particular, each $D_{[s:t]}$ is a smooth conic, and together they cover the $\mathbb{F}_q$-points of $\mathbb{P}^2$. Indeed, on one hand, they all pass through $P_4 \in \mathbb{P}^2(\mathbb{F}_q)$; on the other hand, for each $P \in \mathbb{P}^2(\mathbb{F}_q) \setminus \{P_4\}$, the conic $D_{[-F_2(P):F_1(P)]}$ passes through P. We re-label the elements of the pencil,

$$\{D_{[s:t]} \mid [s,t] \in \mathbb{P}^1(\mathbb{F}_q)\} = \{D_1, D_2, ..., D_{q+1}\}$$

So $D_1, ..., D_{q+1}$ are smooth conics which together cover the set $\mathbb{P}^2(\mathbb{F}_q)$. Finally, we let $C_i = (D_i)^*$ to be the corresponding dual curve for each $1 \leq i \leq q+1$. By reflexivity, we have $D_i = (C_i)^*$, and so the tangent lines to C_i for $1 \leq i \leq q+1$ together cover all the $\mathbb{F}_q$-lines of $\mathbb{P}^2$, i.e. the collection of smooth conics $C_1, ..., C_{q+1}$ admit no common transverse $\mathbb{F}_q$-line.

3. Main Result

In order to establish Theorem 1.2, we will need the following lemma.

Lemma 3.1. *Consider a pencil of hypersurfaces generated by X_1 and X_2 in $\mathbb{P}^n$ defined over k. Given a hyperplane $H \subset \mathbb{P}^n$, either H is non-transverse to every $\overline{k}$-member of the pencil, or H is non-transverse to exactly $n(d-1)^{n-1}$ members of the pencil, counted with appropriate multiplicities.*

Proof. We have $X_1 = \{F_1 = 0\}$ and $X_2 = \{F_2 = 0\}$ where $F_1, F_2 \in \mathbb{F}_q[x_0, ..., x_n]$ are homogeneous polynomials of degree d. By definition, the elements of the pencil are of the form $X_{[s:t]} = \{sF_1 + tF_2 = 0\}$ as $[s:t]$ varies in $\mathbb{P}^1$. Suppose that H is an arbitrary hyperplane in $\mathbb{P}^n$. After a linear change of coordinates, we may assume that $H = \{x_n = 0\}$. We can restrict the original pencil to the hyperplane H to obtain a new pencil whose elements are of the form,

$$\tilde{X}_{[s:t]} = \{sF_1(x_0, x_1, ..., x_{n-1}, 0) + tF_2(x_0, x_1, ..., x_{n-1}, 0) = 0\}$$

which can be viewed as a pencil of hypersurfaces in $\mathbb{P}^{n-1}$. Note that H is transverse to $X_{[s:t]}$ if and only if $\tilde{X}_{[s:t]} = X_{[s:t]} \cap H$ is smooth. Thus, our task has been reduced to understanding how many of $\tilde{X}_{[s:t]}$ are singular. Let $\mathcal{D}_{d,n-1}$ be the discriminant hypersurface parametrizing singular hypersurfaces of degree d in $\mathbb{P}^{n-1}$, and $\mathcal{P} \cong \mathbb{P}^1$ be the pencil whose members are $\tilde{X}_{[s:t]}$. Either $\mathcal{P} \subset \mathcal{D}_{d,n-1}$ or $\mathcal{P} \not\subset \mathcal{D}_{d,n-1}$. In the first case, H is non-transverse to every member $X_{[s:t]}$ of the original pencil. In the second case, the number of the singular members of $\mathcal{P}$ is given by the degree of the discriminant $\mathcal{D}_{d,n-1}$, which is $n(d-1)^{n-1}$ according to [EH16, Proposition 7.4]. Thus, H is non-transverse to exactly $n(d-1)^{n-1}$ members of the original pencil, counted with multiplicity. □

We are now ready to present the proof of the main result.

Proof of Theorem 1.2. We have a pencil of hypersurfaces generated by X_1 and X_2 such that the generic member of the pencil is smooth. Given $\zeta \in \mathbb{P}^1$, we will denote by X_ζ to be the corresponding member of the pencil. Consider the variety,

$$V = \{(H, \zeta) \mid H \text{ is not transverse to } X_\zeta\} \subset (\mathbb{P}^n)^* \times \mathbb{P}^1$$

We claim that V is a geometrically irreducible variety. To see this, we observe that

$$V = \overline{\{(H, \zeta) \mid H \text{ is tangent to } X_\zeta \text{ at a smooth point}\}} \subset (\mathbb{P}^n)^* \times \mathbb{P}^1$$

With this presentation, V is the image of the projection of the following variety onto its first two factors.

$$\mathcal{I} = \overline{\{(H, \zeta, P) \mid H \text{ is tangent to } X_\zeta \text{ at its smooth point } P\}} \subset (\mathbb{P}^n)^* \times \mathbb{P}^1 \times \mathbb{P}^n$$

The variety $\mathcal{I}$ is the relative version of the conormal variety [Kle86] associated with the given pencil. To see that $\mathcal{I}$ is geometrically irreducible, we write $\mathcal{I} = \overline{\mathcal{U}}$ where

$$\mathcal{U} = \{(H, \zeta, P) \mid H \text{ is tangent to } X_\zeta \text{ at its smooth point } P\}$$

The projection map $\pi_{2,3} \colon \mathcal{U} \to \mathbb{P}^1 \times \mathbb{P}^n$ yields an isomorphism of U onto its irreducible image. Thus, U is geometrically irreducible and so is $\mathcal{I} = \overline{\mathcal{U}}$. Therefore, the variety V, realized as the image of $\mathcal{I}$, is also geometrically irreducible.

Now, we consider the projection $\pi_1 \colon V \to (\mathbb{P}^n)^*$. Note that π_1 is surjective, because any chosen hyperplane is non-transverse to at least one element of the pencil by Lemma 3.1. In fact, Lemma 3.1 shows that a fiber of π_1 either consists of $n(d-1)^{n-1}$ points (which is the generic case) or is an entire $\mathbb{P}^1$. Let

$$Z = \{P \in (\mathbb{P}^n)^* \mid \pi_1^{-1}(P) = \mathbb{P}^1\}$$

consist of those hyperplanes P that are simultaneously non-transverse to all the members of the pencil. In particular, such a hyperplane $P \in X_1^* \cap X_2^*$ for any two smooth members X_1, X_2 of the pencil. This shows that $Z \subset X_1^* \cap X_2^*$ and therefore $\dim(Z) \leq n-2$. In particular, Z is a proper Zariski-closed subset in $(\mathbb{P}^n)^*$. Since V is geometrically irreducible, we can apply [PS20, Theorem 1.8] to deduce that the locus

$$M_{\text{bad}} = \{\text{hyperplanes } H \subset (\mathbb{P}^n)^* \mid \pi_1^{-1}(H) \text{ is not geometrically irreducible}\}$$

differs from a proper Zariski-closed subset by at most a constructible set of dimension 1. As a result, $M_{\text{bad}} \neq (\mathbb{P}^n)^*$. Thus, there exists a hyperplane $\mathcal{H} \hookrightarrow (\mathbb{P}^n)^*$ such that $\mathcal{H} \notin M_{\text{bad}}$. Thus, we obtain a map $\pi_1 \colon \pi_1^{-1}(\mathcal{H}) \to \mathcal{H}$. We apply [PS20, Theorem 1.8] again to this new morphism, and continue inductively until we find a line $B = \mathbb{P}^1 \subset \mathbb{P}^{n-1}$ such that $W := \pi_1^{-1}(B)$ is a geometrically irreducible curve. Let k_1/k be a finite field extension such that B and W are defined over k_1. We claim that $[k_1 : k]$ depends only on n and d. Indeed, M_{bad} is a proper closed set whose degree and dimension are bounded by n and d. It is clear that $\dim(M_{\text{bad}}) \leq n$ and to see that the degree of M_{bad} only depends on d (and not on the specific pencil), we can run the argument with the generic pencil where the coefficients of generators are indeterminates, and then specialize the coefficients. Thus, Lang-Weil theorem ensures the existence of an $\mathbb{F}_q$-point in $(\mathbb{P}^n)^* \setminus M_{\text{bad}}$ for q sufficiently large with respect to n and d. The same observation is true for each iteration of the inductive process, explaining why the degree $[k_1 : k]$ depends only on n and d.

We obtain a finite map $f : W \to B \cong \mathbb{P}^1$ of geometrically irreducible curves over the field k_1; its degree is $m := \deg(\pi_1) = n(d-1)^{n-1}$ by Lemma 3.1, which is larger than 1. Furthermore, the map is separable due to the hypothesis $p \nmid n(d-1)$. Note that $B \subset (\mathbb{P}^n)^*$, so a point $P \in B$ will correspond to a hyperplane P in $\mathbb{P}^n$. The fiber $f^{-1}(P)$ above a given point $P \in B$ will be:

$$f^{-1}(P) = \left\{\zeta \in \mathbb{P}^1 \mid P \text{ is non-transverse to } X_\zeta\right\}$$

which is a finite set inside $\mathbb{P}^1$.

Using the formulation above, we observe that a given $\mathbb{F}_q$-hyperplane $P \in B$ is simultaneously transverse to all the $\mathbb{F}_q$-members of the pencil generated by X_1 and

X_2 if and only if the fiber $f^{-1}(P)$ contains no $\mathbb{F}_q$-points of $\mathbb{P}^1$. In order to show the existence of such a point P, we will apply the Twisting Lemma of Dèbes and Legrand [DL12] to the cover W/B after applying a suitable base extension. Note that $f : W \to B$ is a cover of geometrically irreducible curves; so, there exists a finite extension k'/k_1 such that the base extension of the cover $W_{k'}/B_{k'}$ has a regular Galois cover $Z_{k'}/B_{k'}$. More explicitly, k' is the closure of k_1 inside the function field $Z(k_1)$. We also note that for any finite field $\mathbb{F}_q \supseteq k'$, it is still true that $Z_{\mathbb{F}_q}/B_{\mathbb{F}_q}$ is a regular Galois cover.

We claim that k'/k depends only on n and d. Indeed, k' is the algebraic closure of k_1 inside $k_1(Z)$ and so, $[k' : k_1]$ is bounded above by $[k_1(Z) : k_1(B)]$ because $k_1(B)$ is the rational function field over k_1 (since B is isomorphic to $\mathbb{P}^1$) and so, k_1 is closed inside $k_1(B)$. Moreover, Z/B is the Galois closure of W/B. As W/B has degree $n(d-1)^{n-1}$, it follows that Z/B has degree bounded above by $(n(d-1)^{n-1})!$. We deduce that $[k' : k_1]$ is uniformly bounded solely in terms of n and d. This shows that the extension k'/k_1 and therefore also k'/k depends only on n and d.

For the rest of the proof, let $\mathbb{F}_q \supseteq k'$ be any finite field. Let G be the Galois group of $Z_{\mathbb{F}_q}/B_{\mathbb{F}_q}$; we view G as a subgroup of S_m.

We will apply [DL12, Lemma 3.4] to the map $f : W_{\mathbb{F}_q} \to B_{\mathbb{F}_q}$ in order to obtain a point $P \in B(\mathbb{F}_q)$ with the property that no point in $f^{-1}(P)$ is contained in $W(\mathbb{F}_q)$.

We need first a cyclic subgroup H of G generated by an element $\sigma \in S_m$ with the property that σ fixes no element in $\{1, \dots, m\}$ (note that $m > 1$). Indeed, for any Galois group G (seen as a subgroup of S_m), there exists an element $\sigma \in G$ which has no fixed point in $\{1, \dots, m\}$ because G is a transitive group, which means that the stabilizers of the elements in $\{1, \dots, m\}$ are all conjugated and finally, no group is a union of conjugates of a given proper subgroup.

So, we let H be a cyclic subgroup of G generated by an element σ which has no fixed points (as above); we let r be the number of all cycles appearing in $\sigma \in S_m$. We consider the étale $\mathbb{F}_q$-algebra $\prod_{\ell=1}^r E_\ell$, where the E_ℓ's are field extensions of $\mathbb{F}_q$ of degrees equal to the orders of the cycles appearing in the permutation σ. Then we apply [DL12, Lemma 3.4] to the étale algebra $\prod_{\ell=1}^r E_\ell/\mathbb{F}_q$ to obtain a point $P \in B(\mathbb{F}_q)$ with the property that $f^{-1}(P)$ splits into r Galois orbits of order $[E_\ell : \mathbb{F}_q]$; in particular, none of the points in $f^{-1}(P)$ would be contained in $W(\mathbb{F}_q)$ since each of these Galois orbits would have cardinality larger than 1 (because σ does not have fixed points).

Now, the hypothesis in applying [DL12, Lemma 3.4] is satisfied because the (const/comp) condition from [DL12, Section 3.1.1] is automatically satisfied for regular covers. We need to check the following two conditions, namely [DL12, Lemma 3.4, conditions (ii)-1 and (ii)-2]:

(1) This condition is automatically satisfied for large q, because the Lang-Weil bounds for the number of points of curves defined over finite fields guarantees the existence of many rational points on the corresponding twisted covers of Z, which are curves of the same genus as the genus of Z (see also the proof of [DL12, Corollary 4.3]). Note that q can be made to be sufficiently large by extending the field k' even further in a way so that $[k' : k]$ would still only depend on n and d; indeed, Lang-Weil bounds apply once q is larger than some function of the genus of Z. Since Z is a degree δ cover of $\mathbb{P}^1$, where δ is bounded above solely in terms of d and n, it follows that the genus of Z is also bounded solely in terms of d and n.

(2) This condition is satisfied as explained in the discussion regarding cyclic specializations (since our group H is cyclic) on [DL12, p. 153].

Therefore, [DL12, Lemma 3.4] yields the existence of a point $P \in B(\mathbb{F}_q)$ such that no point in $f^{-1}(P)$ is contained in $W(\mathbb{F}_q)$, concluding the proof of Theorem 1.2. □

Remark 3.2. In our proof of Theorem 1.2 we used that the ground field k may have to be replaced by k' when considering the Galois closure Z/B for the cover W/B since we want that Z be geometrically irreducible (over k'). Note that there are covers of degree larger than 1 of geometrically irreducible curves W/B (over k) for which each k-point of B has a preimage contained in $W(k)$, thus contradicting the conclusion we seek for the strategy of our proof of Theorem 1.2.

Indeed, we let $k = \mathbb{F}_q$ and $W = B = \mathbb{P}^1_{\mathbb{F}_q}$ for some prime power q satisfying the congruence equation $q \equiv 2 \pmod 3$ and then let $f : \mathbb{P}^1 \longrightarrow \mathbb{P}^1$ be given by $x \mapsto x^3$. Clearly, f induces a permutation of $\mathbb{P}^1_{\mathbb{F}_q}$; so, each point in $B(\mathbb{F}_q)$ has a preimage contained in $W(\mathbb{F}_q)$. On the other hand, the Galois closure of this cover is $Z = \mathbb{P}^1_{\mathbb{F}_{q^2}}$, i.e., we need to perform a base extension of our ground field in order for the Galois cover be geometrically irreducible. Once we replace q by q^2, then $W_{\mathbb{F}_{q^2}}/B_{\mathbb{F}_{q^2}}$ is actually a regular Galois cover and then it is true that there exist points $P \in B(\mathbb{F}_{q^2})$ such that no point in $f^{-1}(P)$ is contained in $W(\mathbb{F}_{q^2})$.

We do not know whether one can choose $k' = k$ in Theorem 1.2 in general, as our proof strategy requires a base extension (see Remark 3.2). It might be reasonable to expect that if the cardinality of the ground field k is sufficiently large (depending only on n and d), then one does not require an additional field extension. For example, the following result establishes that $k' = k$ works for the case of pencil of plane conics (as long as $\#k \geq 16$).

Proposition 3.3. *Suppose that we have a pencil of reduced conics in $\mathbb{P}^2$ defined over $\mathbb{F}_q$ such that the pencil admits at least one smooth member over $\overline{\mathbb{F}_q}$. Provided that $q \geq 16$, we can find an $\mathbb{F}_q$-line L that is simultaneously transverse to all the conics defined over $\mathbb{F}_q$ in the pencil.*

Proof. Suppose that $C_1 = \{F_1 = 0\}$ and $C_2 = \{F_2 = 0\}$ are the two conics that generate the pencil.

We start with some general considerations regarding our proof strategy. First, we observe that if C is a non-smooth reduced conic, then it means that C is a union of two lines $L_1 \cup L_2$ (over $\overline{\mathbb{F}_q}$) and therefore, we have at most $q+1$ lines defined over $\mathbb{F}_q$ which are non-transverse to C (they would correspond to all the $\mathbb{F}_q$-lines passing through the $\mathbb{F}_q$-point of $L_1 \cap L_2$). Second, we note that if C is any smooth conic defined over $\mathbb{F}_q$, then the only possibility for an $\mathbb{F}_q$-line L be non-transverse to C is for L be tangent to C at an $\mathbb{F}_q$-point (since otherwise, we would have that L is tangent to C at two $\overline{\mathbb{F}_q}$-points, contradiction). In particular, if C is a smooth conic which has no $\mathbb{F}_q$-point, then any $\mathbb{F}_q$-line is transverse to C. On the other hand, the number of $\mathbb{F}_q$-points on a smooth $\mathbb{F}_q$-conic (which has at least one $\mathbb{F}_q$-point) is $q+1$ (since then the conic would be isomorphic to $\mathbb{P}^1$ over $\mathbb{F}_q$); furthermore, each such $\mathbb{F}_q$-point has a tangent line defined over $\mathbb{F}_q$. This provides at most $(q+1)\cdot(q+1)$ lines defined over $\mathbb{F}_q$, which are non-transverse to at least one element of the given $\mathbb{F}_q$-pencil. This number is an overestimate since there are only q^2+q+1 lines defined over $\mathbb{F}_q$, and so there is overcounting that needs to be addressed. In order to refine the counting for the number of non-transverse $\mathbb{F}_q$-lines, we need to take into account the fact that a given $\mathbb{F}_q$-line L will be non-transverse to more than one conic.

In the set-up of the proof for the Theorem 1.2, we have the map $\pi_1 : V \to (\mathbb{P}^2)^*$. Given a line $L \in (\mathbb{P}^2)^*$, the fiber $\pi_1^{-1}(L)$ is either a $\mathbb{P}^1$ or consists of 2 conics according to Lemma 3.1. In the first case, the line L is non-transverse to every element of pencil, and in the second case L is non-transverse to exactly 2 conics (counted with multiplicity). In most cases, we see that each non-transverse $\mathbb{F}_q$-line is counted at least twice. However, there is a locus $\mathcal{B} \subset (\mathbb{P}^2)^*$ consisting of those lines $L \in (\mathbb{P}^2)^*$ which are tangent to exactly one conic (with multiplicity 2) in the pencil. We claim that $\mathcal{B}$ is a plane curve of degree 4.

The variety $V \subset \mathbb{P}^1 \times (\mathbb{P}^2)^*$ can be described as the locus $\{R(s,t,a,b,c) = 0\}$ which has bidegree $(2,2)$, that is, degree 2 in variables s,t and degree 2 in variables a,b,c. The two roots $[s:t] \in \mathbb{P}^1$ satisfying $R(s,t,a,b,c) = 0$ exactly correspond to those members of the pencil to which a given line $L = \{ax+by+cz=0\}$ is non-transverse. The condition that these two roots coincide is controlled by the vanishing of the discriminant D of $R(s,t,a,b,c)$ when R is viewed as a homogeneous quadratic polynomial in s and t. Note that $D = D(a,b,c)$ is a degree 4 homogeneous polynomial in a,b,c. By definition, $\mathcal{B} = \{D = 0\}$ and so $\deg(\mathcal{B}) = 4$. By Lemma 2.1, we have $\#\mathcal{B}(\mathbb{F}_q) \leq 4q+1$, and so there are at most $4q+1$ lines over $\mathbb{F}_q$ which are non-transverse to a single conic (with multiplicity 2) in the pencil.

Finally, there are at most three distinct singular conics in a given pencil of conics by [EH16, Proposition 7.4]. Each such conic is a union of two lines, and the only lines that are not transverse are the $\mathbb{F}_q$-lines passing through the singular point. Thus, there are at most $3(q+1)$ non-transverse lines arising from the singular conics in the pencil.

In total, the number of non-transverse $\mathbb{F}_q$-lines to the $\mathbb{F}_q$-members of the pencil is at most $\frac{(q+1)^2}{2} + 4q + 1 + 3(q+1)$. Since the number of $\mathbb{F}_q$-lines is q^2+q+1, we get a simultaneously transverse $\mathbb{F}_q$-line provided that,

$$q^2 + q + 1 > \frac{(q+1)^2}{2} + 4q + 1 + 3(q+1)$$

The inequality above is equivalent to $q^2 > 14q + 7$ which is true for $q \geq 16$. $\square$

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