A MORDELL-LANG TYPE PROBLEM FOR $\mathrm{GL_m}$

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ABSTRACT. We prove a non-abelian variant of the classical Mordell-Lang conjecture in the context of finite-dimensional central simple algebras.

1. INTRODUCTION

Many important classes of Diophantine problems can be formulated in the context of starting with a set of invertible matrices $B_1, \dots, B_r \in \mathrm{GL}_m$ (over some algebraically closed field K) and then given a subvariety $V \subseteq \mathrm{GL}_m(K)$, one studies the set

$$\{(n_1, \dots, n_r) \in \mathbb{Z}^r \colon B_1^{n_1} \cdots B_r^{n_r} \in V\}. \tag{1.1}$$

To give just a small sample of questions that fall under this framework, when $r = 1$, $K = \mathbb{Q}$, and V is the set of matrices whose $(1,1)$-entry is zero then this question is equivalent to Skolem's problem, which asks whether there is a decision procedure to determine if an integer linear recurrence has a zero. By increasing r and using block diagonal matrices, we can similarly study solutions $(n_1, \dots, n_r) \in \mathbb{N}^r$ to $f_1(n_1) + \cdots + f_r(n_r) = 0$ where $f_1, \dots, f_r$ are sequences satisfying linear recurrences. This was studied by Cerlienco, Mignotte, and Piras [CMP87] and it was later shown by Derksen and Masser [DM15] that determining whether there is a solution is undecidable.

The above construction can be further generalized by considering some ambient algebraic variety G endowed with finitely many self-maps $\varphi_1, \dots, \varphi_r$ and then given some starting point $\alpha \in G$ and some subvariety $V \subset G$, one studies the structure of the set:

$$\{(n_1, \dots, n_r) \in \mathbb{N}^r \colon \left(\varphi_1^{\circ n_1} \circ \varphi_2^{\circ n_2} \circ \cdots \circ \varphi_r^{\circ n_r}\right)(\alpha) \in V\}. \tag{1.2}$$

The Mordell-Lang conjecture (proven by Laurent [Lau84] in the case of algebraic tori, by Faltings [Fal91] in the case of abelian varieties and by Vojta [Voj96] for arbitrary semiabelian varieties) asserts that if G is a semiabelian variety (defined over an algebraically closed field K of characteristic 0), then the intersection of any finitely generated subgroup Γ of $G(K)$ with a subvariety $V \subset G$ is a finite union of cosets of subgroups of Γ. This famed conjecture can easily be translated into a question of type (1.2) by considering the above self-maps φ_i on G be translations by elements from a finite set of generators for Γ. This alternative interpretation of the classical Mordell-Lang question led to the formulation of the Dynamical Mordell-Lang Conjecture (see [BGT16] for a comprehensive treatment of this problem).

Recently, two of the authors (see [Hua20] and [GH24]) have investigated S-unit equations in finite-dimensional division rings and by embedding these rings into matrix rings one can phrase many of these "noncommutative S-unit questions" into the general framework above.

As it turns out, there is a natural dichotomy that arises when studying the original problem (1.1) inside GL_m: when the matrices are diagonalizable then the solution sets are well behaved and a version of the Mordell-Lang conjecture holds in this context (see [GTZ11, Theorem 1.3]). Even though the result from [GTZ11] is stated in the context of endomorphisms of semiabelian varieties, the problem translates immediately to a question regarding finitely generated subgroups of GL_m of diagonalizable matrices. Indeed, given diagonalizable matrices B_i, i.e.,

$$B_i = C_i^{-1} \cdot D_i \cdot C_i \text{ for some } C_i \in \mathrm{GL}_m \text{ and some diagonalizable matrix } D_i,$$

then the question of finding all r-tuples $(n_1, \dots, n_r) \in \mathbb{Z}^r$ with the property that

$$B_1^{n_1} \cdots B_r^{n_r} \in V \text{ (for some given subvariety } V \text{ of } \mathrm{GL}_m)$$

reduces to the classical Mordell-Lang problem for algebraic tori, solved by Laurent [Lau84]. On the other hand, when one allows arbitrary matrices B_i, the above general question becomes undecidable and work of Scanlon and Yasufuku [SY14] shows that any Diophantine subset of $\mathbb{N}^r$ can be realized as the set of solutions to an equation $B_1^{n_1} \cdots B_r^{n_r} \in V$ for a subvariety V of $\mathrm{GL}_m(K)$. The fundamental difference between these two situations is that in the diagonalizable case the analysis reduces to the study of equations

$$P(\lambda_{1,1}^{n_1}, \dots, \lambda_{1,m}^{n_1}, \dots, \lambda_{r,1}^{n_r}, \dots, \lambda_{r,m}^{n_r}) = 0$$

(with P a polynomial and $\lambda_{i,j}$ eigenvalues of the matrices), which, although difficult, can be greatly aided with the help of theorems on S-unit equations (see [Lau84, Sch90]).

In the non-diagnonalizable case, however, one now must contend with polynomial-exponential equations and the famed DPRM Theorem (see [Mat93]) shows that every recursively enumerable subset of $\mathbb{N}^r$ (that is, every subset that can be enumerated by a Turing machine) can be realized as the zero set of such an equation.

Somewhat surprisingly we are able to show that the pathologies that arise in the non-diagonalizable case can be handled when one imposes conditions on the eigenvalues of the matrix generators and only considers subvarieties V of $M_n(K)$ that do not pass through the origin. Given a field K, we recall that a collection of elements $s_1, \dots, s_r \in \overline{K}^\times$ is **multiplicatively independent** if $n_1, \dots, n_r \in \mathbb{Z}$ and $s_1^{n_1} \cdot \dots \cdot s_r^{n_r} = 1$ imply that $n_1 = \cdots = n_r = 0$. We say a collection of elements $B_1, \dots, B_r \in M_r(K)$ has **multiplicatively independent eigenvalues** if the set of eigenvalues (counted without multiplicity) of $B_1, \dots, B_r$ are pairwise disjoint and their union is multiplicatively independent. Our main result is the following general Mordell-Lang variant.

Theorem 1.1. *Let K be an algebraically closed field of characteristic zero, let $B_1, \dots, B_r \in \mathrm{GL}_m(K)$ be matrices with multiplicatively independent eigenvalues, and let V be a closed subvariety of $\mathrm{GL}_m(K)$ not passing through zero. If*

$$\Gamma := \{B_1^{n_1} \cdot \dots \cdot B_r^{n_r} : n_1, \dots, n_r \in \mathbb{Z}\}, \tag{1.3}$$

then $|V(K) \cap \Gamma| < \infty$.

In fact, we prove a slightly more general (although equivalent) version of Theorem 1.1 for central simple algebras (see Theorem 2.2), which has the advantage of being immediately applicable to recent "noncommutative" variants of the Mordell-Lang problem.

We note that the finiteness we obtain can be viewed as the "generic" situation of the Mordell-Lang conjecture; that is, one generally expects the intersection of a finitely generated group in a semiabelian variety with a subvariety to be finite unless there is some additional geometric structure that explains the infinite intersection. One cannot expect to get a general finiteness result of the form given in Theorem 2.2 without some mild constraints; we give examples in § 2.3 which show that attempts to weaken the hypotheses can give rise to pathological intersection sets.

The outline of this paper is as follows. In § 2 we state a couple of technical results, which are both variants of the classical Mordell-Lang problem in two different settings. Theorem 2.2 is an equivalent formulation of Theorem 1.1 in the context of finite-dimensional central simple algebras, while Theorem 2.7 is a variant of the classical Mordell-Lang theorem in the context of commutative linear algebraic groups. Theorem 2.7 is also a key ingredient in our proof of Theorem 2.2. We prove Theorem 2.7 in § 3 and we conclude our proof of Theorem 2.2 in § 4.

Acknowledgments. We thank Zinovy Reichstein for numerous helpful conversations.

2. A couple of variants of the classical Mordell-Lang conjecture

We start by setting the notation for our paper.

2.1. **Notation.** Throughout this paper, let $\mathbb{N}$ denote the set of nonnegative integers, K be a field of characteristic 0, and A be a finite-dimensional central simple algebra over K. We let $\ell := [A : K]$. Then, geometrically, A can be identified with the K-points of the ℓ-dimensional affine space $\mathbb{A}^\ell(K)$. Moreover, since the multiplicative group $A^\times$ embeds into $\mathrm{GL}_\ell(K)$ by $f \mapsto L_f$, where $L_f : A \to A$ is left-multiplication map by f, we see that $A^\times$ can be identified with the K-points of a linear algebraic group. For $f \in A$, we define Λ_f to be the set of eigenvalues of L_f over the algebraic closure $\overline{K}$, not counting multiplicities.

Definition 2.1. We say a collection of elements $f_1, \dots, f_r \in A^\times$ has **multiplicatively independent eigenvalues** if $\Lambda_{f_1}, \dots, \Lambda_{f_r}$ are disjoint and their union is multiplicatively independent.

In this paper, unless otherwise noted, each algebraic group is connected.

2.2. **A variant of the Mordell-Lang problem for central simple algebras.** The following result is an equivalent re-statement of Theorem 1.1 in the setting of finite-dimensional central simple algebras.

Theorem 2.2. *Let K, A be as above, V be a closed K-subvariety of A not passing through zero, $f_1, \dots, f_r \in A^\times$, and Γ be the set:*

$$\Gamma = \{f_1^{n_1} \cdot \dots \cdot f_r^{n_r} : n_1, \dots, n_r \in \mathbb{Z}\} \subseteq A^\times. \tag{2.1}$$

If $f_1, \dots, f_r$ have multiplicatively independent eigenvalues, then $|V(K) \cap \Gamma| < \infty$.

Next we present various examples showing the relevance of the hypotheses from Theorem 2.2.

2.3. **Examples.** We first recall that for any nilpotent matrix x in $\mathrm{Mat}_n(K)$ (so that $x^n = 0$), there is a well-defined unipotent matrix

$$\exp(x) := \sum_{k=0}^{\infty} x^k/k! = \sum_{k=0}^{n-1} x^k/k!. \tag{2.2}$$

Moreover, $\exp(x + y) = \exp(x)\exp(y)$ if $xy = yx$.

We denote by ε_n the nilpotent matrix

$$\varepsilon_n := \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix} \in \mathrm{Mat}_n(K). \tag{2.3}$$

The following example shows the necessity of the assumption of Theorem 2.2 that V does not pass through 0.

Example 2.3. Let $A = \mathrm{Mat}_3(K)$, $\varepsilon := \varepsilon_3$, $f_1 = 2\exp(\varepsilon)$, and $f_2 = 3\exp(-\varepsilon^2)$. Then for $n_1, n_2 \in \mathbb{Z}$, we have that

$$f_1^{n_1} f_2^{n_2} = 2^{n_1} 3^{n_2} \exp\left(n_1\varepsilon - n_2\varepsilon^2\right) = 2^{n_1}3^{n_2}\left(1 + n_1\varepsilon + \left(-n_2 + \frac{n_1^2}{2}\right)\varepsilon^2\right). \tag{2.4}$$

Hence, if $V \subseteq A$ is cut out by the condition that the top-right entry is zero, then $f_1^{n_1} f_2^{n_2} \in V$ if and only if $n_2 = \frac{n_1^2}{2}$. The corresponding subset in $\mathbb{Z}^2$ is not only infinite, but also not a finite union of cosets of subgroups of $\mathbb{Z}^2$.

In this example, f_1, f_2 have multiplicatively independent eigenvalues, but $0 \in V$.

In view of [GH24], it is tempting to replace the assumption of Theorem 2.2 that $f_1, \ldots, f_r$ have multiplicatively independent eigenvalues by the assumption that $\det(f_1), \ldots, \det(f_r)$ are multiplicatively independent. The following example shows we cannot do so.

Example 2.4. Let $A = \mathrm{Mat}_4(K)$, $\varepsilon := \varepsilon_3$, $f_1 = \mathrm{diag}(2\exp(\varepsilon), 1) \in \mathrm{Mat}_{3+1}(K)$, and $f_2 = \mathrm{diag}(3\exp(-\varepsilon^2), 1)$. Let $V \subseteq A$ be cut out by the condition that the $(3,1)$-entry is zero and the $(4,4)$-entry is 1. Then by the same argument as the preceeding example, $f_1^{n_1} f_2^{n_2} \in V$ if and only if $n_2 = \frac{n_1^2}{2}$.

In this example, $0 \notin V$, and $\det(f_1) = 2^3, \det(f_2) = 3^3$ are multiplicatively independent. However, f_1 has eigenvalues $\{2, 1\}$, and the presence of 1 implies that f_1, f_2 do not have multiplicatively independent eigenvalues.

Finally, the next example shows that if one were to replace the set Γ from Theorem 2.2 with the *subgroup* generated by $f_1, \ldots, f_r$, then the conclusion would fail.

Example 2.5. Let $A = \mathrm{Mat}_2(K)$, $f_1 = \mathrm{diag}(2,3)$, and $f_2 = \mathrm{diag}(5,7) + \varepsilon$, where $\varepsilon = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then the subgroup Γ generated by f_1, f_2 contains $u := f_1 f_2 f_1^{-1} f_2^{-1} = I_2 - \frac{1}{21}\varepsilon$. In particular, u^n for $n \in \mathbb{N}$ give infinitely many elements in $V(K) \cap \Gamma$, where

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a = d = 1, c = 0 \right\}. \tag{2.5}$$

2.4. **A variant of the Mordell-Lang problem in an arbitrary commutative algebraic group.** Theorem 2.7 is our key ingredient for proving Theorem 2.2 and it is itself another variant of the classical Mordell-Lang problem (this time in the context of the commutative linear algebraic groups).

One of the most important special case of the classical Mordell-Lang conjecture says that given a semiabelian variety G (defined over an algebraically closed field K of characteristic 0) and given a finitely generated subgroup $\Gamma \subset G(K)$, if

$$V \text{ contains no translate of a nontrivial algebraic subgroup of } G, \tag{2.6}$$

then $|V(K) \cap \Gamma| < \infty$. It is natural to consider extensions of the classical Mordell-Lang conjecture to arbitrary commutative algebraic groups G. The first such case to consider would be for affine groups, i.e., when G is isomorphic to $\mathbb{G}_m^\ell \times \mathbb{G}_a^k$ for some $k, \ell \in \mathbb{N}$, and once again study this problem under the assumption (2.6). However, [GHST19, Examples 1.1 and 1.2] show that the presence of multiple copies of $\mathbb{G}_a$ will generate counterexamples of the Mordell-Lang principle. Our next Theorem 2.7 provides a setting where the aforementioned variant of the Mordell-Lang conjecture holds even in the presence of arbitrarily many copies of $\mathbb{G}_a$ in the linear algebraic group G.

In order to state Theorem 2.7, we need the following definition.

Definition 2.6. Let G be a commutative linear algebraic group over a field K of characteristic 0, and Γ be a finitely generated subgroup of $G(K)$. We say $\Gamma \subseteq G(K)$ is **strongly independent** if there are $r, u, \ell_1, \dots, \ell_r \in \mathbb{Z}_{\geq 0}$ and an isomorphism $G \simeq \mathbb{G}_m^{\ell_1+\cdots+\ell_r} \times \mathbb{G}_a^u$, under which a generating set $g_1, \dots, g_r$ of Γ takes the form

$$\begin{aligned} g_1 &= (\lambda_{1,1}, \dots, \lambda_{1,\ell_1}) \times (1, \dots, 1) \times \cdots \times (1, \dots, 1) \times \bar{b}_1, \\ g_2 &= (1, \dots, 1) \times (\lambda_{2,1}, \dots, \lambda_{2,\ell_2}) \times \cdots \times (1, \dots, 1) \times \bar{b}_2, \\ &\cdots \\ g_r &= (1, \dots, 1) \times (1, \dots, 1) \times \cdots \times (\lambda_{r,1}, \dots, \lambda_{r,\ell_r}) \times \bar{b}_r, \end{aligned} \tag{2.7}$$

where $\lambda_{i,j} \in K^\times$ are multiplicatively independent and $\bar{b}_1, \dots, \bar{b}_r \in K^u$ (i.e., they are K-vectors with u entries). In this case, we let $T_i := \mathbb{G}_m^{\ell_i}$ and $U = \mathbb{G}_a^u$ and we say $\Gamma \subseteq G(K)$ is strongly independent with respect to the decomposition $G = T_1 \times \cdots \times T_r \times U$.

Theorem 2.7. *Let G be a commutative linear algebraic group over a field K of characteristic zero, $V \subseteq G$ be a closed subvariety, and Γ be a finitely generated subgroup of $G(K)$. Assume*

(i) *Γ is strongly independent in $G(K)$ with respect to some decomposition $G = T_1 \times \cdots \times T_r \times U$, and*
(ii) *there is no $i \in \{1, \dots, r\}$ and no geometric point $v \in \left(\prod_{j \neq i} T_j\right) \times U$ such that*

$$T_i \times \{v\} \subseteq V; \tag{2.8}$$

then $|V(K) \cap \Gamma| < \infty$.

2.5. **Remarks regarding Theorem 2.7.** We show next that the conclusion in Theorem 2.7 fails if one removes the hypotheses regarding V and Γ.

First of all, condition (2.8) is a weaker version of condition (2.6) and in its absence, Theorem 2.7 would fail as the next example shows it.

Example 2.8. Consider $G = \mathbb{G}_m^2$ and Γ be generated by $(2, 1)$ and $(1, 3)$, while $V = \{2\} \times \mathbb{G}_m$. Then clearly, $V \cap \Gamma$ is infinite. Note that in this example, Γ is strongly independent, but V does not satisfy hypothesis (2.8) from Theorem 2.7.

The relevance of the hypothesis that Γ is strongly independent is more subtle. On one hand, the next example shows that in its absence, one would definitely have to strengthen the hypothesis (2.8) to the original hypothesis (2.6) from the classical Mordell-Lang problem.

Example 2.9. Consider the case when $G = \mathbb{G}_m^2$ and $\Gamma \subset G(K)$ is spanned by $(2,1)$ and $(1,4)$. Then Γ is contained in the subvariety V given by the equation $x_2 = x_1^2$. We note that V meets the hypothesis (2.8) from Theorem 2.7, but since Γ is not strongly independent, the intersection $V \cap \Gamma$ is infinite in this case.

However, the following example shows that even if we were to strengthen the hypothesis (2.8) to (2.6), one would still not obtain the desired conclusion in Theorem 2.7 in the absence of the hypothesis that Γ is strongly independent.

Example 2.10. Consider the case when $G = \mathbb{G}_m \times \mathbb{G}_a$ and Γ is generated by $(2,1)$ and $(2,0)$. Then Γ has infinite intersection with the diagonal subvariety $V \subset G$ given by the equation $x_1 = x_2$. Indeed, Γ consists of all points of the form

$$\left\{\left(2^{m+n}, m\right) : m, n \in \mathbb{Z}\right\}$$

and so, $V \cap \Gamma$ consists of the set

$$\left\{(2^s, 2^s) : s \in \mathbb{N}\right\},$$

which is not a finite union of cosets of a subgroups of Γ. In this case, V is a curve, which is not a coset of subgroup of Γ, but nevertheless the intersection $V \cap \Gamma$ is infinite; however, note that Γ is not strongly independent in this example.

3. Proof of Theorem 2.7

In this Section we work with the notation and hypotheses from Theorem 2.7 in order to prove it.

It suffices to prove Theorem 2.7 assuming K is algebraically closed. Therefore, we may assume $G = \mathbb{G}_m^\ell \times \mathbb{G}_a^k$ for some $\ell, k \in \mathbb{N}$. Furthermore, we can write

$$\ell := \ell_1 + \ell_2 + \cdots + \ell_r,$$

where the group Γ is generated by

$$\begin{aligned} g_1 &= (\lambda_{1,1}, \dots, \lambda_{1,\ell_1}) \times (1, \dots, 1) \times \cdots \times (1, \dots, 1) \times \bar{b}_1, \\ g_2 &= (1, \dots, 1) \times (\lambda_{2,1}, \dots, \lambda_{2,\ell_2}) \times \cdots \times (1, \dots, 1) \times \bar{b}_2, \\ &\dots \\ g_r &= (1, \dots, 1) \times (1, \dots, 1) \times \cdots \times (\lambda_{r,1}, \dots, \lambda_{r,\ell_r}) \times \bar{b}_r, \end{aligned} \tag{3.1}$$

for some $\bar{b}_1, \dots, \bar{b}_r \in \mathbb{G}_a^k(K)$; moreover, the elements $\lambda_{i,j} \in K^\times$ are multiplicatively independent. We write each

$$\bar{b}_i := (b_{i,1}, \dots, b_{i,k}) \text{ for } i = 1, \dots, r, \tag{3.2}$$

where each $b_{i,j} \in K$ (for $1 \le i \le r$ and $1 \le j \le k$).

Taking into account that $\ell = \ell_1 + \cdots + \ell_r$, we represent each polynomial f in the vanishing ideal $\mathcal{I}(V)$ of V as a polynomial in

$$K[x_{1,1}, \dots, x_{1,\ell_1}, x_{2,1}, \dots, x_{2,\ell_2}, \dots, x_{r,1}, \dots, x_{r,\ell_r}, y_1, \dots, y_k].$$

So, we let $f_1, \dots, f_m$ be a given set of generators for $\mathcal{I}(V)$ and we consider next f_s, for some $1 \le s \le m$. Then we can write f_s as

$$f_s := \sum_{\substack{\bar{j}=(j_{i,u}) \\ 1\le i\le r \\ 1\le u\le \ell_i}} \prod_{i=1}^{r} \prod_{u=1}^{\ell_i} x_{i,u}^{j_{i,u}} \cdot P_{s,\bar{j}}(y_1, \dots, y_k), \tag{3.3}$$

where the sum in (3.3) runs over a finite set J_s of tuples

$$\bar{j} := (j_{1,1}, \dots, j_{1,\ell_1}, j_{2,1}, \dots, j_{2,\ell_2}, \dots, j_{r,1}, \dots, j_{r,\ell_r}) \in \mathbb{N}^{\ell},$$

while $P_{s,\bar{j}} \in K[y_1, \dots, y_k]$ is some nonzero polynomial.

For each $\bar{j} \in J_s$, there is a polynomial $Q_{s,\bar{j}} \in K[z_1, \dots, z_r]$ (depending on the $\bar{b}_i$'s, according to equation (3.2)) with the property that

$$Q_{s,\bar{j}}(n_1, \dots, n_r) := P_{s,\bar{j}}\left(\sum_{i=1}^{r} n_i \cdot \bar{b}_i\right) \text{ for each } n_1, \dots, n_r \in \mathbb{Z}. \tag{3.4}$$

Using equations (3.1), (3.2) and (3.4), we see that f_s vanishes at the point $g_1^{n_1} \cdots g_r^{n_r} \in \mathbb{G}_m^{\ell} \times \mathbb{G}_a^{k}$ (for some integers $n_1, \dots, n_r$) if and only if

$$\sum_{\bar{j}\in J_s} \prod_{i=1}^{r} \prod_{u=1}^{\ell_i} \lambda_{i,u}^{j_{i,u}\cdot n_i} \cdot Q_{s,\bar{j}}(n_1, \dots, n_r) = 0. \tag{3.5}$$

So, $\prod_{i=1}^{r} g_i^{n_i} \in V$ if and only if $\bar{n} := (n_1, \dots, n_r)$ satisfies the system of equations (3.5) for $s = 1, \dots, m$. We argue by contradiction and assume there exists an infinite set of solutions $\mathcal{S} := \left\{\bar{n}^{(i)}\right\}_{i\ge 1}$ to the above system of polynomial-exponential equations (3.5). In order to simplify the index notation from (3.5), we re-write that equation as follows:

$$\sum_{\bar{j}\in J_s} \left(\bar{\lambda}^{\bar{j}}\right)^{\bar{n}} \cdot Q_{s,\bar{j}}\left(\bar{n}\right) = 0, \tag{3.6}$$

where $\bar{\lambda} := (\lambda_{i,u})_{\substack{1\le i\le r \\ 1\le u\le \ell_i}}$ and

$$\left(\bar{\lambda}^{\bar{j}}\right)^{\bar{n}} := \prod_{i=1}^{r} \prod_{u=1}^{\ell_i} \lambda_{i,u}^{j_{i,u}\cdot n_i}.$$

In order to analyze the system of m polynomial-exponential equations (3.5), we will apply the method from [Lau84, § 8]. Since there are finitely many partitions for each set of indices J_s, there exists a given collection of partitions

$$\mathcal{P} := (\mathcal{P}_s)_{1\le s\le m} \tag{3.7}$$

where $\mathcal{P}_s$ is a partition of J_s for each $s = 1, \dots, m$, which is **maximally compatible** (as defined in [Lau84, p. 320]) for infinitely many solutions $\bar{n} \in \mathcal{S}$ for the system of polynomial-exponential equations (3.5). At the expense of replacing $\mathcal{S}$ by a suitable infinite subset, we may assume that *each* of the solutions $\bar{n} \in \mathcal{S}$ is maximally compatible with the given partition (3.7). The **compatibility** of the collection of partitions $\mathcal{P}$ with respect to each solution $\overline{n}\in\mathcal{S}$ refers to the fact that for each $s=1,\dots,m$, considering the partition $\mathcal{P}_s$ of J_s given by

$$J_s:=J_{s,1}\cup\cdots\cup J_{s,v_s}\text{ (for some positive integer }v_s), \tag{3.8}$$

we have that for each $s=1,\dots,m$:

$$\sum_{\overline{j}\in J_{s,i}}\left(\overline{\lambda}^{\overline{j}}\right)^{\overline{n}}Q_{s,\overline{j}}\left(\overline{n}\right)=0\text{ for each }i=1,\dots,v_s. \tag{3.9}$$

The **maximality** of the collection $\mathcal{P}$ refers to the fact that there is no further refined collection of partitions (3.8) such that equations (3.9) hold for each subpart of each corresponding partition of J_s for $s=1,\dots,m$.

We let $H_{\mathcal{P}}$ be the subgroup of $\mathbb{Z}^r$ defined as in [Lau84, p. 319], i.e., $H_{\mathcal{P}}$ consists of all $\overline{n}\in\mathbb{Z}^r$ with the property that for each $s=1,\dots,m$ and for each $i=1,\dots,v_s$, we have that

$$\left(\overline{\lambda}^{\overline{j_1}}\right)^{\overline{n}}=\left(\overline{\lambda}^{\overline{j_2}}\right)^{\overline{n}}\text{ for each }\overline{j}_1,\overline{j}_2\in J_{s,i}. \tag{3.10}$$

According to [Lau84, Theorem 6], we cannot have that $H_{\mathcal{P}}$ is the trivial subgroup of $\mathbb{Z}^r$ since we assumed there exists an infinite set $\mathcal{S}$ of solutions $\overline{n}$ maximally compatible with respect to $\mathcal{P}$. So, from now on, we assume there exists some nontrivial $\overline{n}^{(0)}\in H_{\mathcal{P}}$ (i.e., not all the entries of $\overline{n}^{(0)}$ are equal to 0). Without loss of generality, we assume that

$$\overline{n}^{(0)}:=\left(n_1^{(0)},\dots,n_r^{(0)}\right)\text{ with }n_1^{(0)}\neq 0. \tag{3.11}$$

Equation (3.10) says that for each $s=1,\dots,m$ and for each $i=1,\dots,v_s$,

$$\left(\overline{\lambda}^{\overline{j}}\right)^{\overline{n}^{(0)}}\text{ is the same as we vary }\overline{j}\in J_{s,i}. \tag{3.12}$$

We fix some $s\in\{1,\dots,m\}$ and also some $i\in\{1,\dots,v_s\}$; then we write each $\overline{j}\in J_{s,i}$ as we did before: $\overline{j}:=(j_{1,1},j_{1,2},\dots,j_{1,\ell_1},j_{2,1},j_{2,2},\dots,j_{2,\ell_2},\dots,j_{r,1},j_{r,2},\dots,j_{r,\ell_r})$. Re-writing (3.12) using the index notation as in (3.5), we get that

$$\prod_{p=1}^{r}\prod_{u=1}^{\ell_p}\lambda_{p,u}^{j_{p,u}\cdot n_p^{(0)}}\text{ is constant as we vary }\overline{j}\in J_{s,i}. \tag{3.13}$$

Since the $\lambda_{p,u}$'s are multiplicatively independent and also, $n_1^{(0)}\neq 0$, it follows from (3.13) that for each $s=1,\dots,m$ and for each $i=1,\dots,v_s$, the vector:

$$\overline{j}^{(1)}:=(j_{1,1},j_{1,2},\dots,j_{1,\ell_1})\text{ is constant as we vary }\overline{j}\in J_{s,i}. \tag{3.14}$$

We let $\overline{n}^{(1)}\in\mathcal{S}$ and write (3.9) for $\overline{n}:=\overline{n}^{(1)}$; we get for each $s\in\{1,\dots,m\}$ and for each $i\in\{1,\dots,v_s\}$ that

$$\sum_{\overline{j}\in J_{s,i}}\prod_{p=1}^{r}\prod_{u=1}^{\ell_p}\lambda_{p,u}^{j_{p,u}\cdot n_p^{(1)}}Q_{s,\overline{j}}\left(\overline{n}^{(1)}\right)=0, \tag{3.15}$$

where $\overline{n}^{(1)} := \left(n_1^{(1)}, \dots, n_r^{(1)}\right)$. Using (3.14), we divide (3.9) by

$$\prod_{u=1}^{\ell_1} \lambda_{1,u}^{j_{1,u}\cdot n_1^{(1)}}$$

and therefore, we get

$$\sum_{\overline{j}\in J_{s,i}} \prod_{p=2}^{r} \prod_{u=1}^{\ell_p} \lambda_{p,u}^{j_{p,u}\cdot n_p^{(1)}} \cdot Q_{s,\overline{j}}\left(\overline{n}^{(1)}\right) = 0. \tag{3.16}$$

We write $\overline{y}^{(1)} := \sum_{i=1}^r n_i^{(1)} \cdot \overline{b}_i \in \mathbb{G}_a^k(K)$. Specializing $\overline{y}$ to $\overline{y}^{(1)}$ in equation (3.3), we re-write that equation (for each $s = 1, \dots, m$) according to the given partition $\mathcal{P}_s$ of J_s as follows (see also the way we re-wrote equation (3.5) as equation (3.6)):

$$f_s\left(\overline{x}, \overline{y}^{(1)}\right) = \sum_{i=1}^{v_s} \sum_{\overline{j}\in J_{s,i}} \left(\overline{x}^{\overline{j}}\right) \cdot P_{s,\overline{j}}\left(\overline{y}^{(1)}\right). \tag{3.17}$$

In equation (3.17), we have (as before) $(\overline{x})^{\overline{j}} := \prod_{p=1}^r \prod_{u=1}^{\ell_p} x_{p,u}^{j_{p,u}}$. Using equations (3.14) and (3.16) (for each $s = 1, \dots, m$ and each $i = 1, \dots, v_s$) and specializing further in equation (3.17) each

$$x_{p,j} := \lambda_{p,j}^{n_p^{(1)}} \text{ for } p = 2, \dots, r \text{ and } j = 1, \dots, \ell_p,$$

we get that

$$f_s\left(x_{1,1}, \dots, x_{1,\ell_1}, \lambda_{2,1}^{n_2^{(1)}}, \lambda_{2,2}^{n_2^{(1)}}, \dots, \lambda_{2,\ell_2}^{n_2^{(1)}}, \dots, \lambda_{r,1}^{n_r^{(1)}}, \dots, \lambda_{r,\ell_r}^{n_r^{(1)}}, \overline{y}^{(1)}\right) = 0, \tag{3.18}$$

for *any* $x_{1,1}, \dots, x_{1,\ell_1}$. Thus, equation (3.18) tells us that V contains the entire subvariety of $\mathbb{G}_m^\ell \times \mathbb{G}_a^k = T_1 \times \cdots T_r \times \mathbb{G}_a^k$ given by

$$T_1 \times \left\{\left(\lambda_{2,1}^{n_2^{(1)}}, \lambda_{2,2}^{n_2^{(1)}}, \dots, \lambda_{2,\ell_2}^{n_2^{(1)}}, \dots, \lambda_{r,1}^{n_r^{(1)}}, \dots, \lambda_{r,\ell_r}^{n_r^{(1)}}, \overline{y}^{(1)}\right)\right\},$$

thus contradicting the hypothesis from Theorem 2.7. Therefore, indeed, we only have finitely many solutions to the system of polynomial-exponential equations (3.6). Hence, this concludes our proof of Theorem 2.7.

4. Proof of Theorem 2.2

In this section, we work with the notation and the hypotheses from Theorem 2.2 in order to prove it.

By base-changing from K to the algebraic closure $\overline{K}$, we may assume K is algebraically closed, and $A = \mathrm{Mat}_n(K)$ is the matrix algebra for some $n \geq 1$, so $f_1, \dots, f_r \in \mathrm{GL}_n(K)$. Note that the eigenvalues of L_{f_i} are just the eigenvalues of f_i repeated n times, so Λ_{f_i} is simply the set of eigenvalues of f_i (not counting multiplicities).

Consider the commutative K-subalgebra $K[f_i]$ of $\mathrm{Mat}_n(K)$ generated by f_i. We have commutative linear algebraic groups $G_i := K[f_i]^\times$ and $G := G_1 \times \cdots \times G_r$. Let g_i be the element $(1, \dots, f_i, \dots, 1)$ in G, and let Γ' be the subgroup of G generated by $g_1, \dots, g_r$.

Over an algebraically closed field K, any commutative linear algebraic group is necessarily of the form $\mathbb{G}_m^\ell \times \mathbb{G}_a^u$ with $\ell, u \geq 0$. Indeed, one has a direct product decomposition into the semisimple part (which gives a torus) and the unipotent part. For a commutative unipotent group, the exponential map is an isomorphism of algebraic groups, so the unipotent part is of the form $\mathbb{G}_a^u$.

Next, we will be more explicit regarding each algebraic group G_i. Let $\Lambda_{f_i} = \{\lambda_{i,1}, \dots, \lambda_{i,\ell_i}\}$ and also, let $G_i = T_i \times U_i$, where T_i is the semisimple part and U_i is the unipotent part. Using the Jordan canonical form of f_i, we have an isomorphism $T_i \simeq (K^\times)^{\ell_i}$, such that the image of f_i projected onto T_i is identified with $(\lambda_{i,1}, \dots, \lambda_{i,\ell_i})$.

Let $U := U_1 \times \cdots \times U_r \simeq \mathbb{G}_a^u$ and let $T := T_1 \times \cdots \times T_r$. From the above discussion, we get an isomorphism $G = T \times U \simeq \mathbb{G}_m^{\ell_1+\cdots+\ell_r} \times \mathbb{G}_a^u$ under which $g_1, \dots, g_r$ takes the form of (2.7) for some $v_1, \dots, v_r \in K^u$. Since $f_1, \dots, f_r$ have multiplicatively independent eigenvalues, $\lambda_{i,j}$ are multiplicatively independent. Thus $\Gamma' \subseteq G$ is strongly independent in the sense of Definition 2.6.

Define an algebraic map (not necessarily a group homomorphism)

$$\mu : G = G_1 \times \cdots \times G_r \to \mathrm{GL}_n(K), (z_1, \dots, z_r) \to z_1 \dots z_r, \tag{4.1}$$

where the multiplication takes place in $\mathrm{GL}_n(K)$, and let $W := \mu^{-1}(V)$, which is a closed subvariety of G. We claim that W does not contain a subvariety of the form $T_i \times \{v\}$ for some $i \in \{1, \dots, r\}$ and some $v \in \left(\prod_{j\neq i} T_j\right) \times U$.

We argue by contradiction and so, without loss of generality, assume that $T_1 \times \{v\} \subseteq W$ for some $v \in (\prod_{i=2}^r T_i) \times U$. Consider the canonical map $\tau_1 : K^\times \hookrightarrow G_1 \subseteq \mathrm{GL}_n(K)$ given by the scalar matrices. Then τ_1 in fact maps into $T_1 = (K^\times)^{\ell_1}$ and takes the form $\tau_1(c) = (c, \dots, c)$. A crucial property of μ is that it behaves well with scalar multiplication: for $c \in K^\times$ and $(z_1, \dots, z_r) \in G = G_1 \times \cdots \times G_r$, we have

$$\mu(\tau_1(c) \cdot (z_1, \dots, z_r)) = \mu(cz_1, \dots, z_r) = c\mu(z_1, \dots, z_r). \tag{4.2}$$

Now we pick $(z_1, \dots, z_r) \in G$ that lies in $T_1 \times \{v\}$ under the identification $G = T \times U$. For all $c \in K^\times$, since $T_1 \times \{v\}$ is a T_1-coset of G, and $\tau_1(c) \in T_1 \subseteq T$, it follows that $\tau_1(c) \cdot (z_1, \dots, z_r) \in T_1 \times \{v\}$. Since $T_1 \times \{v\} \subseteq W$, applying μ gives

$$c\mu(z_1, \dots, z_r) \in V \text{ for all } c \in K^\times. \tag{4.3}$$

As V is Zariski closed in $\mathrm{Mat}_n(K)$, we get that $0 \in V$, contradicting thus the hypothesis from Theorem 2.2. This proves the claim.

Finally, since the assumptions of Theorem 2.7 are verified, we have $|W(K) \cap \Gamma'| < \infty$. Applying μ, this means $f_1^{n_1} \dots f_r^{n_r} \in V$ for only finitely many $(n_1, \dots, n_r) \in \mathbb{Z}^r$, so the conclusion of Theorem 2.2 is proved.

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