Science One
Integral Calculus

January 2017

Happy New Year!

Differential Calculus central idea:
The derivative

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Physical interpretation: rate of change of f (with respect to x) at x

Geometrical interpretation: slope of tangent line to graph of f at x

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Physical interpretation: rate of change of f (with respect to x) at x

Geometrical interpretation: slope of tangent line to graph of f at x

What is the definition of f'(x)? It's a limit!

$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \quad \text{or equivalently} \quad \lim_{\Delta x\to 0} \frac{\Delta f}{\Delta x}$$

Integral Calculus central idea:
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Other interpretations: depends on what f(x) represents....if f=v(t) velocity then definitive integral is the distance traveled in time interval Δt=b-a

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What is the definition of $\int_a^b f(x)dx$? It's a limit!

(some of) our goals this term will be to…

- Give a **precise definition** of definite integral
- Find a **fundamental connection** with the derivative (Fundamental Theorem of Calculus)
- Master **integration techniques** to compute complicated antiderivatives
- Apply integration to a variety of **science contexts**

Today's goal: Give a **precise definition** of definite integral

The area problem:
Find the area of the region S that lies under the curve $y= f(x)$ from a to b.

What is *area*?

Easy for regions with straight sides.

Not so easy for regions with curved sides.

We need a precise definition of area.

Example: Find the area under $f(x)=x^2$ on [0,1].

- Worksheet

We found that the sum S_n of areas of n rectangles converges as $n \rightarrow \infty$

We define area as a limit, $S = \lim_{n \rightarrow \infty} S_n$

The definite integral

Consider the region under the curve $y = f(x)$ above $[a, b]$.

Take n vertical strip of equal width $\Delta x = (b-a)/n$

n intervals $[x_0, x_1]$, $[x_1, x_2]$, $[x_2, x_3]$, … $[x_{i-1}, x_i]$, … $[x_{n-1}, x_n]$.

The definite integral

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- Take n vertical strip of equal width $\Delta x = (b-a)/n$
- n intervals $[x_0, x_1], [x_1, x_2], [x_2, x_3], \ldots [x_{i-1}, x_i], \ldots [x_{n-1}, x_n]$.
- Sum areas of all rectangles

$S_n = \Delta x\, f(x_1^*) + \Delta x\, f(x_2^*) + \ldots. + \Delta x\, f(x_i^*) + \ldots + \Delta x\, f(x_n^*) = \sum_{i=1}^{n} f(x_i^*)\, \Delta x$

where sample point x_i^* is *any* number in the interval $[x_{i-1}, x_i]$.

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We define

area S = $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\ \Delta x = \int_a^b f(x)dx$ **Definite Integral**

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Riemann Sum

The definite integral

Consider the region S under the curve y = *f*(x) above [a ,b].

If f(x)>0 on [a ,b],

$$\text{area S} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\ \Delta x\ = \int_a^b f(x)dx$$

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Riemann sum

The Riemann sum

$$\lim_{n \to \infty} \sum_{i=1}^{n} f(x_i^*)\ \Delta \mathrm{x}\ = \int_a^b f(x)dx$$

The Riemann sum does two things for us:

1. It gives a method for computing an approximation of an integral
2. It gives us a way to make that approximation "arbitrarily close" to the exact value of the integral.

Interpret the following limit by first recognizing the sum as a Riemann sum for a function defined on [0,1]:

$$\lim_{n\to\infty} \frac{1}{n}\left(\sqrt{\frac{1}{n}} + \sqrt{\frac{2}{n}} + \sqrt{\frac{3}{n}} + \dots + \sqrt{\frac{n}{n}}\right)$$

Which of the following integrals is defined as the above limit?

a) $\int_0^7 \sqrt{x}\, dx$

b) $\int_0^1 \sqrt{x}\, dx$

c) $\int_0^7 \sqrt{\frac{1}{x}}\, dx$

d) $\int_0^1 \sqrt{\frac{1}{x}}\, dx$

Consider $\int_0^\pi \sin(5x)\,dx$. Which of the following expressions represent the integral as a limit of Riemann sums?

a) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(\pi + \frac{5\pi\, i}{n}\right)$

b) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi\, i}{n}\right)$

c) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(\frac{\pi\, i}{n}\right)$

d) $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(\frac{5\pi\, i}{n}\right)$

e) $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{5\pi\, i}{n}\right)$

Which of the following correctly expresses the limit

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2}{n} cos(1 + \frac{i}{n})$ as a definite integral?

a) $\int_0^2 \cos(x)\, dx$

b) $\int_0^2 \cos(x+1)\, dx$

c) $\int_1^3 \cos(x)\, dx$

d) $\int_1^2 \cos(x)\, dx$

e) $\int_1^2 2\cos(x)\, dx$

Terminology

$$\int_a^b f(x)dx$$

integrand

Terminology

$$\int_a^b f(x)dx$$

limits of integration

Terminology

upper limit

$$\int_a^b f(x)\,dx$$

Terminology

$$\int_{a}^{b} f(x)\,dx$$

lower limit

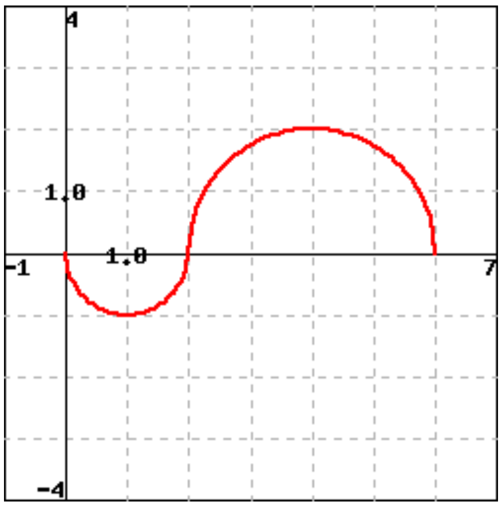
Definite Integral as area if….

…. *f*(x) > 0 on [a ,b].

If *f* (x) < 0 on [a ,b], then the Riemann sum is the sum of the *negatives* of the areas of the rectangles that lie below the x-axis (and above the curve) and

$\int_a^b f(x)dx < 0$ areas cannot be negative!

$\left| \int_a^b f(x)dx \right|$ = area of region below x-axis and above the curve y=f(x) on [a,b]



$$\int_0^2 f(x)dx =$$

a) π
b) $\pi/2$
c) Not defined
d) $-\pi$
e) $-\pi/2$

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4) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5) $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$

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6) $\int_a^b c f(x)dx = c \int_a^b f(x)dx$ where c is a constant

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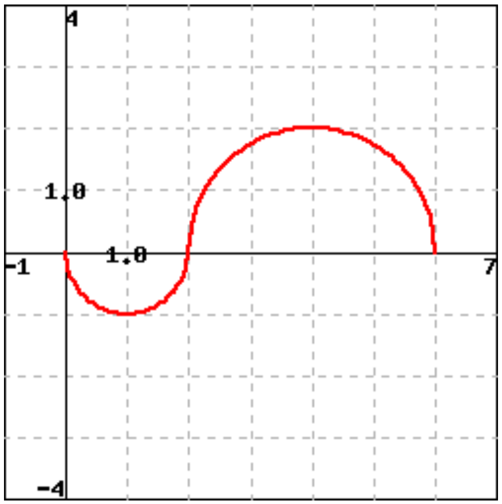
4) $\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$
5) $\int_a^b [f(x) - g(x)]dx = \int_a^b f(x)dx - \int_a^b g(x)dx$
6) $\int_a^b cf(x)dx = c\int_a^b f(x)dx$ where c is a constant
7) $\int_a^c f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)dx$

Comparison properties of the integral

1. If $f(x) \geq 0$ on [a, b], then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x)$ on [a, b], then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.
3. If $m \leq f(x) \leq M$ on [a, b], then

$$m(b-a) \leq \int_a^b f(x)dx \leq M\,(b-a).$$



$$\int_1^4 f(x)dx =$$

a) 4π
b) $3\pi/4$
c) $5\pi/4$
d) 2π

Compute $\int_{-1}^{1} 3+\sqrt{1-x^2}dx$

$$\int_{-1}^{1} 3 + \sqrt{1 - x^2}\,dx = \int_{-1}^{1} 3\,dx + \int_{-1}^{1} \sqrt{1 - x^2}\,dx$$

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$\int_{-1}^{1} 3\,dx =$ *area rectangle of base 2 and height 3 = 2x3=6*

$$\int_{-1}^{1} 3+\sqrt{1-x^2}\,dx = \int_{-1}^{1} 3\,dx + \int_{-1}^{1} \sqrt{1-x^2}\,dx$$

$\int_{-1}^{1} 3\,dx =$ *area rectangle of base 2 and height 3 = 2x3=6*

$\int_{-1}^{1} \sqrt{1-x^2}\,dx =$ *area half circle of radius 1 centred at origin =*

½ π

$$\int_{-1}^{1} 3+\sqrt{1-x^2}\,dx = 6+\pi/2$$

What have we learned so far?

- The integral is defined as a limit of a Riemann sum
- Does the limit of Riemann sums always exist?

Definition: If the limit

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\ \Delta\text{x} = \int_a^b f(x)dx$$

exists, we say *f* is **integrable** on [a, b].

Are all functions integrable?

Theorem:

If f is **continuous** on [a, b] or has a finite number of removable discontinuity, then f is integrable on [a, b].

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Piecewise functions are integrable:

Compute $\int_1^4 | x - 3 | dx$.

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Theorem:

If f is **continuous** on [a, b] or has a finite number of removable discontinuity, then f is integrable on [a, b].

Piecewise functions are integrable:

$\int_1^4 |\ x-3\ |\, dx = \int_1^3 (3-x)\ dx + \int_3^4 (x-3)\ dx =$

= ½ (2)(2) + ½ (1)(1) = 5/2

Evaluating the limit of Riemann sums is hard!

Good news: there is an easier way to compute integrals

The Fundamental Theorem of Calculus

Let f be continuous on an interval I containing a.

1. Define F(x) = $\int_a^x f(t)dt$ on I. Then *F* is differentiable on I with *F'*(x) = *f*(x).
2. Let G be any antiderivative of *f* on I. Then for any b in I

$$\int_a^b f(t)dt = G(b) - G(a)$$

FTC part I: The area function

Suppose *f* is continuous (and positive) on an interval containing a,
then the area below the curve y = f(t) on [a, x] is

$$F(x) = \int_a^x f(t)dt\,.$$

At what rate is F(x) changing with respect to x?

$$F'(x) = \lim_{h\to 0} \frac{F(x+h)-F(x)}{h}$$

Observation: F(x+h) – F(x) is a difference of areas, can be approximated by a rectangle of area h · *f*(x).

In the limit h → 0, we get F'(x) = f(x).

FTC part II:

If *F(x)* = $\int_a^x f(t)dt$, we know F'(x)= f(x). That is, F is an antiderivative of f.

Suppose G is also an antiderivative of *f* on I. Then F(x) = G(x) + C for some constant C. That is

$\int_a^x f(t)dt$ = F(x) = G(x) + C

If x=a, F(a) = 0, so C = – G(a).

If x =b, $\int_a^b f(t)dt$ = F(b) = G(b) + C = G(b) – G(a).