APMA900 Final Project
Small Amplitude Oscillations In The Length of a Pendulum

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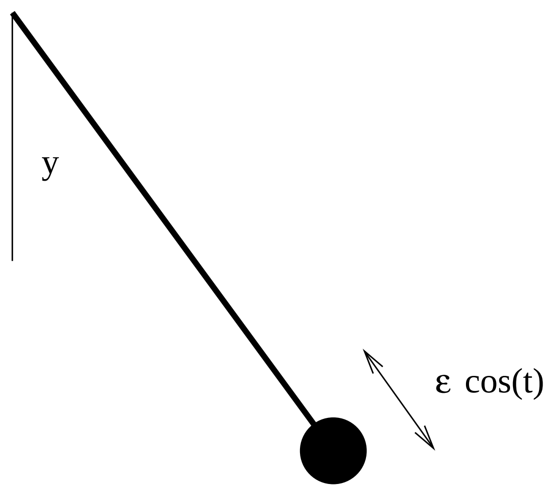
The Problem



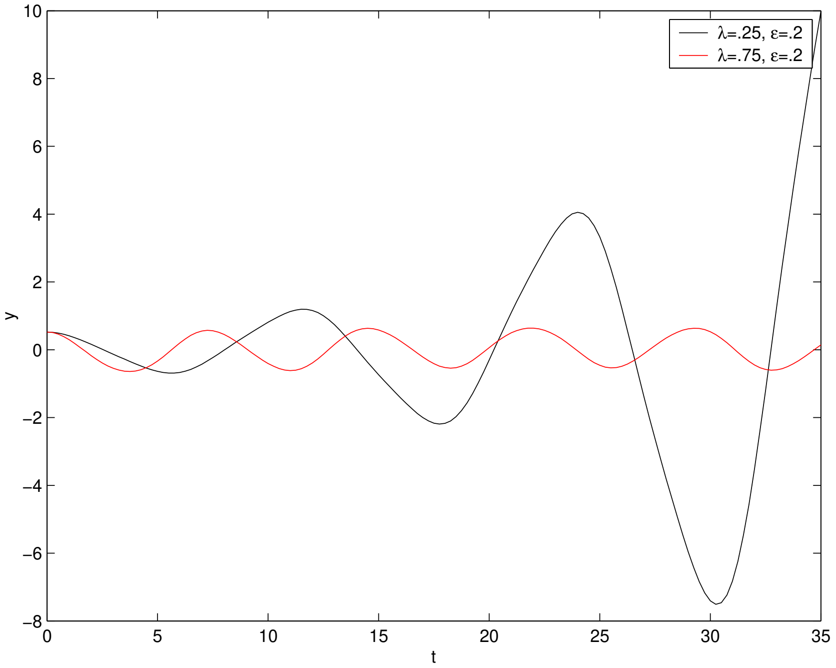
- For example, a child on a swing.

- The equation, $y'' + (\lambda + \varepsilon \cos t) \sin y = 0$, can be linearized to

$$y'' + (\lambda + \varepsilon \cos t) y = 0.$$

When is this problem stable?

- What happens when $\lambda = 0.25, \varepsilon = .2$? $\lambda = .75, \varepsilon = .2$?



- *Question*: For which λ, ε is the problem *stable*?

Leading Order Expansion for Independent λ, ε

- For *small* ε, expand y by

$$y \sim y_0 + \varepsilon y_1 + \varepsilon^2 y_2 \ldots . \tag{1}$$

- *Leading order* satisfies: $y_0'' + \lambda y_0 = 0$
- If $\lambda \leq 0$ then we have *growth* $\implies$ instability.
- If $\lambda = \mu^2$ then

$$y_0 = c_1 \cos(\mu t) + c_2 \sin(\mu t) \tag{2}$$

First Correction

- y_1 satisfies

$$\begin{aligned} y_1'' + \lambda y_1 &= \varepsilon(\cos t) y_0 \\ &= \varepsilon(\cos t)\left(c_1 \cos(\mu t) + c_2 \sin(\mu t)\right) \\ &= \frac{c_1 \varepsilon}{2}\big(\cos((1+\mu)t) + \cos((1-\mu)t)\big) \\ &\qquad + \frac{c_2 \varepsilon}{2}\big(\sin((1+\mu)t) - \sin((1-\mu)t)\big). \end{aligned}$$

- *Secular growth* whenever $\mu = \frac{1}{2} \implies \lambda = \frac{1}{4}$.

Second Correction

- y_2 satisfies

$$
\begin{aligned}
y_2'' + \lambda y_2 &= \varepsilon(\cos t) y_1 \\
&= \frac{c_1 \varepsilon^2 \mu}{-2 + 8\mu^2} \big(\cos((2+\mu)t) + \cos((2-\mu)t) \big) \\
&\quad - \frac{c_2 \varepsilon^2 \mu}{-2 + 8\mu^2} \big(\sin((2+\mu)t) + \sin((2-\mu)t) \big) + \ldots
\end{aligned}
$$

- *Secular growth* whenever $\mu^2 = \lambda = 1$.

- Could continue this process and find secular when $\lambda = \frac{n^2}{4}, n = 0, 1, 2, \ldots$.

What happens near $\lambda = \frac{1}{4}$?

- Now consider $\lambda \sim \frac{1}{4} + \frac{\varepsilon\lambda_1}{2}$ and again $y \sim y_0 + \varepsilon y_1$. We try to choose λ_1 to avoid *secular growth*.

- Leading order is still $y_0 = c_1 \cos\frac{t}{2} + c_2 \sin\frac{t}{2}$.

- First correction must satisfy $y_1'' + \frac{1}{4}y_1 = -\frac{\varepsilon\lambda_1}{2}y_0 - \varepsilon(\cos t)y_0$:

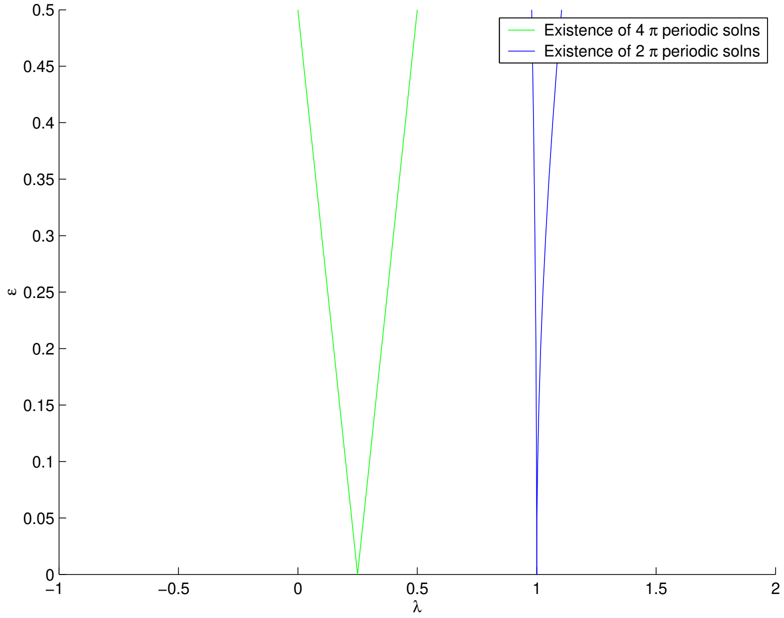
$$y_1'' + \frac{1}{4}y_1 = (-\lambda_1 - 1)\frac{c_1\varepsilon}{2}\cos\frac{t}{2} + (1 - \lambda_1)\frac{c_2\varepsilon}{2}\sin\frac{t}{2} - \frac{c_1\varepsilon}{2}\cos\frac{3t}{2} - \frac{c_2\varepsilon}{2}\sin\frac{3t}{2}.$$

- Can choose $\lambda \pm 1 \implies$ one 4π *periodic* solution, and one secular growth solution.

- We can do something similar for $\lambda \sim 1 + \varepsilon^2 \lambda_1$,

- This time, we choose $\lambda = -\frac{1}{12}$ or $\lambda = \frac{5}{12}$ to obtain one 2π *periodic* solution (again the other solution will exhibit one secular growth solution.)

- Plot these curves in λ, ε space:



Floquet Theory

- Formulate problem as *first-order system* $\vec{x}_t = P\vec{x}$.

- Solve the problem (numerically) for one period of P using the ICs $(1,0)^T$ and $(0,1)^T$ to obtain solutions $\vec{x}_A$ and $\vec{x}_B$ respectively.

- Find the *Floquet multipliers*, μ_1, μ_2, i.e., the eigenvalues of $[\vec{x}_A \quad \vec{x}_B]$.

- Consider four cases to determine stability.

Four Cases for the Floquet Multipliers

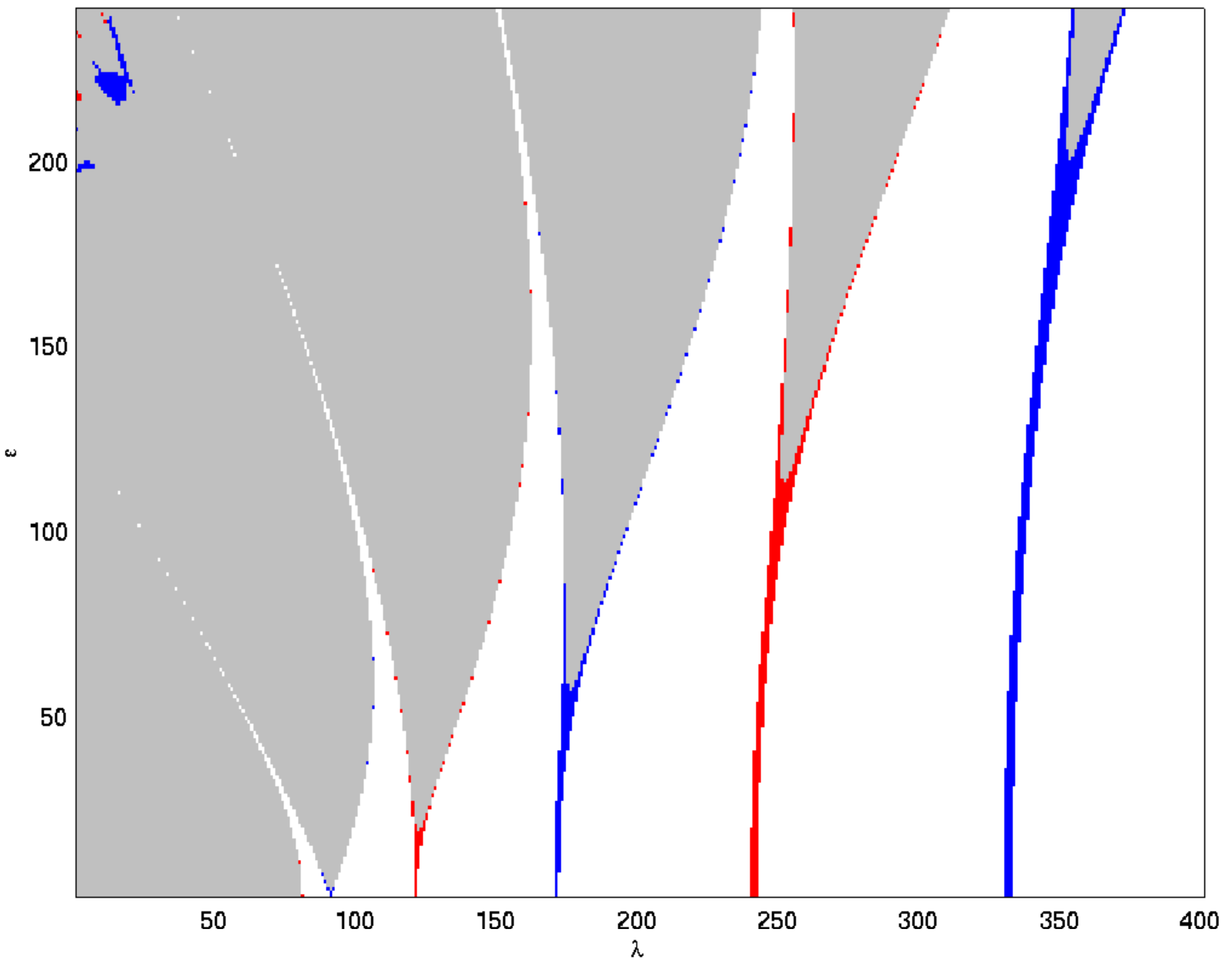
- If μ_1, μ_2 are *real and distinct*, then the solution is *unstable*. [white]

- If μ_1, μ_2 are *negative complex conjugates*, then we have oscillatory bounded solutions (i.e., *stable*). [gray]

- If $\mu_1 = \mu + 2 = 1$ then we have one 2π *periodic* solution and one unbound solution. [red]

- If $\mu_1 = \mu_2 = -1$ then we have one 4π *periodic* solution and one unbounded solution. [blue]

Stability Regions in $-2 \le \lambda \le 8$, $0 \le \varepsilon \le 6$



Acknowledgments

Thanks to:

- Dr. Muraki
- Everyone in APMA900 ;-)

References:

- M. Holmes, *Introduction to Perturbation Methods*.
- D. Jordan, P. Smith, *Nonlinear Ordinary Differential Equations*.