THE UNIVERSITY OF BRITISH COLUMBIA

MATH 253
Midterm 2
13 November 2013

TIME: 50 MINUTES

LAST NAME: Solutions FIRST NAME: ______

STUDENT # : ______ INSTRUCTOR'S NAME: ______

This Examination paper consists of 9 pages (including this one). Make sure you have all 9.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.

MARKING:

Q1		8
Q2		8
Q3		10
Q4		9
TOTAL		35

NAMES OF INSTRUCTORS: Jim Bryan, Dale Peterson, Sabin Cautis, Lior Silberman

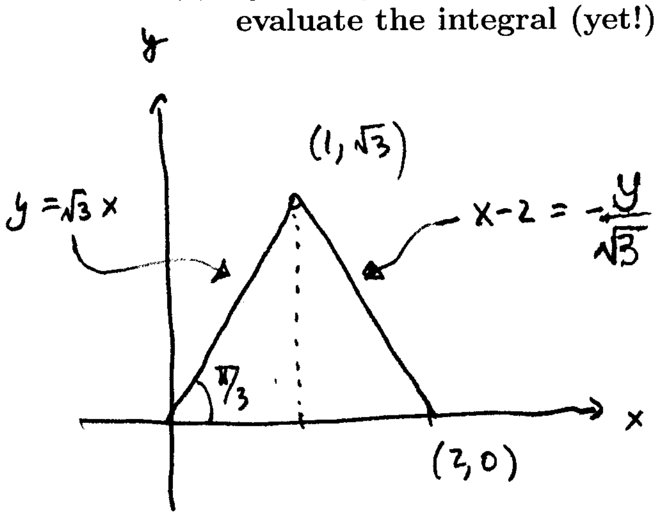
Q1 [8 points]

Consider the integral

$$\iint_T \sqrt{3}\, dA$$

where T is the triangle in the xy-plane with vertices $(0,0)$, $(2,0)$, and $(1,\sqrt{3})$.

(*a*) [2 points] Write the integral as an iterated integral where you integrate x first. **Do not evaluate the integral (yet!).**

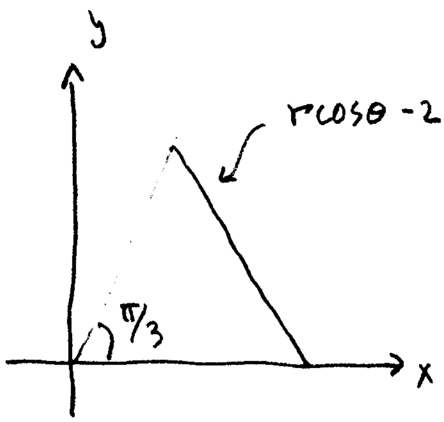


$$\int_{y=0}^{\sqrt{3}} \int_{x=\frac{y}{\sqrt{3}}}^{2-\frac{y}{\sqrt{3}}} \sqrt{3}\, dx\, dy$$

(*b*) [2 points] Write the integral as an iterated integral where you integrate y first. Hint: you may write this integral as the sum of two integrals. **Do not evaluate the integral (yet!).**

$$\iint_T \sqrt{3}\, dA = \int_{x=0}^{1} \int_{y=0}^{\sqrt{3}x} \sqrt{3}\, dy\, dx + \int_{x=1}^{2} \int_{y=0}^{\sqrt{3}(2-x)} \sqrt{3}\, dy\, dx$$

(c) [2 points] Write the integral as an iterated integral in polar coordinates. **Do not evaluate the integral (yet!).**



$r\cos\theta - 2 = -\frac{1}{\sqrt{3}} r\sin\theta$

$$r\left(\cos\theta + \frac{1}{\sqrt{3}}\sin\theta\right) = 2$$

$$r = \frac{2}{\cos\theta + \frac{1}{\sqrt{3}}\sin\theta}$$

$$\iint_T \sqrt{3}\,dA = \int_{\theta=0}^{\pi/3} \int_{r=0}^{\frac{2}{\cos\theta + \frac{1}{\sqrt{3}}\sin\theta}} \sqrt{3}\, r\,dr\,d\theta$$

(d) [2 points] Evaluate the integral using any method.

Easiest: $$\iint_T \sqrt{3}\,dA = \sqrt{3}\,\text{Area}(T) = \sqrt{3}\left(\tfrac{1}{2}\cdot 2\cdot\sqrt{3}\right) = 3$$

Integration in part (a):

$$\int_{y=0}^{\sqrt{3}} \int_{x=\frac{y}{\sqrt{3}}}^{2-\frac{1}{\sqrt{3}}y} \sqrt{3}\,dx\,dy = \sqrt{3}\int_{y=0}^{\sqrt{3}} \left(2 - \frac{1}{\sqrt{3}}y - \frac{1}{\sqrt{3}}y\right)dy$$

$$= 2\sqrt{3}\int_0^{\sqrt{3}} \left(1 - \frac{1}{\sqrt{3}}y\right)dy = 2\sqrt{3}\left[y - \frac{1}{2\sqrt{3}}y^2\right]_0^{\sqrt{3}}$$

$$= 2\sqrt{3}\left[\sqrt{3} - \frac{1}{2\sqrt{3}}(\sqrt{3})^2\right] = 2\sqrt{3}\left(\frac{\sqrt{3}}{2}\right) = 3.$$

Q2 [8 points]

Consider the following iterated integral.

$$\int_{x=0}^{1} \int_{y=1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} x \sin\left(\pi\left(1-y^2+\frac{y^3}{3}\right)\right) dy\, dx$$

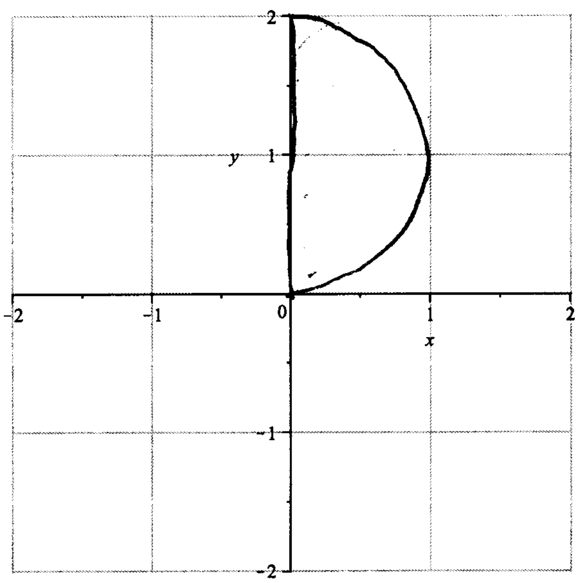
(*a*) [3 points] Sketch the domain of integration on the graph provided below.

$y = 1 \pm \sqrt{1-x^2}$

$\mp\sqrt{1-x^2} = 1-y$

$\Rightarrow\ 1-x^2 = (1-y)^2$

$\Rightarrow\ (y-1)^2 + x^2 = 1$



(*b*) [5 points] Compute the integral.

switch order of integration:

$x = \sqrt{1-(y-1)^2} = \sqrt{2y-y^2}$

$y=0 \leadsto u=\pi$

$y=2 \leadsto u=\pi\left(1-4+\frac{8}{3}\right) = -\frac{\pi}{3}$

$$\int_{y=0}^{2} \int_{x=0}^{\sqrt{2y-y^2}} x \sin\left(\pi\left(1-y^2+\frac{y^3}{3}\right)\right) dx\, dy$$

$$= \int_{y=0}^{2} \left[\frac{x^2}{2}\right]_0^{\sqrt{2y-y^2}} \sin\left(\pi\left(1-y^2+\frac{y^3}{3}\right)\right) dy = \int_{y=0}^{2} \left(y-\frac{1}{2}y^2\right) \sin\left(\pi\left(1-y^2+\frac{y^3}{3}\right)\right) dy$$

$u = \pi\left(1-y^2+\frac{y^3}{3}\right)$

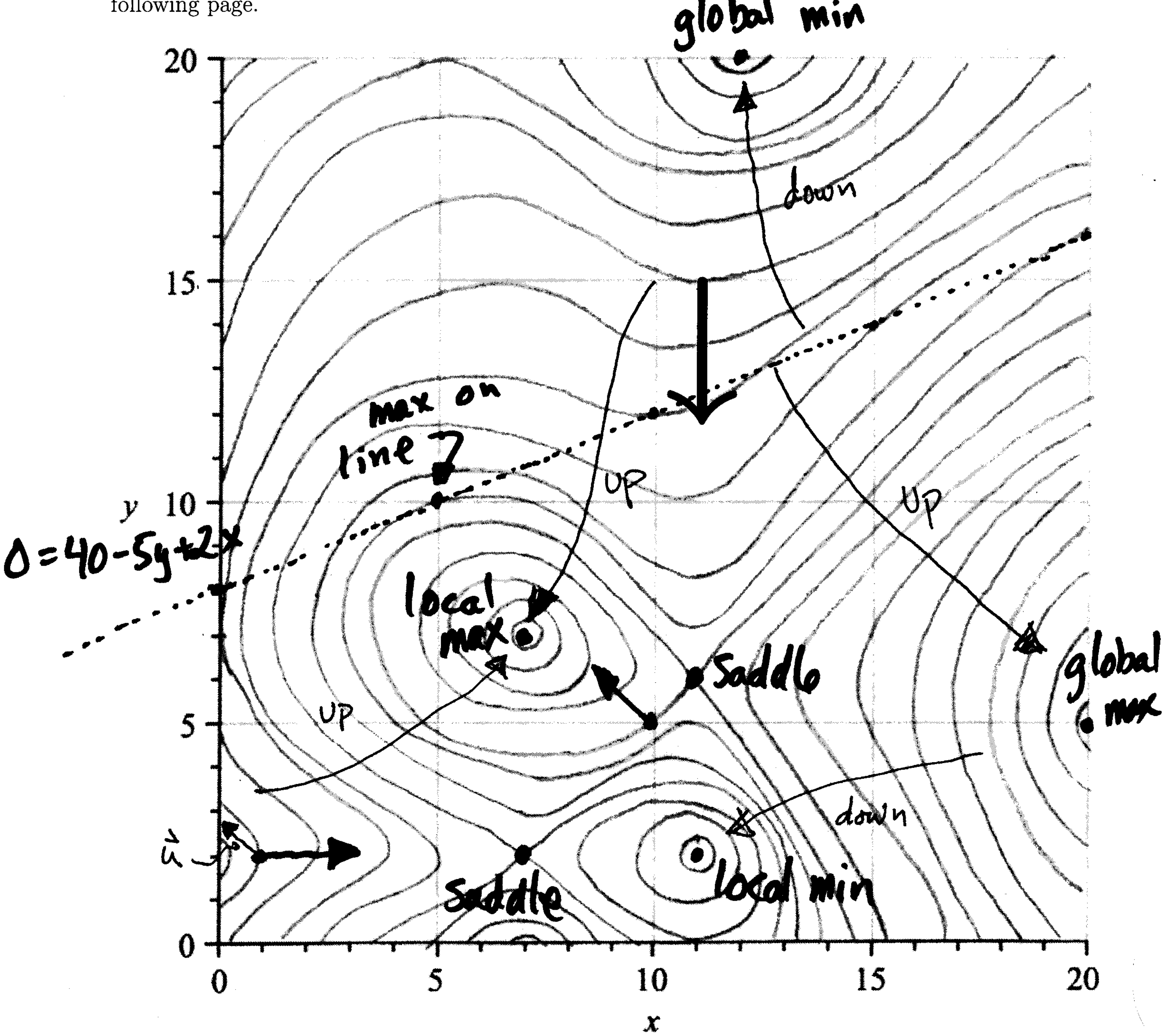
$du = (-2\pi y + \pi y^2)\, dy$

$$= \int_{u=\pi}^{-\pi/3} -\frac{1}{2\pi} \sin u\, du = \frac{1}{2\pi} \cos u \Big]_{\pi}^{-\pi/3}$$

$$= \frac{1}{2\pi}\left[+\frac{1}{2} - (-1)\right] = \boxed{\frac{3}{4\pi}}$$

Q3 [10 points]

The following is the contour plot of a function $f(x, y)$ with domain $\{(x, y) : 0 \le x \le 20, 0 \le y \le 20\}$. The values of the contours are spaced evenly. You may make reasonable assumptions about the function: the gradient does not vanish along an entire contour, the function does not fluctuate wildly on a scale smaller that shown by the contours. As indicated below, **The gradient of f at the point (11,15) is given by** $\langle 0, -3\rangle$. Answer the questions on the following page.



(*a*) (**4 points.**) Find the coordinates of the critical points of f on the interior of its domain. Classify each critical point as a local maximum, a local minimum, a saddle point, or "other".

(7,7) local max (7,2) saddle point
(11,2) local min (11,6) saddle point

(*b*) (**2 points.**) Find the coordinates of the global maximum of f and the global minimum of f. Global max/min occurs at critical points or on boundary. Since contours are evenly spaced, we can count contour lines to determine max/min. global max = (20,5) global min = (13,20)

(*c*) (**1 point.**) Let $u = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. Is the value of $(D_u f)(1,2)$ positive, negative, or zero? Circle the correct answer.

$\vec{u} \cdot (\vec{\nabla} f)(1,2)$ is negative since the angle between them is obtuse

(*d*) (**1 point.**) Is the value of $f_{xx}(7,7)$ positive, negative, or zero? Circle the correct answer.

(7,7) is a local max so $f_{xx} < 0$ negative

(*e*) (**1 point.**) The direction of the gradient of f at the point $(10,5)$ is given by which of the following (circle the correct answer):

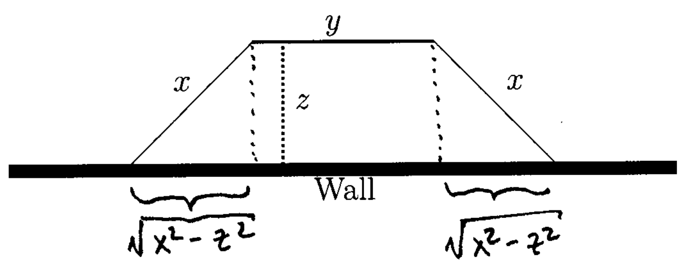
i) $\frac{1}{\sqrt{2}} \langle 1, 1 \rangle$
ii) $\frac{1}{\sqrt{2}} \langle 1, -1 \rangle$
iii) $\frac{1}{\sqrt{2}} \langle -1, 1 \rangle$ as shown in diagram
iv) $\frac{1}{\sqrt{2}} \langle -1, -1 \rangle$

(*f*) (**1 point.**) Let $g(x,y) = 40 - 5y + 2x$. Find the approximate coordinates of the point which maximizes $f(x,y)$ subject to the constraint $g(x,y) = 0$.

maximum occurs on boundary or where $\vec{\nabla} f = \lambda \vec{\nabla} g$ $\Leftrightarrow$ line is tangent to contour. This occurs at (5,10) which is the maximum

Q4 [9 points]

A trapezoidal enclosure is to be constructed by a fence with three sides and an existing wall. Two of the fence sides are the same length x and the third side is length y and is to be parallel to the existing wall. Let z be the distance from the wall to the parallel wall (see the picture). The area of the enclosure is required to be $3\sqrt{3}$ square meters and we wish to determine the fence which uses as little fencing material as possible. Find the values of the side lengths x and y and the distance z for the fence which has the minimal total length of fence.



$$\text{Area} = A(x,y,z) = z\sqrt{x^2-z^2} + yz$$

$$\text{Length} = f(x,y,z) = 2x+y$$

$$\boxed{\begin{aligned} f_x &= \lambda A_x \\ f_y &= \lambda A_y \\ f_z &= \lambda A_z \\ A &= 3\sqrt{3} \end{aligned}} \Rightarrow \boxed{\begin{aligned} 2 &= \frac{\lambda z x}{\sqrt{x^2-z^2}} && (1) \\ 1 &= \lambda z && (2) \\ 0 &= \lambda\left(\sqrt{x^2-z^2} - \frac{z^2}{\sqrt{x^2-z^2}} + y\right) && (3) \\ 3\sqrt{3} &= z\sqrt{x^2-z^2} + yz && (4) \end{aligned}}$$

$$(1) \,\&\, (2) \Rightarrow 2 = \frac{x}{\sqrt{x^2-z^2}} \Rightarrow 4(x^2-z^2) = x^2 \Rightarrow 3x^2 = 4z^2 \Rightarrow \boxed{z = \frac{\sqrt{3}}{2}x} \;(*)$$

substituting$^{(*)}$ into (3) and observing from (2) that $\lambda \neq 0$ we get

$$0 = \sqrt{x^2 - \tfrac{3}{4}x^2} - \frac{\frac{3x^2}{4}}{\sqrt{x^2 - \frac{3x^2}{4}}} + y \Rightarrow 0 = \frac{x}{2} - \frac{3x^2}{4}\left(\frac{2}{x}\right) + y$$

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$0 = \frac{x}{2} - \frac{3x}{2} + y \Rightarrow \boxed{x=y}$ substituting this and (*) into

(4) we get $3\sqrt{3} = \frac{\sqrt{3}}{2} x \sqrt{x^2 - \frac{3}{4}x^2} + x \cdot \frac{\sqrt{3}}{2} x$

$\Rightarrow 3 = \frac{x}{2} \cdot \frac{x}{2} + \frac{x^2}{2} = \frac{3}{4} x^2 \Rightarrow x^2 = 4 \Rightarrow x = 2$

so $\boxed{x=2,\ y=2,\ z=\sqrt{3}}$ $f(x,y,z) = 6$

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boundary cases: $y=0$ (enclosed region is a triangle)
$x=z$ (enclosed region is a rectangle).

each case is solved as a Lagrange multiplier problem with one fewer variable. We verify below that the minimum of f does not occur on the boundary

$\underline{y=0}$: $2 = \frac{\lambda z x}{\sqrt{x^2-z^2}}$ (1), $0 = \lambda\left(\sqrt{x^2-z^2} - \frac{z^2}{\sqrt{x^2-z^2}}\right)$ (2), $3\sqrt{3} = z\sqrt{x^2-z^2}$ (3)

(1) $\Rightarrow \lambda \neq 0$ substituting $\sqrt{x^2-z^2} = \frac{3\sqrt{3}}{z}$ into (2) we get

$0 = \frac{3\sqrt{3}}{z} - \frac{z^3}{3\sqrt{3}} \Rightarrow (3\sqrt{3})^2 = z^4 \Rightarrow z = \sqrt{3\sqrt{3}} \Rightarrow \sqrt{3\sqrt{3}}\sqrt{x^2 - 3\sqrt{3}} = 3\sqrt{3}$

$\Rightarrow x^2 - 3\sqrt{3} = 3\sqrt{3} \Rightarrow x = \sqrt{2}\sqrt{3\sqrt{3}} \Rightarrow f = 2\sqrt{2}\sqrt{3\sqrt{3}}$ which is bigger

than 6 since $\left(2\sqrt{2}\sqrt{3\sqrt{3}}\right)^4 = \underbrace{16 \cdot 4 \cdot 9 \cdot 3}_{4 \cdot 2^4 \cdot 3^3} > \underbrace{6^4}_{3 \cdot 2^4 \cdot 3^3}$

$\underline{x=z}$ $A = yz = 3\sqrt{3}$ $f(y,z) = 2z + y$

$\left.\begin{array}{l} f_z = \lambda A_z \Rightarrow 2 = \lambda y \\ f_y = \lambda A_y \Rightarrow 1 = \lambda z \end{array}\right\} \Rightarrow 2 = \frac{y}{z},\ yz = 3\sqrt{3} \Rightarrow 2z^2 = 3\sqrt{3} \Rightarrow z = \frac{1}{\sqrt{2}}\sqrt{3\sqrt{3}}$
$y = \sqrt{2}\sqrt{3\sqrt{3}}$

$\hookrightarrow f = \frac{2}{\sqrt{2}}\sqrt{3\sqrt{3}} + \sqrt{2}\sqrt{3\sqrt{3}} = 2\sqrt{2}\sqrt{3\sqrt{3}}$ which as before is bigger than 6