THE UNIVERSITY OF BRITISH COLUMBIA

MATH 253
Midterm 1
10 October 2012

TIME: 50 MINUTES

FIRST NAME: Solutions LAST NAME : ____________

STUDENT #: ____________

This Examination paper consists of 6 pages (including this one). Make sure you have all 6.

INSTRUCTIONS:

No memory aids allowed. No calculators allowed. No communication devices allowed.
PLEASE CIRCLE YOUR INSTRUCTOR'S NAME BELOW

MARKING:

Q1	/10
Q2	/10
Q3	/10
Q4	/10
TOTAL	/40

NAMES OF INSTRUCTORS: Jim Bryan, Dale Peterson, Ian Hewitt, Yariv Dror-Mizrahi, Ed Richmond

Q1 [10 marks]

Find the partial derivatives f_x, f_y, and f_{xy} of the following functions:

(a)

$$f(x,y) = xe^{xy}$$

$f_x = e^{xy} + xye^{xy} = (1+xy)e^{xy}$

$f_y = x^2e^{xy}$

$f_{xy} = xe^{xy} + (1+xy)xe^{xy} = (2x+x^2y)e^{xy}$

(b)

$$f(x,y) = x\sin(e^y)$$

$f_x = \sin(e^y)$ $f_y = xe^y\cos(e^y)$

$f_{xy} = e^y\cos(e^y)$

(c)

$$f(x,y) = \int_y^x t\sin(e^t)dt$$

$f(x,y) = F(x) - F(y)$ where $F'(t) = t\sin(e^t)$

so $f_x = F'(x) = x\sin(e^x)$

$f_y = -F'(y) = -y\sin(e^y)$

$f_{xy} = 0$

Q2 [10 marks]

Match each function with its contour plot (labeled A-I).

$f(x, y) = \sin(2x) + \sin(y)$ I

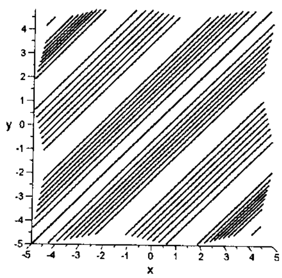
$f(x, y) = \cos(x + y)$ B

$f(x, y) = 3x - y^2$ H

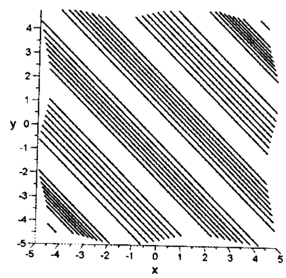
$f(x, y) = (x - 2)(y + 1)$ C

$f(x, y) = x^2 - y^2$ G

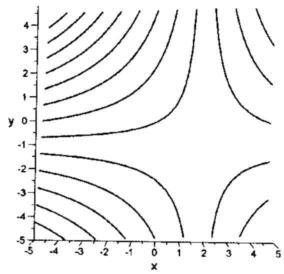
A.



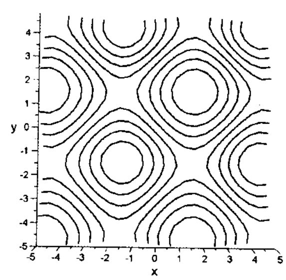
B.



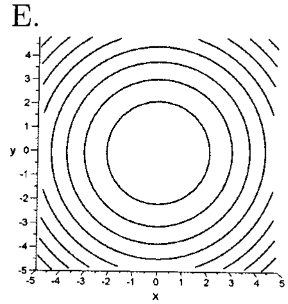
C.



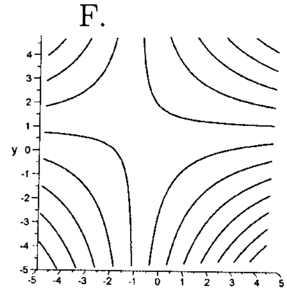
D.



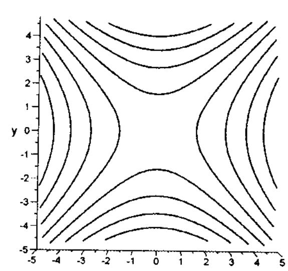
E.



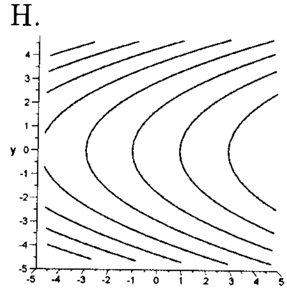
F.



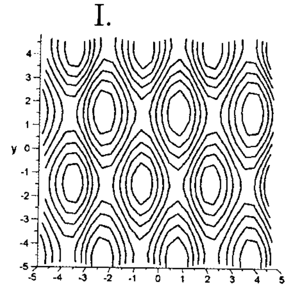
G.



H.



I.



Q3 [10 marks]

Consider the surface $z = x^2 - 6xy + 2y^3$.

$$z = z_0 + \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0)$$

$$\frac{\partial z}{\partial x} = 2x - 6y \qquad \frac{\partial z}{\partial y} = -6x + 6y^2$$

(*a*) Find an equation for the tangent plane to the surface at $(1, 2, 5)$.

$$z = 5 + (2-12)(x-1) + (-6+6\cdot 4)(y-2)$$

$$\boxed{z = 5 - 10(x-1) + 18(y-2)}$$

(*b*) On the surface near $(1, 2, 5)$, there is a point $(x, 1.99, 5.02)$. Find an approximate value for x.

near $(1,2,5)$ $\quad z \approx 5 - 10(x-1) + 18(y-2)$

so $\quad 5.02 \approx 5 - 10(x-1) + 18(1.99-2)$

$$0.02 \approx -10(x-1) - .18$$

$\Rightarrow \quad 0.2 \approx -10(x-1) \Rightarrow x - 1 \approx -0.02 \qquad \boxed{x \approx .98}$

(*c*) Find all points on the surface where the tangent plane is parallel to the plane $2x+6y+z = 4$.

tangent plane $\quad 0 = \frac{\partial z}{\partial x}(x - x_0) + \frac{\partial z}{\partial y}(y - y_0) - (z - z_0)$ is

normal to $\quad \left\langle \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, -1 \right\rangle = \langle 2x_0 - 6y_0, -6x_0 + 6y_0^2, -1 \rangle$

which needs to be parallel to $\langle 2, 6, 1 \rangle$

$\Rightarrow \quad -2 = 2x_0 - 6y_0 \qquad -6 = -6x_0 + 6y_0^2 \Rightarrow 1 = x_0 - y_0^2$

$\Rightarrow x_0 = 1 + y_0^2 \quad$ so $\quad -2 = 2(1+y_0^2) - 6y_0 \Rightarrow 0 = 1 + 1 + y_0^2 - 3y_0$

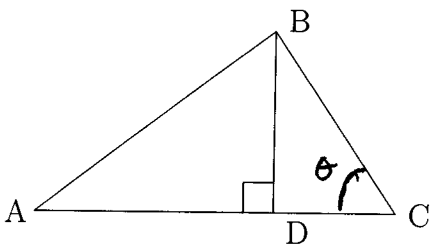
$y_0^2 - 3y_0 + 2 = 0 \Rightarrow (y_0 - 2)(y_0 - 1) = 0 \Rightarrow y_0 = 2$ or $1 \Rightarrow x_0 = 5$ or 2

so $\boxed{(x_0, y_0, z_0) = (5, 2, -19) \text{ or } (2, 1, -6)}$

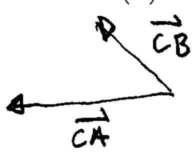
$$z_0 = 5^2 - 6\cdot 5\cdot 2 + 2\cdot 8$$
$$= 25 - 60 + 16$$

Q4 [10 marks]

Consider the triangle formed by the three points $A = (4, \frac{3}{\sqrt{2}}, 0)$, $B = (0, 0, \frac{3}{\sqrt{2}})$, and $C = (-3, \frac{3}{\sqrt{2}}, 0)$. Let D be the point obtained by dropping a perpendicular line from B to the side AC as indicated in the following picture. **Please note that the angles and distances of the triangle in this drawing are not necessarily accurate.**



(*a*) Find the angle between the sides AC and BC.



$$\vec{CA} = \langle 4, \tfrac{3}{\sqrt{2}}, 0\rangle - \langle -3, \tfrac{3}{\sqrt{2}}, 0\rangle = \langle 7, 0, 0\rangle$$

$$\vec{CB} = \langle 0, 0, \tfrac{3}{\sqrt{2}}\rangle - \langle -3, \tfrac{3}{\sqrt{2}}, 0\rangle = \langle 3, -\tfrac{3}{\sqrt{2}}, \tfrac{3}{\sqrt{2}}\rangle$$

$$\vec{CA} \cdot \vec{CB} = 21 = |\vec{CA}|\,|\vec{CB}| \cos\theta = 7 \cdot 3 \left|\langle 1, -\tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle\right| \cos\theta$$

$$= 21\sqrt{1 + \tfrac{1}{2} + \tfrac{1}{2}}\cos\theta \quad \text{so} \quad \cos\theta = \frac{1}{\sqrt{2}} \Rightarrow \boxed{\theta = 45^\circ \text{ or } \pi/4}$$

(*b*) Find the area of the triangle ABC.

$$\text{Area} = \frac{1}{2}\left|\vec{CB} \times \vec{CA}\right| = \frac{1}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & -\tfrac{3}{\sqrt{2}} & \tfrac{3}{\sqrt{2}} \\ 7 & 0 & 0 \end{vmatrix} = \frac{21}{2}\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -\tfrac{1}{\sqrt{2}} & \tfrac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{vmatrix}$$

$$= \frac{21}{2}\left|\langle 0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\rangle\right| = \frac{21}{2}\sqrt{\tfrac{1}{2} + \tfrac{1}{2}} = \boxed{\frac{21}{2}}$$

(c) Find the equation of the plane containing the points A, B, and C.

$\vec{CB} \times \vec{CA} = \frac{21}{2} \langle 0, 1/\sqrt{2}, 1/\sqrt{2} \rangle$ is normal to the plane

so $\langle 0,1,1 \rangle$ is also normal let $(x_0, y_0, z_0) = (0, 0, \frac{3}{\sqrt{2}})$

using $A(x-x_0) + B(y-y_0) + C(z-z_0) = 0$ we get

$$\boxed{y + z - \frac{3}{\sqrt{2}} = 0}$$

(d) Find a unit vector which is normal to the plane.

make $\langle 0,1,1 \rangle$ into a unit vector:

$$\boxed{\vec{N} = \langle 0, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle}$$

(e) Find the coordinates of the point D.

$$\vec{CD} = \text{Proj}_{\vec{CA}} \vec{CB} = \vec{CA} \frac{\vec{CA} \cdot \vec{CB}}{|\vec{CA}|^2} = \langle 7,0,0 \rangle \frac{21}{7^2} = \langle 3,0,0 \rangle$$

The coords of D is given by $\vec{C} + \vec{CD} = \langle -3, \frac{3}{\sqrt{2}}, 0 \rangle + \langle 3,0,0 \rangle$

$= \langle 0, \frac{3}{\sqrt{2}}, 0 \rangle$

$$\boxed{D = (0, \frac{3}{\sqrt{2}}, 0)}$$