Triple integrals $\iiint_E f(x,y,z)\,dV$

last day

Type 1 : $\iint_D \boxed{\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz}\,dA$

"shadow"

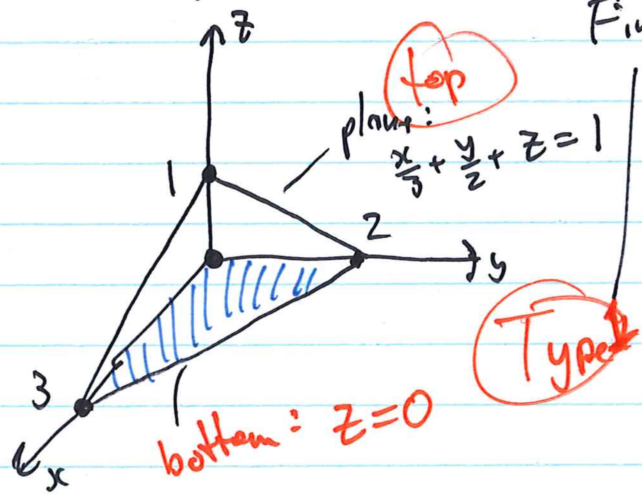
where $E = \{(x,y,z) : (x,y) \in D$ and $u_1(x,y) \le z \le u_2(x,y)\}$

Type 2: $E = \{(x,y,z) : (y,z) \in D,\ u_1(y,z) \le x \le u_2(y,z)\}$

Type 3: $E = \{(x,y,z) : (x,z) \in D,\ u_1(x,z) \le y \le u_2(x,z)\}$

$\hookrightarrow \iint_D \boxed{\int_{u_1(x,z)}^{u_2(x,z)} f(x,y,z)\,dy}\,dA$

Ex Let E be a tetrahedron through the points shown with density $\rho(x,y,z) = e^{x+y+z}$. Find the mass.

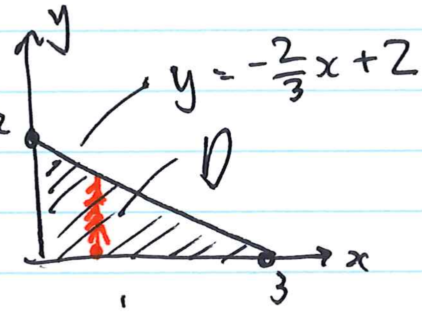


$$m = \iiint_E e^{x+y+z}\,dV$$

$$= \iint_D \int_{z=0}^{1 - x/3 - y/2} e^{x+y+z}\,dz\,dA$$

$u_2(x,y)$ $u_1(x,y)$

Region D in x-y plane:



$$m = \int_{x=0}^{3} e^{x} \int_{y=0}^{-\frac{2}{3}x+2} e^{y} \underbrace{\int_{z=0}^{1-x/3-y/2} \cancel{e^{x}}\,\cancel{e^{y}}\,e^{z}\,dz}_{\parallel} \,dy\,dx$$

$$e^{1-x/3-y/2} - e^{0} \quad dy\,dx$$

$$= \ldots = (e-1)^3$$

__Centre of mass__ (previously for laminae)

$G(\bar{x}, \bar{y}, \bar{z})$

density: $\rho(x,y,z)$ kg/m^3

total mass $m = \iiint_E \rho(x,y,z)\,dV$

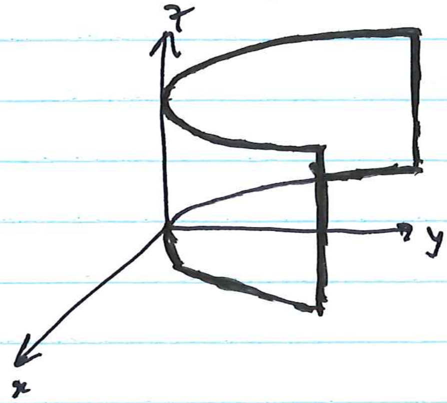
$$\bar{x} = \frac{\iiint_E x\rho(x,y,z)\,dV}{\iiint_E \rho(x,y,z)\,dV}$$

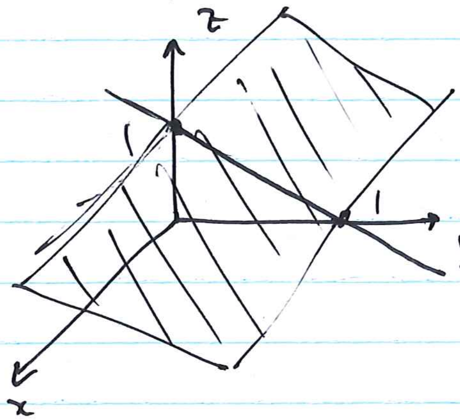
← moment about y-z plane

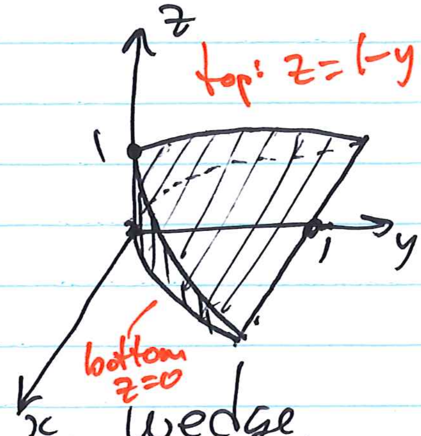
← total mass

$$\bar{y} = \iiint_E y\rho\,dV \Big/ m, \quad \bar{z} = \iiint_E z\rho\,dV \Big/ m$$

Ex Find centre of mass of region with uniform density $\rho = \rho_0$ bounded by a parabolic cylinder $y = x^2$, the x-y plane and the plane $y + z = 1$.





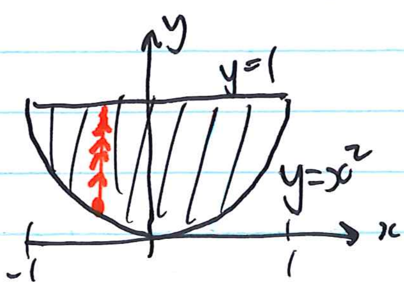


E wedge, door stop

Note: ρ_0 cancels from numerator/demominator, so we need:

$$\bar{x} = \frac{\cancel{\rho_0}\iiint_E x\,\cancel{\rho_0}\,dV}{\cancel{\rho_0}\iiint_E \cancel{\rho_0}\,dV} \leftarrow \text{vol}(E)$$

$$\text{vol}(E) = \iiint_E dV = \iint_D \int_{z=0}^{z=1-y} dz\,dA = \iint_D (1-y)\,dA$$



$$= \int_{x=-1}^{1} \int_{y=x^2}^{1} (1-y)\,dy\,dx$$

$$= \dots = 8/15$$

odd

$$\bar{x} = \frac{15}{8}\iiint_E x \, dV = 0 \quad \text{by symmetry}$$

E ← symmetric about $x=0$

$$\bar{y} = \frac{15}{8}\iiint_E y \, dV = \int_{-1}^{1}\int_{x^2}^{1} y(1-y)\, dy\, dx$$

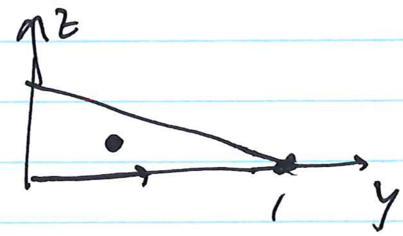
$$= \int_{-1}^{1}\int_{x^2}^{1}\int_{0}^{1-y} y \, dz\, dy\, dx$$

$$= \dots = 3/7$$

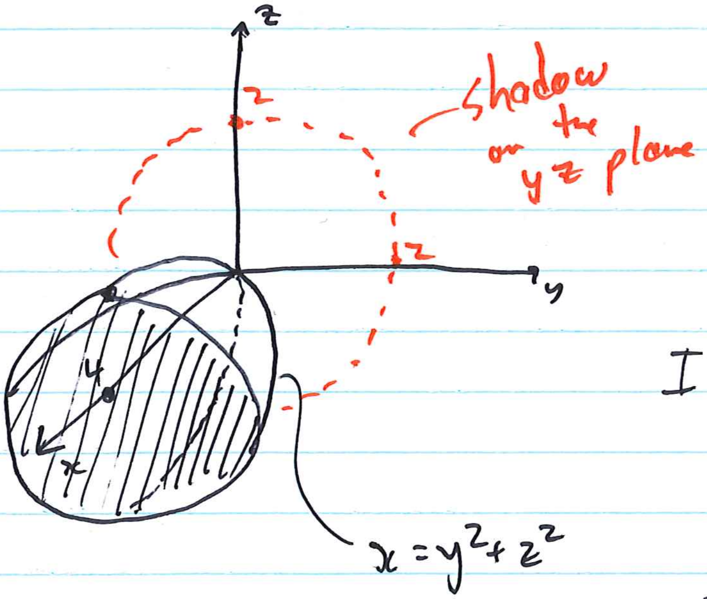
$$\bar{z} = \frac{15}{8}\iiint_E z \, dV = \int_{-1}^{1}\int_{x^2}^{1}\underbrace{\int_{0}^{1-y} z \, dz}_{\frac{1}{2}z^2\Big|_0^{1-y}}\, dy\, dx$$

$$= \dots = 2/7$$

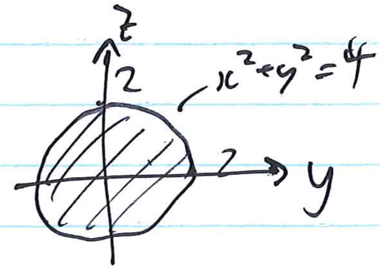
Centre of mass is $(0, 3/7, 2/7)$.



Ex Let E be the solid region (paraboloid) b/w $x=4$ and $x=y^2+z^2$. Setup limits of integration for $I = \iiint_E f(x,y,z)\,dV$.



Type 2

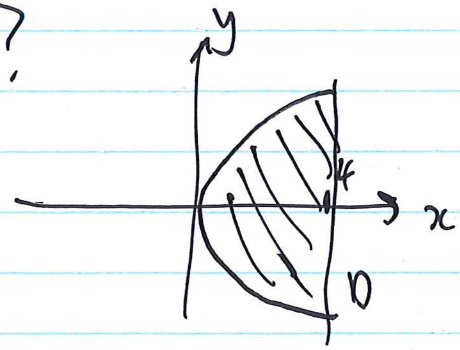


$$I = \iint_D \int_{x=y^2+z^2}^{x=4} f(x,y,z)\,dx\,dA$$

$u_2(y,z)$, $u_1(y,z)$

$$= \int_{y=-2}^{2} \int_{z=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} \int_{x=y^2+z^2}^{4} f(x,y,z)\,dx\,dz\,dy$$

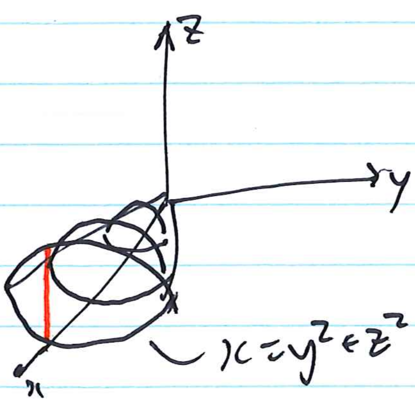
Type 1?



$$\iint_D \int_{z=-\sqrt{x-y^2}}^{\sqrt{x-y^2}} f(x,y,z)\,dz\,dA$$

$$= \int_{y=-2}^{2} \int_{x=y^2}^{4} \int_{-\sqrt{x-y^2}}^{\sqrt{x-y^2}} f\,dz\,dx\,dy$$

~~$x=\sqrt{y}$~~ ?



$x=y^2+z^2 \Rightarrow z = \pm\sqrt{x-y^2}$