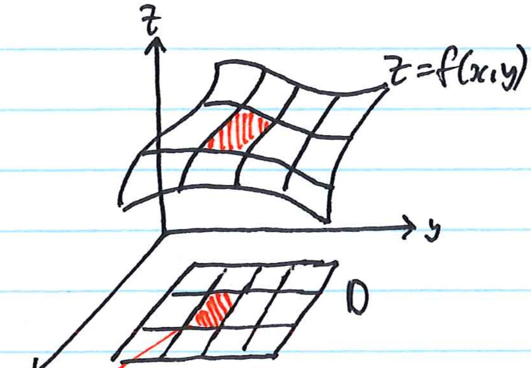
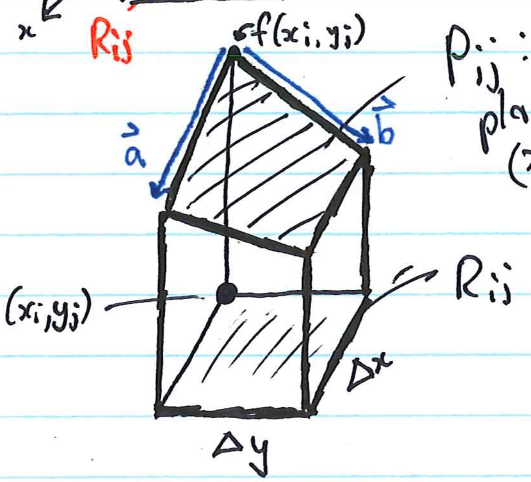
§13.5 Surface area of surface $z = f(x,y)$

$$\iint_D \; ? \; dA$$

? ← application of dbl integral

D ← goal: area of surface $z=f(x,y)$ above D.





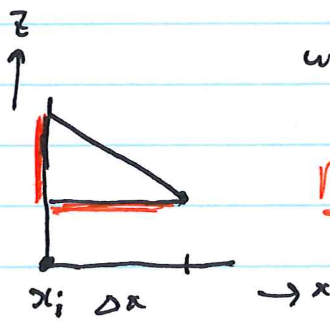
P_{ij}: tangent plane at (x_i, y_j) } the bit above R_{ij}

- use area(P_{ij}) to approx surface area of the part of $z=f(x,y)$ above R_{ij}
- Riemann
- ...
- $\iint_D ? \, dA$

let $\vec{a}$ and $\vec{b}$ be vectors as shown. then:

area$(P_{ij}) = \| \vec{a} \times \vec{b} \|$ = area of parallelogram with sides $\vec{a}, \vec{b}$

$$\vec{a} = \Delta x \left(1\vec{i} + f_x(x_i, y_j)\vec{k} \right)$$
$$= \Delta x \langle 1, 0, f_x \rangle$$



$\frac{\text{rise}}{\text{run}} = f_x$

slicing at $y = y_j$

Similarly:

$$\vec{b} = \Delta y \langle 0, 1, f_y \rangle$$

evaluated at (x_i, y_j)

$$\vec{a} \times \vec{b} = \Delta x \Delta y \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = \Delta x \Delta y \underbrace{\langle -f_x, -f_y, 1 \rangle}_{\text{evaluated at } x_i, y_j}$$

$$\text{area}(P_{ij}) = \| \vec{a} \times \vec{b} \| = \Delta x \Delta y \sqrt{1 + f_x^2 + f_y^2}$$

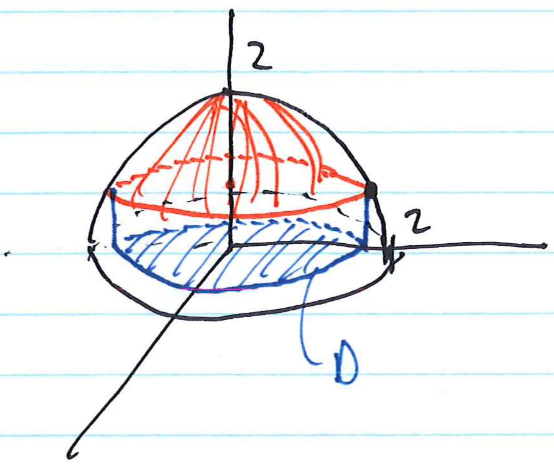
↳ Riemann sum

↳ lim

↳ ...

$$\boxed{S.A. = \iint_D \underbrace{\sqrt{1 + (f_x)^2 + (f_y)^2}}_{\text{special integrand, application of double integral}} \, dA}$$

Ex [Stewart §15.6 #10] Find the surface area of part of a sphere $x^2+y^2+z^2=4$ above $z=1$



$D = x^2+y^2+1^2=4$

$\Rightarrow x^2+y^2=(\sqrt{3})^2$

D is circle in x-y plane radius $\sqrt{3}$

$z=f(x,y)=\sqrt{4-x^2-y^2}$

$$\frac{\partial f}{\partial x} = \frac{-x}{\sqrt{4-x^2-y^2}}, \quad \frac{\partial f}{\partial y} = \frac{-y}{\sqrt{4-x^2-y^2}}$$

$$S.A. = \iint_D \sqrt{1+\frac{x^2}{4-x^2-y^2}+\frac{y^2}{4-x^2-y^2}}\, dA$$

$$= \iint_D \frac{2}{\sqrt{4-x^2-y^2}}\, dA$$

polar!

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2}{\sqrt{4-r^2}}\, r\, dr\, d\theta$$

let $u=r^2$

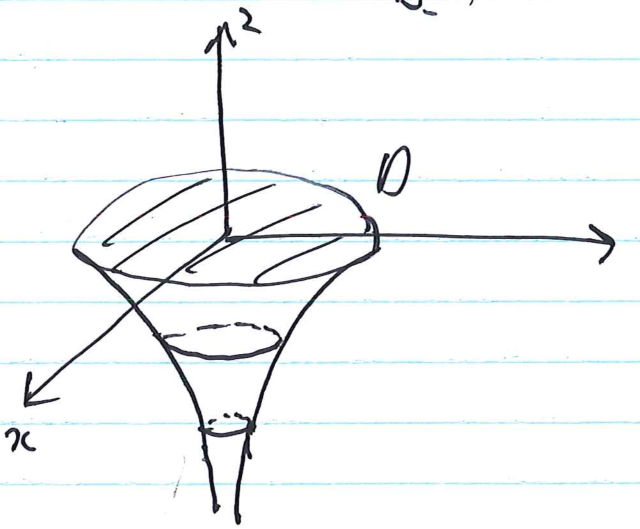
$$= \ldots = \int_0^{2\pi} 2\, d\theta = 4\pi$$

todo: redo with $z = \frac{1}{\sqrt{x^2+y^2}} - 1$

Ex Gabriel's horn $z = \underbrace{\frac{-1}{\sqrt{x^2+y^2}}}_{f(x,y)} + 1$

a) Find volume below $\underbrace{x^2+y^2 \le 1}_{D}$

b) Find the surface area below D.



a) $-V = \iint_D \frac{-1}{\sqrt{x^2+y^2}} + 1\, dA$

$= \int_0^{2\pi}\int_0^1 -\frac{1}{\cancel{r}}\,\cancel{r}\,dr\,d\theta + \cancel{\iint_D 1\, dA}^{\pi}$

$= -2\pi + \pi$

$$\boxed{V = \pi \quad m^3}$$

+ $S.A. = \iint_D \sqrt{1+f_x^2+f_y^2}\, dA$

this is positive, made mistake in lecture

$\frac{\partial f}{\partial x} = \frac{x}{(x^2+y^2)^{3/2}}$

$\frac{\partial f}{\partial y} = \frac{y}{(x^2+y^2)^{3/2}}$

Integrand is : $\sqrt{1 + \frac{x^2}{(x^2+y^2)^3} + \frac{y^2}{(x^2+y^2)^3}}$

$= \sqrt{1 + \frac{\cancel{x^2+y^2}}{(x^2+y^2)^{\cancel{3}}}} = \sqrt{1+\frac{1}{r^4}} = \sqrt{\frac{r^4+1}{r^4}}$

+ $S.A. = \int_0^{2\pi}\int_0^1 \sqrt{\frac{r^4+1}{r^4}}\, r\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \frac{1}{r^2}\underbrace{\sqrt{1+r^4}}_{\geq 1}\, r\,dr\,d\theta$

~~$$-S.A. \ngeq \int_0^{2\pi}\int_0^1 \frac{1}{r^2}\cancel{r}\,dr\,d\theta = \int_0^{2\pi} \ln r\Big|_0^1$$~~

~~$\boxed{S.A. = \infty}$~~ ~~$\to -\infty\ m^2$~~

Gabriel's Horn Paradox

(finite volume but infinite surface area)

↳ not enough paint in the con to paint the outside of the con :)

$$S.A. \geq \int_0^{2\pi}\int_0^1 \frac{1}{r^{\cancel{2}}}\cancel{r}\,dr\,d\theta = \int_0^{2\pi} \ln r\Big|_0^1 d\theta$$

$$= \int_0^{2\pi}(\ln 1 - \ln 0)\,d\theta$$

$$= \int_0^{2\pi}(0 - -\infty)\,d\theta$$

$$= \int_0^{2\pi} \infty\, d\theta$$

$$= \infty$$

(or more carefully $\lim_{c\to 0}\int_0^{2\pi}\int_c^1 \frac{1}{r}\,dr\,d\theta$

$$= \ldots = \lim_{c\to 0} -2\pi \ln c \to \infty$$