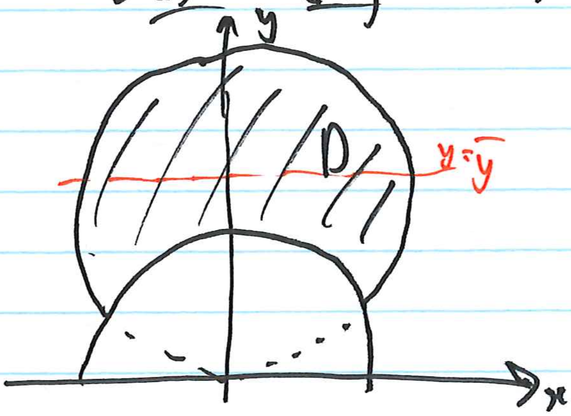
Handout today (or online)

Last day: moments problem



$$\rho(x,y) = \frac{k}{\sqrt{x^2+y^2}}$$

$$\bar{y} = \frac{\iint_D y\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA} \simeq 1.26$$

— m

$\bar{x} = 0$? (intuition)

$$\bar{x} = \frac{\iint_D x\,\overbrace{\rho(x,y)}^{\text{even}}\,dA}{m} = 0$$

odd

Symmetric about $x=0$

odd

$f(-x) = -f(x)$

even

$f(-x) = f(x)$

Math 253 Notes on Moments of Inertia

November 14, 2017

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1 Moments of Inertia

We've previously seen *moments* when calculating centre of mass of a lamina. This involved two double integrals:

$$M_y = \int\int_D x\rho(x,y)dA$$
$$M_x = \int\int_D y\rho(x,y)dA$$

These can also be called the "first moments"; here we look at the "second moments" or "moments of inertia".

1.1 Kinetic Energy of a spinning lamina

Suppose our lamina (which lies in the x-y plane) is rotating around the z-axis (note this is orthogonal to the lamina) at a constant angular rotational speed ω radians/s. (E.g., 60 rpm = 1 rev/s = 2π rad/s). Find the *Kinetic Energy* of the lamina. [Draw diagram!]

Riemann sum idea: as before, we consider a small rectangular piece R_{ij} with area $\Delta x \Delta y$. The kinetic energy of a point mass is $\frac{1}{2}mv^2$. Its going to be small in the limit so we use this to get:

$$\frac{1}{2}\rho(x_i, y_j)\Delta x \Delta y |\vec{v}_{ij}|^2.$$

The piece R_{ij} moves faster the further it is from the axis of rotation (z-axis, $(x,y) = (0,0)$). Different pieces move at different speeds. Our piece has kinetic energy:

$$\frac{1}{2}\rho(x_i, y_j)\Delta x \Delta y \omega^2 \left(x_i^2 + y_j^2\right)).$$

So take the Riemann sum over all pieces of the lamina and we get:

$$K = \frac{1}{2}\omega^2 \int\int_D (x^2+y^2)\rho(x,y)dA.$$

We define I_0 the **moment of inertia** about the z-axis as just the integral part:

$$I_0 = \int\int_D (x^2+y^2)\rho(x,y)dA.$$

Larger I_0 means more energy (work) to rotate the lamina about the z-axis.

1.2 About some other axis?

A similar argument shows how to compute the moment of inertia about some other axis parallel to the z-axis, centred at $(x, y) = (a, b)$:

$$I_0 = \int\int_D ((x-a)^2 + (y-b)^2)\rho(x,y)dA.$$

And in particular about the centre of mass $(x, y) = (\bar{x}, \bar{y})$, this would be:

$$I_{0,c} =$$

1.3 Rotation around x or y axes

What about rotating around the x-axis and y-axis? This gives the moment of inertia about the y-axis denoted I_y and the moment of inertia about the x-axis denoted I_x. [Draw diagrams]

$$I_y =$$

$$I_x =$$

Note relationship to previous,

$$I_0 =$$

1.4 Changing the axis of rotation

Suppose we have $I_{0,c}$ and want rotation around z-axis? Let M be overall mass of lamina. We get:

$$I_0 =$$

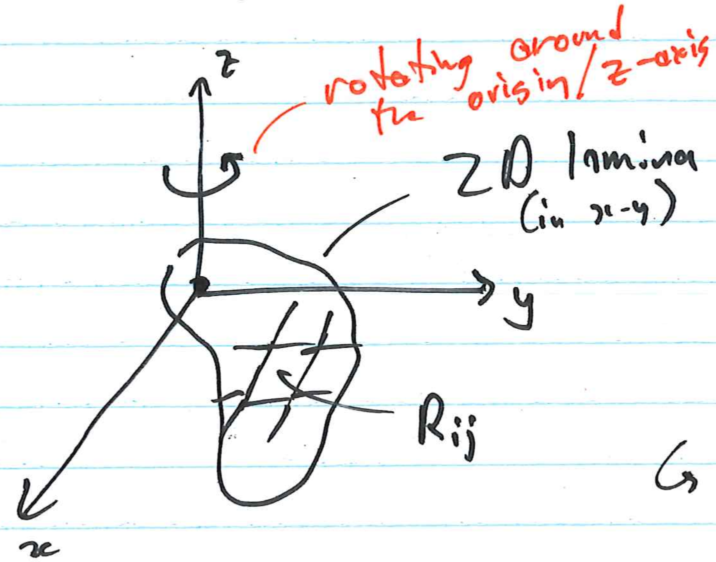
1.5 Examples

1. Find moment of inertia about the z-axis of a uniform circular disc of radius R and total mass M, centred at the origin.
2. Find same, but with disc centred at point (a, b).
3. Find same, for a uniform rectangular plate, mass M, axis through centre, size $a \times b$.

Applications of $\iint$: Moments on inertia

(exercise in §13.4)
(and handout)

Still talking laminae: suppose it is rotating with angular speed ω ~~rad~~ rad/s



E.g. 60 rpm = 1 rev/s
= 2π rad/s

Want to find the kinetic energy of the lamina.

↳ Riemann Sum:

① find the K.E. of R_{ij}

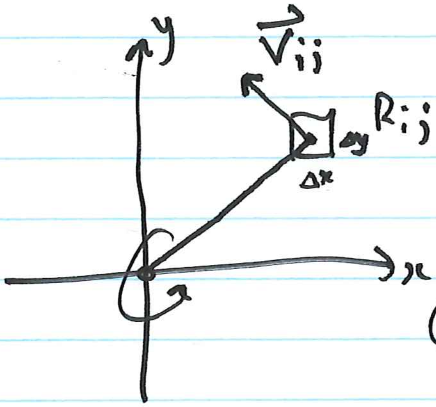
② $\sum\sum$ over all R_{ij}

③ $\lim_{\Delta x, \Delta y \to 0} \sum\sum \;?\; = \iint_D \;?\; dA$

K.E. of R_{ij} : so pretend its a point mass → K.E. is $\frac{1}{2} m v^2$

mass speed

b/c R_{ij} is small



① $$K.E. = \frac{1}{2} \overbrace{\rho(x_i, y_j)\,\Delta x\,\Delta y}^{\text{mass}} \underbrace{|\vec{v}_{ij}|^2}_{\omega^2 (x_i^2 + y_j^2)}$$

(why? $|\vec{v}_{ij}| = \omega r_{ij}$

②③ $$K = \frac{1}{2}\omega^2 \boxed{\iint_D (x^2+y^2)\,\rho(x,y)\, dA} \to I_0$$

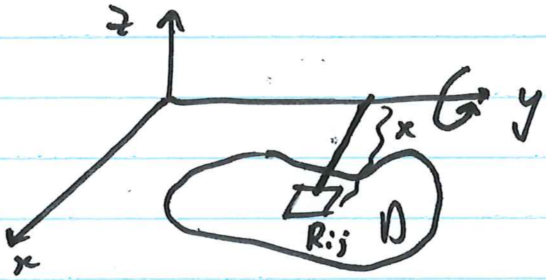
Moment of Inertia about the z-axis (or "about the origin").

Similarly, about the centre of mass,

$$I_{0,c} = \iint_D \left[(x-\bar{x})^2 + (y-\bar{y})^2 \right] \rho(x,y)\, dA$$

centre of mass

Moment of Inertia about the y-axis (i.e. $x=0$)



$$\boxed{I_y = \iint_D x^2 \rho(x,y)\, dA}$$

2nd moment

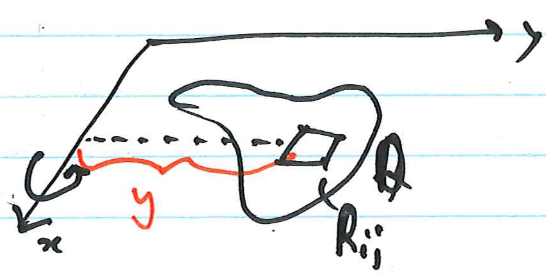
distance^2 to the y-axis / $x=0$

Recall: $M_y = \iint_D x\rho(x,y)\, dA$

1st moment

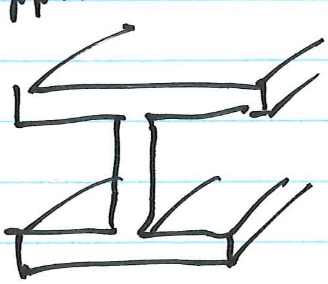
Moment of Inertia about the x-axis: ($y=0$)

$$\boxed{I_x = \iint_D y^2 \rho(x,y)\, dA}$$

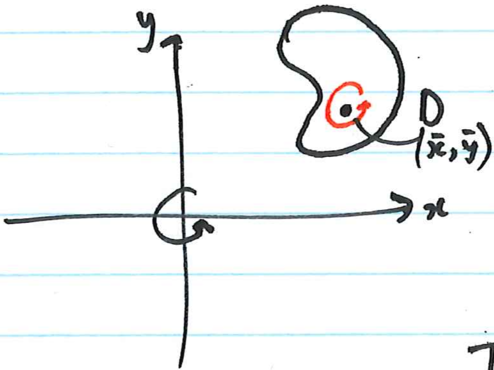


<u>Note</u>: $I_0 = I_x + I_y$

Applications: gyroscopes, spacecraft rigidity of beams



Changing axis of rotation



Suppose we have $I_{o,c}$ calculated

Want I_o, w/o integrating "from scratch".

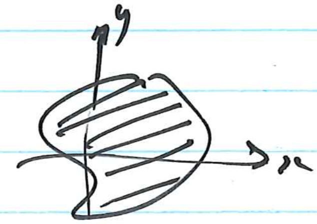
$$I_o = \underbrace{\overset{\text{total mass}}{m}(\bar{x}^2 + \bar{y}^2)}_{\text{moment of inertia of a point mass (equivalent to lamina) about z-axis.}} + \underbrace{I_{o,c}}_{\text{moment of inertia about centre of mass}}$$

(Derivation requires $\bar{x}, \bar{y}$ to be centre of mass).

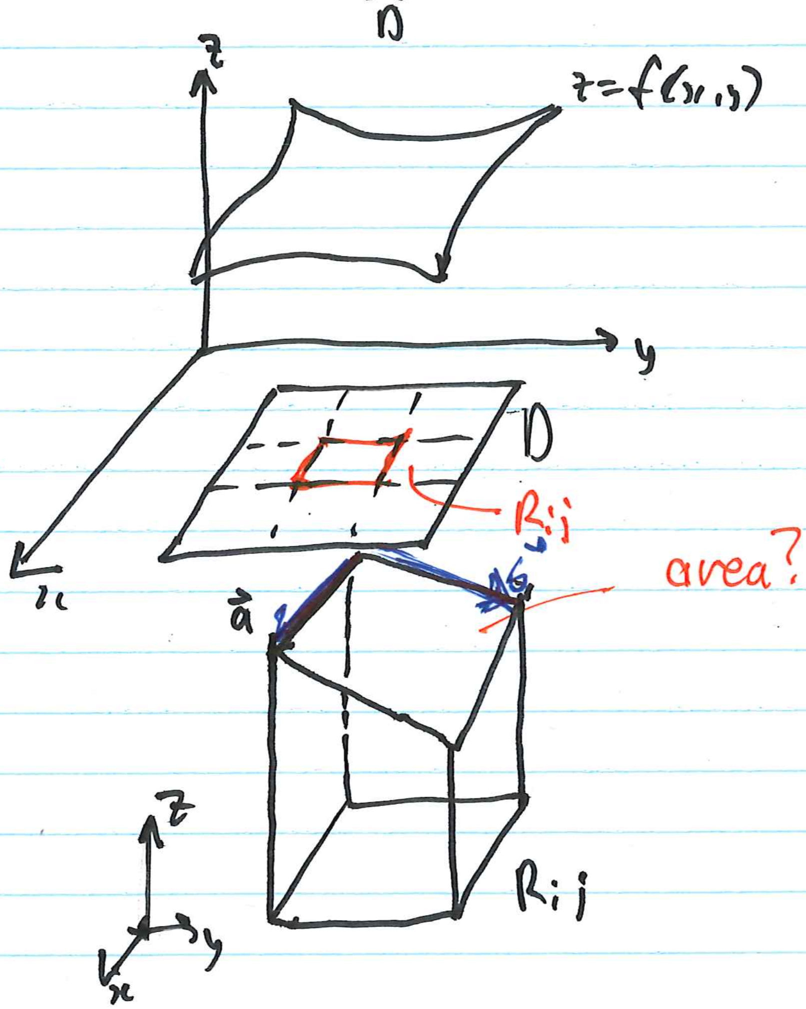
~~Exer~~ (Exercises on handout).

Applications of $\iint$: Surface area §13.5

Recall $\iint_D 1\,dA$ = area of D.



$\iint_D f(x,y)\,dA$ = volume under surface $z=f(x,y)$



What about the surface area of $z=f(x,y)$ (above D)?

↳ how much paint do I need?