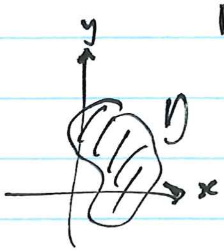
§13.4 Applications of Double Integrals

$$\iint_D f(x,y)\, dA$$

$f(x,y)$ ↑ applications



We've seen:

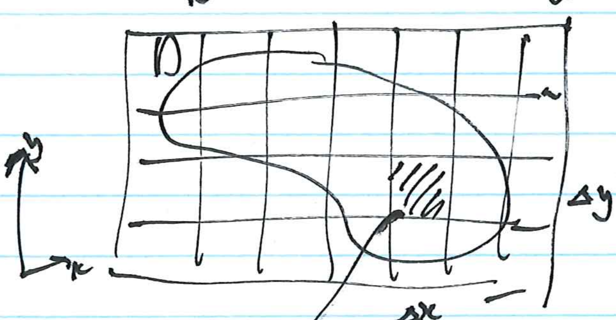
① $f(x,y) \equiv 1$
$\Rightarrow$ area of D

② $f(x,y)$ as height "above" D.
$(z = f(x,y))$
$\Rightarrow$ volume "under" surface $z = f(x,y)$ and "above" D.

Today: mass and moments.

Recall Riemann Sum:

$$\iint_D f(x,y)\, dA = \lim_{n,m\to\infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x_i, y_j)\, \Delta x\, \Delta y$$



Suppose D is a "lamina" (a thin piece of material) with variable density
$\rightarrow$ mass density $\rho(x,y)$

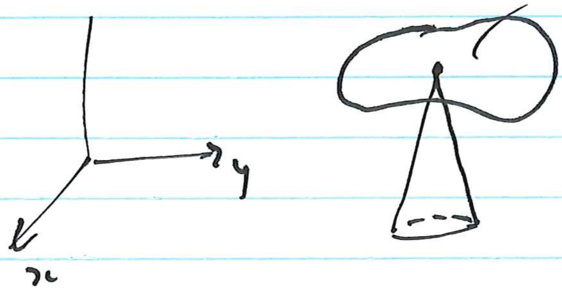
$R_{ij} \rightarrow$ Mass of small $R_{ij} \approx \rho(x_i, y_j)\, \Delta x\, \Delta y$

Total mass of the lamina is $m = \iint_D \rho(x,y)\, dA$

Units: $\boxed{\rho = \text{mass}/\text{area}}$

Examples: charge density
population density
light/photon density
probability density
etc

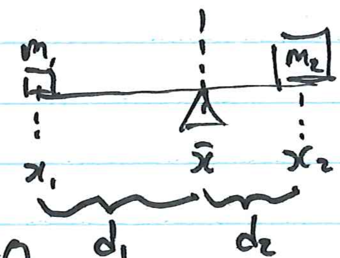
Center of Mass



balance point $(\bar{x}, \bar{y})$

a.k.a. centre of mass.

Think in 1D, discrete $\rightarrow$ teeter-totter



balance when $m_1 d_1 = m_2 d_2$

i.e. $m_1(x_1 - \bar{x}) + m_2(x_2 - \bar{x}) = 0$

In general: $\sum_{i=1}^{n} m_i (x_i - \bar{x}) = 0$

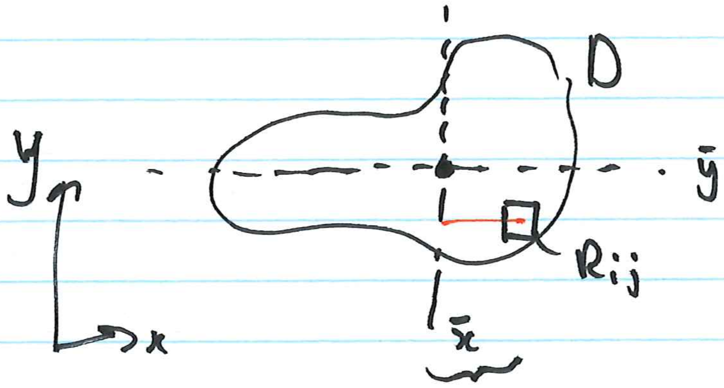
$$\sum_{i=1}^{n} m_i x_i = \bar{x} \sum_{i=1}^{n} m_i$$

$$\bar{x} = \frac{\sum m_i x_i}{\sum m_i}$$

... Riemann ...
(density $\rho(x)$ for a 1D rod $a \leq x \leq b$)

$$\bar{x} = \frac{\int_a^b x\rho(x)\,dx}{\int_a^b \rho(x)\,dx} \qquad \frac{\text{"moment"}}{\text{mass}}$$

Back to our lamina (in 2D)



moment of R_{ij} about $x = \bar{x}$ $=$ $\underbrace{\text{(signed) dist}}_{x_i - \bar{x}} \cdot \rho(x_i, y_j)\Delta x \Delta y$ $\rightarrow$ Riemann Sum

↓

Solve for $\bar{x}$

$$\frac{M_y}{m} = \boxed{\bar{x} = \frac{\iint_D x\,\rho(x,y)\,dA}{\iint_D \rho(x,y)\,dA}}$$

moment of the lamina about the y-axis $(x=0)$

→ notation: M_y

mass

Repeat:

$$\boxed{\bar{y} = \frac{\iint_D y\,\rho\,dA}{\iint_D \rho\,dA}}$$

moment of the lamina about the x-axis $(y=0)$ $\rightarrow M_x$

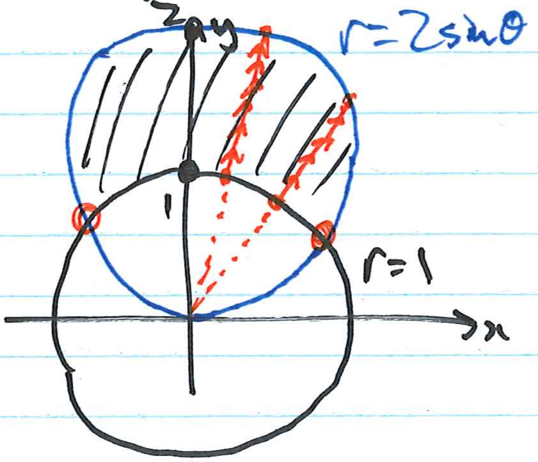
Note: 3 integrals over the ~~a~~ region D.

Ex Find the centre of mass of a region inside $x^2+y^2=2y$ and outside $x^2+y^2=1$, with density inversely proportional to distance from the origin.

[Stewart §15.5 #16]

$$\rho(x,y) = \frac{k}{\sqrt{x^2+y^2}} = \frac{k}{r}$$ const.

Region



$\boxed{r=1}$

$x^2+y^2=2y$

$r^2 = 2r\sin\theta$

$\boxed{r = 2\sin\theta}$

intersections:

$1 = 2\sin\theta$

$\theta = \frac{\pi}{6}, \ \pi-\frac{\pi}{6}$

$$m = \iint \rho(x,y)\,dA$$

$$= \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} \underbrace{\frac{k}{\cancel{r}}}_{\rho}\ \underbrace{\cancel{r}\,dr\,d\theta}_{dA}$$

$$= k\int_{\pi/6}^{5\pi/6}(2\sin\theta - 1)\,d\theta$$

$$= -k\Big[2\cos\theta + \theta\Big]_{\pi/6}^{5\pi/6}$$

$$= -k\left[2\left(-\frac{\sqrt{3}}{2}-\frac{\sqrt{3}}{2}\right)+\frac{4\pi}{6}\right]$$

$$= k\left(2\sqrt{3}-\frac{2\pi}{3}\right)$$

moment about x-axis $\quad M_x = \iint_D y\,\rho(x,y)\,dA = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} \underbrace{r\sin\theta}_{y}\ \underbrace{\frac{k}{r}}_{\rho}\ \underbrace{r\,dr\,d\theta}_{dA}$

$$= k\int_{\pi/6}^{5\pi/6} \sin\theta \underbrace{\int_1^{2\sin\theta} r\,dr}_{\frac{1}{2}r^2\Big|_1^{2\sin\theta}} d\theta$$

$$= \frac{k}{2}\int_{\pi/6}^{5\pi/6} \sin\theta\left[4\sin^2\theta - 1\right]d\theta$$

$$= \ldots = \sqrt{3}\,k$$

$$\Rightarrow \quad \bar{y} = \frac{M_x}{m} = \frac{\sqrt{3}\,\cancel{k}}{\cancel{k}\left(2\sqrt{3} - \frac{2\pi}{3}\right)} \approx 1.26$$

$\bar{x}$ next day $(=0)$.