Last day: $V = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \, r \, dr \, d\theta$

~~Mnemonic~~ trick to remember

$dA = r\,dr\,d\theta = \underbrace{dr}_{m} \underbrace{r\,d\theta}_{m}$

$V = \underset{R}{\iint} \underset{m}{f} \underset{m^2}{dA}$

$V = \int_{-\pi/2}^{\pi/2} \frac{1}{4} r^4 \Big|_0^{2\cos\theta} d\theta$

$= \int_{-\pi/2}^{\pi/2} 4 \underbrace{\cos^4\theta}_{\hookrightarrow (\cos^2\theta)^2} d\theta$

$= \left[\left(\frac{1}{2}(1+\cos 2\theta)\right)\right]^2$

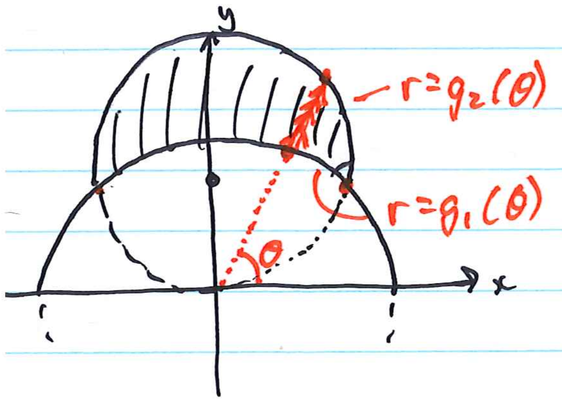
$= \frac{1}{4}\left(1 + 2\cos 2\theta + \cos^2 2\theta\right)$

$\cos^2 2\theta \to \frac{1}{2}(1+\cos 4\theta)$

$V = \int_{-\pi/2}^{\pi/2} \cancel{4}\left(\frac{1}{\cancel{4}}\left(1 + 2\cos 2\theta + \frac{1}{2}(1+\cos 4\theta)\right)\right) d\theta$

$= \ldots = 3\pi/2$

Ex Find the area of a region outside circle radius $\sqrt{2}$ centred at the origin and inside unit circle centred at $(0,1)$.



$$A = \iint_R 1\,dA = \int_?^? \int_{?\ (r=g_1(\theta))}^{?\ (r=g_2(\theta))} r\,dr\,d\theta$$

$g_1(\theta)$: $r = \sqrt{2}$

$g_2(\theta)$: ?

guess based on $2\cos\theta$ example that $\boxed{g_2(\theta) = 2\sin\theta}$

~~check~~ check

$$r = 2\sin\theta$$
$$r^2 = 2r\sin\theta$$
$$x^2+y^2 = 2y$$
$$x^2 + (y-1)^2 = 1$$

Need α, β for $\int_\alpha^\beta d\theta$:

$g_1(\theta) = g_2(\theta) \Rightarrow$

$$\sqrt{2} = 2\sin\theta$$
$$\theta = \arcsin\left(\sqrt{2}/2\right)$$
$$\theta = \pi/4,\ 3\pi/4$$

$$A = \int_{\pi/4}^{3\pi/4} \int_{\sqrt{2}}^{2\sin\theta} r\,dr\,d\theta$$

$$A = \int_{\pi/4}^{3\pi/4} \frac{1}{2} r^2 \Big|_{\sqrt{2}}^{2\sin\theta} d\theta$$

$$= \int_{\pi/4}^{3\pi/4} \underbrace{\frac{1}{2}\left(4\sin^2\theta - 2\right)}_{} d\theta$$

$$2\sin^2\theta - 1$$

$$= \cancel{2}\left(\cancel{\tfrac{1}{2}}(1-\cos 2\theta)\right) - \cancel{1}$$

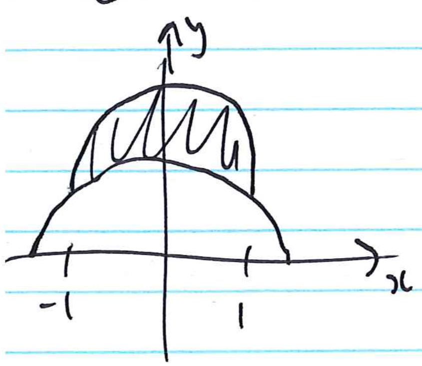
$$= -\cos 2\theta$$

$$A = \int_{\pi/4}^{3\pi/4} -\cos 2\theta \, d\theta = -\frac{1}{2}\sin 2\theta \Big|_{\pi/4}^{3\pi/4}$$

$$= -\frac{1}{2}(-1-1)$$

$$\boxed{= 1.}$$

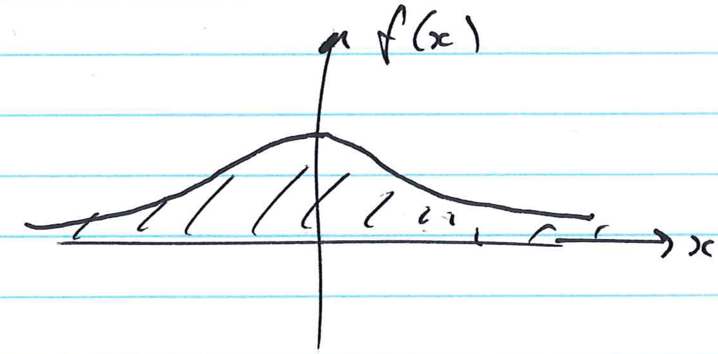
Could use 1-var calculus:



$$A = \int_{-1}^{1} \underbrace{1+\sqrt{1-x^2}}_{\text{top}} - \underbrace{\sqrt{(\sqrt{2})^2 - x^2}}_{\text{bottom.}} \, dx$$

Ex You won't believe this one weird trick to compute the area under the bell curve.

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$



$$A = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

tricky, antiderivative is "erf"

$$A^2 = \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx\right)\left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy\right)$$

y dummy var of integration

$$A^2 = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy\right) \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi} e^{(-x^2-y^2)/2}\, dx\, dy$$

$$= \iint_R \frac{1}{2\pi} e^{(-x^2-y^2)/2}\, dA$$

entire xy-plane

$$= \int_0^{2\pi}\int_0^{\infty} \frac{1}{2\pi} e^{-r^2/2}\, r\, dr\, d\theta$$

let $u = r^2/2$

$du = r\, dr$

$$= \int_0^{2\pi} \frac{1}{2\pi} \left[\int_0^{\infty} e^{-u}\, du\right] d\theta$$

$$-e^{-u}\Big|_0^{\infty}$$

$$= \int_0^{2\pi} \frac{1}{2\pi}\, 1\, d\theta = \frac{2\pi}{2\pi} = 1$$

$$\boxed{A^2 = 1 \Rightarrow A = 1}$$

Summary of min/max problems

(I) min $f(x,y)$ for some x,y, e.g. $x^2+y^2 \leq 1$

inequality constraint (not Lagrange M.)

(1) find c.p.
(2) classify c.p. (if asked) or easier to evaluate $f(\text{c.p.})$

(2) solve the problem on the boundary.

parametrize the boundary:
$x=\cos t$, $y=\sin t$
→ define 1-var calc problem in t

L.M.
min $f(x,y)$.
subject to
$g(x,y)=0$
$(x^2+y^2-1=0)$

(II) min $f(x,y,z)$ subject to

$g(x,y,z)=0$

equality

→ use Lagrange Mult.