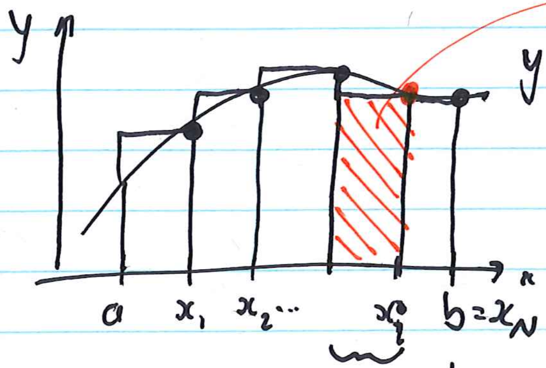
Integration §13.1, §13.2

1-var review : Riemann sum



$\Delta x = \dfrac{b-a}{N}$

Riemann sum: $\displaystyle\sum_{i=1}^{N} f(x_i)\,\Delta x$

Def'n of integral

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{i=1}^{N} f(x_i)\,\Delta x$$

(Signed) area under the curve and over $[a,b]$

if lim exists

any $x_i^* \in [x_{i-1}, x_i]$

Other interpretation: $f_{avg} \approx \dfrac{1}{N} \sum f(x_i)$

$$= \frac{1}{b-a} \sum f(x_i) \frac{b-a}{N}$$

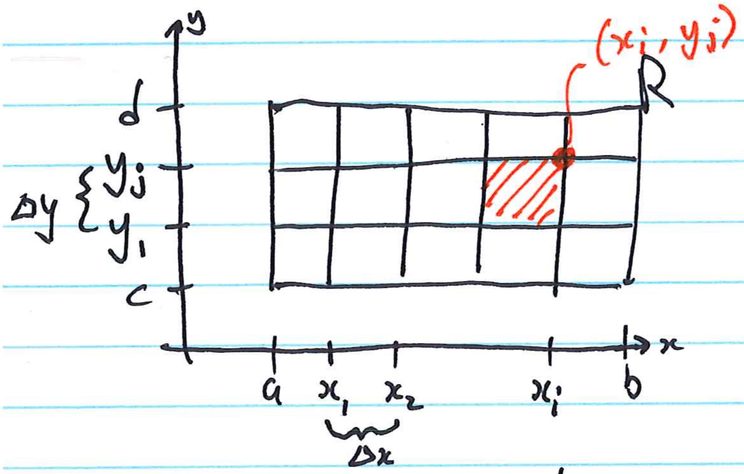
Δx

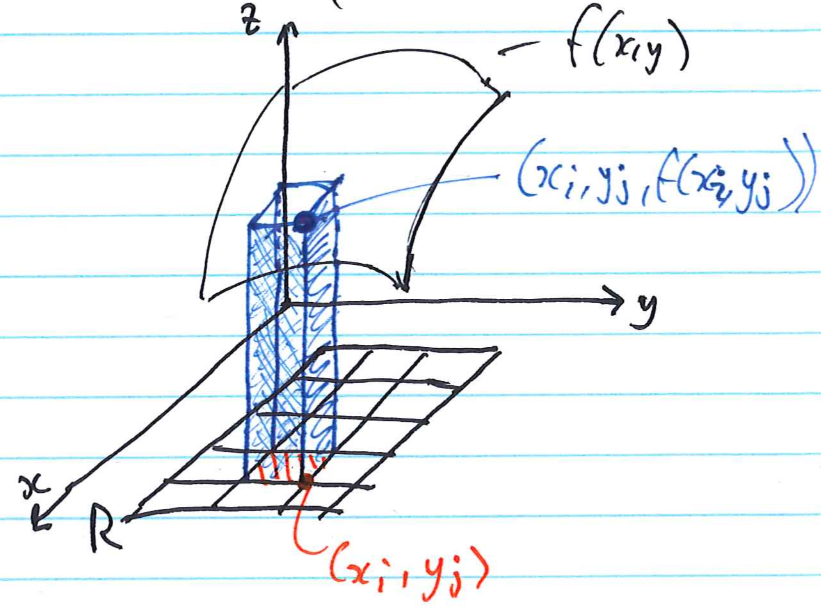
... define $f_{avg} = \dfrac{1}{b-a} \displaystyle\int_a^b f(x)\,dx$

length of segment.

Same ideas in two-variables... $f(x,y)$

Rectangle $(x,y) \in R = [a,b] \times [c,d]$ $\quad (a \le x \le b,\ c \le y \le d)$





$\Delta x = \frac{b-a}{m}, \quad \Delta y = \frac{d-c}{n}$

Volume of the column:

$f(x_i, y_j)\,\Delta x\,\Delta y$

Volume under the graph of $z = f(x,y)$ above R

$$\approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta x\,\Delta y$$

Def'n:

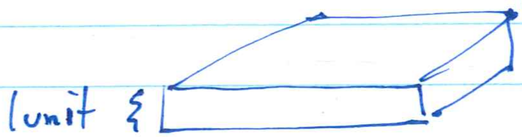
$$\iint_R f(x,y)\,\underbrace{dA}_{\text{often } dx\,dy} := \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\,\Delta x\,\Delta y$$

Interpretation:
- signed volume "over" R and "under" $z = f(x,y)$,
- negative volume for $f(x,y) < 0$.

- Called the "double integral"
- can also compute area (!)

$$\iint_R 1 \, dA = \text{area of } R$$

volume of a unit thickness pancake.



- Average value: $f_{avg} = \frac{1}{\text{area}(R)} \iint_R f(x,y)\, dA$
- Properties ① linearity $\iint_R f(x,y) + c\, g(x,y)\, dA$ (scalar, const)

$$= \iint_R f \, dA + c \iint_R g \, dA$$

② monotonicity: if $f(x,y) \le g(x,y) \;\; \forall x,y \in R$ ("for all")

$$\iint_R f \, dA \le \iint_R g \, dA$$

So far, theoretical — only have Riemann Sum to compute things.

Iterated Integral §13.1

Recall single variable calc: usually used

~~Fu~~ Fundamental Thm of Calc: if $F' = f$

then $\int_a^b f\,dx = F(b) - F(a)$

i.e., we tried to find the antiderivative F.

(tables, software, substitution, int by parts).

nice to reuse all that!

2 vars: $f(x,y)$ over rectangle $R = [a,b] \times [c,d]$

Fix x, compute $\underbrace{A(x)}_{\text{no more } y!} = \int_{y=c}^{y=d} f(x,y)\,dy$

partial integral.

~~Take~~ Iterated integral $\int_{x=a}^{x=b} A(x)\,dx$

$$\int_c^d \underbrace{\left[\int_a^b f(x,y)\,dx\right]}_{B(y)} dy = \iint_R f(x,y)\,dA = \int_a^b \underbrace{\left[\int_c^d f(x,y)\,dy\right]}_{A(x)} dx$$

<u>Fubini's Thm</u>: either iterated integral gives the same value, which is a <u>number</u>

Mistake earlier !! $f_{xy} = f_{yx}$ ← ~~Cl...~~ Schwartz Thm