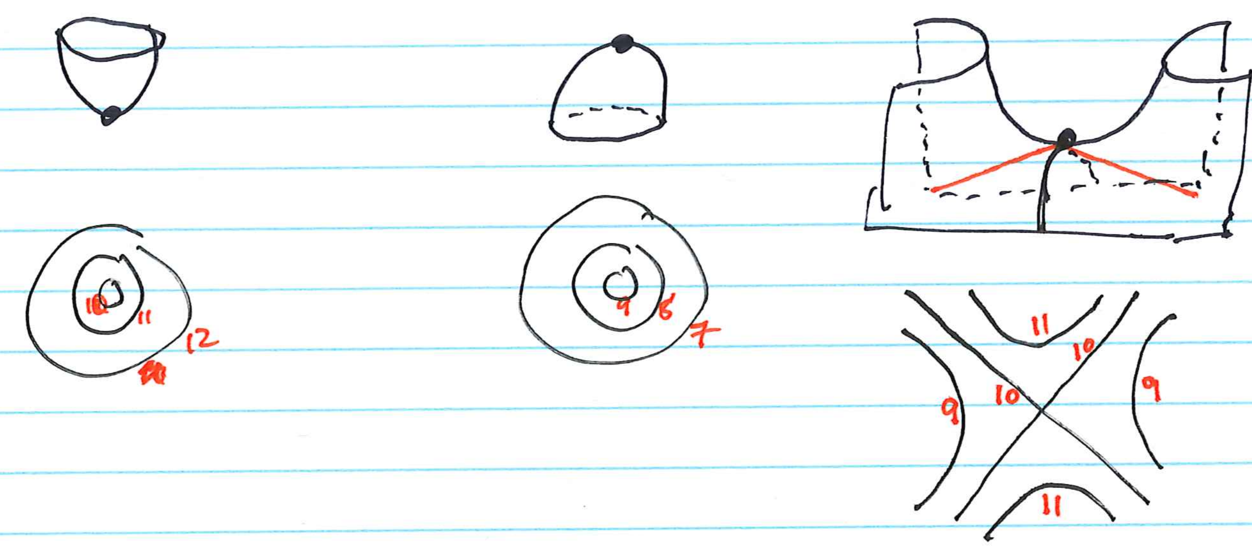
Last day : critical pts

$\nabla f \underbrace{(a,b)}_{\text{c.p.}} = \vec{0}$, horizontal tangent plane.

3 most-common critical pts : (many others)

local min   local max   saddle



2nd derivative test

Let $D(x,y) = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2 =$ scalar

Suppose (a,b) is a critical point

could use f_{yy}

Then (a,b) is $\begin{cases} \text{a local min if } D(a,b) > 0, \; f_{xx} > 0 \\ \text{a local max if } D(a,b) > 0, \; f_{xx} < 0 \\ \text{a saddle if } D(a,b) < 0 \end{cases}$

- $D > 0$: uniform concavity. up or down
- If $D(a,b) = 0$, ~~the~~ test is inconclusive $\Rightarrow$ (a,b) could be a max, min, saddle, none, etc.
- Note test needs $\nabla f(a,b) = \vec{0}$
- What about $D(a,b) > 0$ but $f_{xx} = 0$?

  impossible b/c $f_{xx} = 0 \Rightarrow D \leq 0$

  $D = -(f_{xy})^2 < 0$ $\Rightarrow\Leftarrow$

- Note could have $f_{xx} > 0$ and $f_{yy} > 0$ but $f_{xy}^2 > f_{xx} f_{yy}$ so $D < 0$ $\Rightarrow$ saddle.

__Example__ Find and classify the critical points of $f(x,y) = (2x - x^2)(2y - y^2)$

$f_x = \cancel{2xy}(2-2x)(2y-y^2) = 0 \Rightarrow x=1,\ y=0,\ y=2$

$f_y = (2x-x^2)(2-2y) = 0 \Rightarrow x=0,\ x=2,\ y=1$

C.P. are where $f_x = f_y = 0$

$f_x = 0$	$f_y = 0$	C.P.
$x=1$	$y=1$ $\cancel{x=0}$ $\cancel{x=2}$	$(x,y) = (1,1)$
$y=0$	$\cancel{y=1}$ $x=0$ $x=2$	$(x,y) = (0,0)$ $(x,y) = (2,0)$
$y=2$	$\cancel{y=1}$ $x=0$ $x=2$	$(x,y) = (0,2)$ $(x,y) = (2,2)$

<u>Classify</u>: $f_{xx} = -2(2y - y^2)$
$f_{yy} = -2(2x - x^2)$
$f_{xy} = (2-2x)(2-2y)$

$f_{xy}(1,1) = 0$
$f_{xx}(1,1) = -2$
$f_{yy}(1,1) = -2$

$D(1,1) = \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} = 4$

$D(1,1) > 0$ and $f_{xx}(1,1) = -2 < 0$

<u>local max</u>

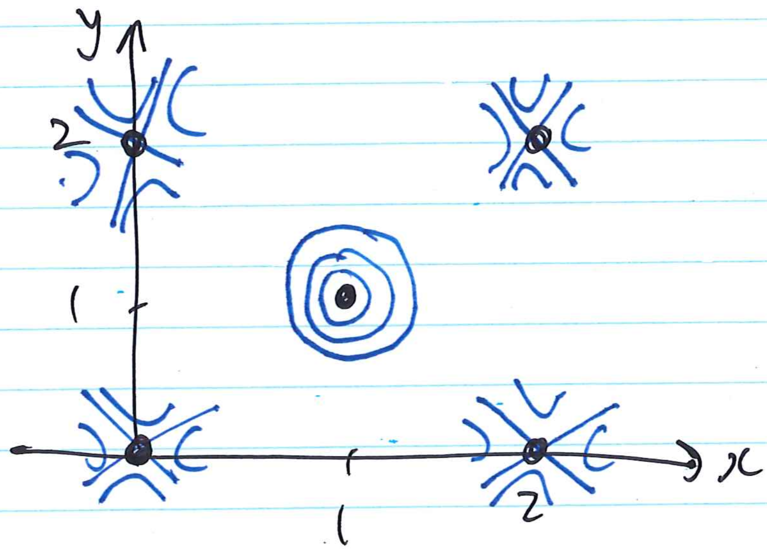
...

$D(0,0) = -16$
$D(2,0) = -16$
$D(0,2) = -16$
$D(2,2) = -16$

$\underbrace{}_{\text{4 saddles}}$

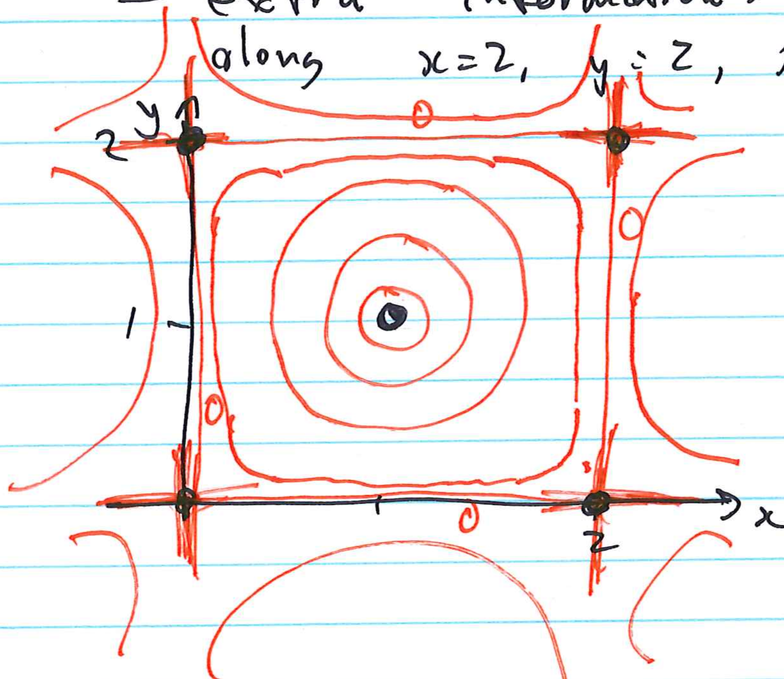
<u>4 saddles</u>

Sketch "cartoons" of each c.p.



Join up the local sketches:

– extra information: f is zero along $x=2$, $y=2$, $x=0$, $x=2$



Example: Same f, what is global max/min on the domain $0 \le x \le 2$ and $0 \le y \le 2$.

1. find/classify the c.p. (those inside the domain.
2. check behaviour on ~~the~~ boundary.
   ↳ Here $f(x,y) = 0$ on boundary.

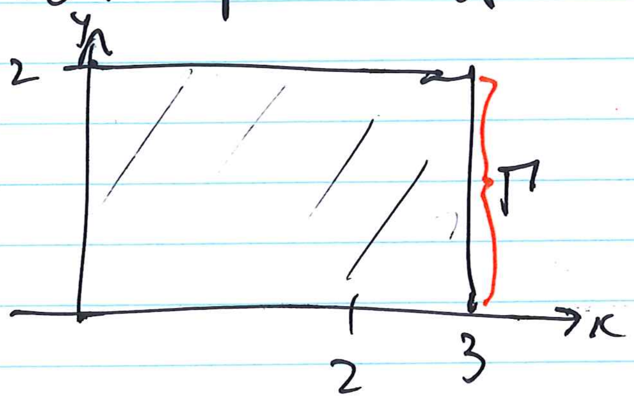
global max is $f(1,1) = \cancel{\ } 1$

global min is 0 and ~~occ~~ occurs on boundary

Example: same f. domain is $0 \le x \le 3$

$0 \le y \le 2$.

In general, we must solve a 1-variable calculus problem on each piece of the boundary.



parameterize Γ:

$$\Gamma = \{(x,y) : x = 3, \ 0 \le y \le 2\}$$

$$g(y) := f(3,y)$$

$$g(y) = -3(2y - y^2) = -6y + 3y^2$$

min/max on the interval $0 \le y \le 2$