Tangent Planes revisited (end of §12.7)

Last day: gradient of a fcn of 3-vars $F(x, y, z)$

$\vec{\nabla} F = \langle F_x, F_y, F_z \rangle$ a 3-vector

$\vec{\nabla} F$ is the normal vector to the <u>level surfaces</u> $F(x, y, z) = k$.

Tangent plane to the implicit surface <u>$F(x, y, z) = k$</u> at a point $\underbrace{(x_0, y_0, z_0)}$:

(Recall) "$\vec{n} \cdot (\underbrace{\vec{r}}_{\langle x, y, z \rangle} - \underbrace{\vec{r}_0}_{\langle x_0, y_0, z_0 \rangle}) = 0$"

$$\underbrace{F_x(x_0, y_0, z_0)}_{\#}\underbrace{(x - x_0)}_{\#} + \underbrace{F_y(x_0, y_0, z_0)}_{\#}\underbrace{(y - y_0)}_{\#} + \underbrace{F_z(x_0, y_0, z_0)}_{\#}\underbrace{(z - z_0)}_{\#} = 0$$

Special case: $z = f(x,y)$ ← explicit surface

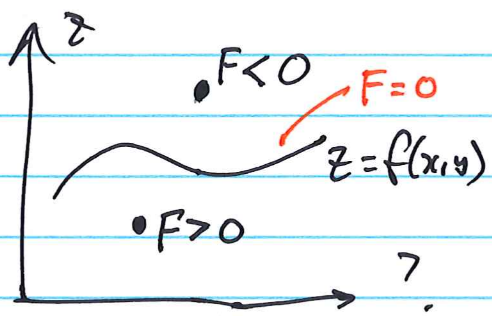
"Embed" this surface as the 0-level surface of a new $F(x,y,z)$ (defined for all $\mathbb{R}^3$)

$$F(x,y,z) := f(x,y) - z$$

- For example, to find the tangent plane:

$$\nabla F(x_0,y_0,z_0) = \langle f_x(x_0,y_0),\ f_y(x_0,y_0),\ -1 \rangle$$

$$f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0) - (z-z_0) = 0$$



(same formula we had earlier).

- Idea is also useful to find the normal vector of an explicit surface.

Ex Show that every tangent plane of the surface $z^2 = x^2+y^2$ passes through the origin.

A: defining $F(x,y,z) = x^2+y^2-z^2$ $\left(\begin{array}{l}\text{Surface}\\ \text{is } F=0\end{array}\right)$

$\vec{\nabla} F = \langle 2x, 2y, -2z \rangle$, $\boxed{\text{Say } (x_0, y_0, z_0) \text{ is on the surface.}}$

$\Rightarrow F(x_0, y_0, z_0) = 0$

Tangent

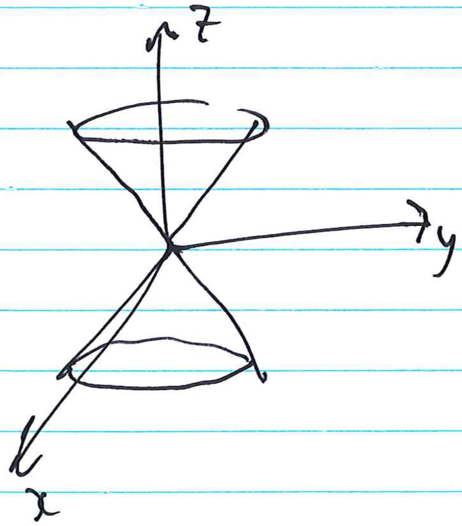
$$2x_0(x-x_0) + 2y_0(y-y_0) - 2z_0(z-z_0) = 0$$

Check tangent plane goes through ~~to~~ the origin
$\rightarrow$ sub $(0,0,0)$ into the LHS for (x,y,z)

$$LHS = -2x_0^2 - 2y_0^2 + 2z_0^2 = -2\big(F(x_0,y_0,z_0)\big)$$

$$= 0$$

$\boxtimes$

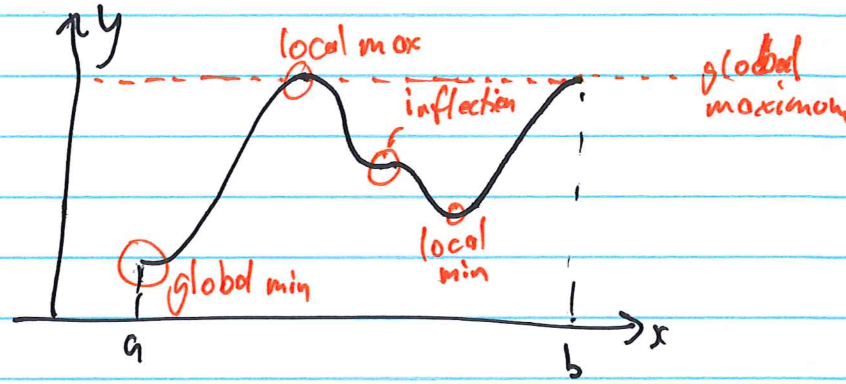


Min / Max of fcns of 2 vars §12.8

Review of 1D calc

$y = f(x)$

$x \in [a, b]$



To find max/min of $f(x)$ on $x \in [a,b]$

① find critical pts where $f'(x) = 0$, classify

② check boundary pts a, b

local max $f'' < 0$

" min $f'' > 0$

either/neither $f'' = 0$

(test fails)

Now consider $f(x, y)$

Def'n f has a local min at (a, b) if there exists some neighbourhood Ω with $f(a, b) \leq f(x, y)$ $\forall (x, y) \in \Omega$

Similarly for max

Thm If f differentiable in a disc around (a, b) and if f has a local min or max at (a, b) then

$$f_x(a, b) = 0 = f_y(a, b).$$

(Proof uses trace curves.)

Def'n (a, b) is a critical point of f if

$$f_x(a, b) = 0 = f_y(a, b)$$

① tangent plane of $z = f(x, y)$ is horizontal at (a, b)

② $\vec{\nabla} f(a, b) = \vec{0}$

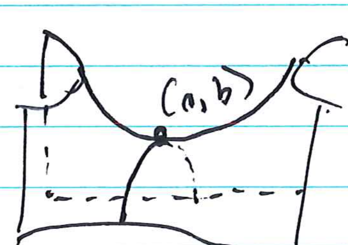
3 common critical points:

eg. $z = x^2 + y^2$

local min

eg $z = -(x^2 + y^2)$

local max



Saddle pt