§12.5 Chain Rule (cont)

Implicit differentiation and the chain rule (pg 717)

Suppose $f(x,y)=0$ defines a ~~curve~~ curve $y(x)$ implicitly. Suppose we want $\frac{dy}{dx}$

$$\frac{\partial}{\partial x} f(x,y) = \frac{\partial}{\partial x} 0 = 0$$

Chain rule 1 (?)

$$\frac{\partial f}{\partial x}\cancel{\frac{dx}{dx}}^{1} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$\boxed{\underbrace{\frac{dy}{dx}}_{y(x)\text{ unknown!}} = -\underbrace{\frac{\partial f/\partial x}{\partial f/\partial y}}_{\text{two "easy" to compute partials}}}$$

Aside: $f(x(t), y(t))$ with $t=x$ see, chain rule 1.

Ex $f(x,y) = x^2y - xy^3 - 3 = 0$

$f_x = 2xy - y^3$, $f_y = x^2 - 3xy^2$

$$\frac{dy}{dx} = -\frac{(2xy-y^3)}{x^2-3xy^2}$$

check: $\frac{\partial}{\partial x}\left[x^2y - xy^3 - 3 = 0\right]$

Solve for $\frac{dy}{dx}$, get same

dep var $\quad$ 3 indep vars. $\quad$ e.g. sphere

Suppose $F(\underbrace{x, y, z}_{\text{3 indep vars.}}) = 0$ implicitly defines a surface $z = f(x,y)$. We want partials of f (without / find f explicitly).

($z=$) f is dep var, with x, y indep.

Find the partial $\frac{\partial f}{\partial x}$ $\left(= \frac{\partial z}{\partial x} \text{ where } \underline{z(x,y)} \right)$

Chain rule 2

$$\frac{\partial}{\partial x} F(x,y,z) = \frac{\partial}{\partial x} 0$$

$$\frac{\partial F}{\partial x}\cancelto{1}{\frac{\partial x}{\partial x}} + \frac{\partial F}{\partial y}\cancelto{0}{\frac{\partial y}{\partial x}} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

nonzero, $\frac{\partial f}{\partial x}$

$$\boxed{\frac{\partial z}{\partial x} = \frac{\partial f}{\partial x} = \frac{-\partial F/\partial x}{\partial F/\partial z}}$$

Similarly $\boxed{\frac{\partial f}{\partial y} = \frac{-\partial F/\partial y}{\partial F/\partial z}}$

<u>Aside</u> : are you worried that $\frac{\partial y}{\partial x} = 0$ but $\frac{\partial z}{\partial x}$ nonzero?

Suppose $F(u, v, w) = 0$ where $u(x,y) = x$
$v(x,y) = y$
$w(x,y) = f(x,y)$

$$\cancel{\frac{\partial F}{\partial u}} \frac{\partial}{\partial x}\left[F(u,v,w) \right] = \frac{\partial}{\partial x} 0$$

$$\frac{\partial F}{\partial u} \cancelto{1}{\frac{\partial u}{\partial x}} + \frac{\partial F}{\partial v} \cancelto{0}{\frac{\partial v}{\partial x}} + \frac{\partial F}{\partial w} \underbrace{\frac{\partial w}{\partial x}}_{\frac{\partial f}{\partial x}} = 0$$

partial of F w.r.t. its 1st argument.

Ex $x^3 - y^4 z + e^{xz^2} = 0$ implicitly defines a surface $z = f(x,y)$. Find the tangent plane of that surface at $(x_0, y_0, z_0) = (0,1,1)$.

A tangent: $z = \cancel{z_0}^{\,1} + f_x(0,1)(x - \cancel{x_0}^{\,0}) + f_y(0,1)(y - \cancel{y_0}^{\,1})$

need these

define $F(x,y,z) := x^3 - y^4 z + e^{xz^2}$

$F_x(x,y,z) = F_x = 3x^2 + z^2 e^{xz^2} \Rightarrow F_x(\underbrace{0,1,1}_{3 \text{ vars}}) = 1$

$F_y = -4y^3 z \Rightarrow F_y(0,1,1) = -4$

$F_z = -y^4 + 2xz\, e^{xz^2} \Rightarrow F_z(0,1,1) = -1$

$f_x(\underbrace{0,1}_{2 \text{ vars}}) = -\dfrac{F_x(0,1,1)}{F_z(0,1,1)} = \dfrac{-1}{-1} = 1$

$f_y(0,1) = -\dfrac{F_y(0,1,1)}{F_z(0,1,1)} = \dfrac{4}{-1} = -4$

tangent plane: $z = 1 + 1\cdot(x-0) - 4\cdot(y-1)$

$$\boxed{z = 1 + x - 4(y-1)}$$