Last day: The $\underline{\text{tangent}}$ $\underline{\text{plane}}$ at $(x,y)=(a,b)$

$$\underset{\text{different surface than } z=f(x,y)}{z} = \underbrace{f(a,b)}_{\#} + \underbrace{f_x(a,b)}_{\#}(x-a) + \underbrace{f_y(a,b)}_{\#}(y-b)$$

§12.4 $\quad$ linear approx $\quad$ (no tangent plane in §12.4 in APEX)

Let:
$$\left.\begin{aligned}\Delta x &= x-a \\ \Delta y &= y-b\end{aligned}\right\} \text{ indep var change}$$
$$\left.\Delta z = f(x,y)-f(a,b)\right\} \text{ dep. var change.}$$

$$\Delta z \approx f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$$

a.k.a. $\quad f(x,y) \approx f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y$

Ex a) find the linear approx to $f(x,y)=\sqrt{x^2+y^2}$ at $(3,4)$

b) use it to approx $\sqrt{(3.1)^2+(3.9)^2}$

$$f(3,4)=\sqrt{3^2+4^2}=5$$

$$f_x = \cancel{\tfrac{1}{2}}(x^2+y^2)^{-1/2}(\cancel{2}x) = x/f(x,y)$$

$$f_x(3,4)=3/5$$

Similarly $f_y(3,4)=4/5$

$$\boxed{f(x,y)\approx 5+\tfrac{3}{5}(x-3)+\tfrac{4}{5}(y-4)}$$

b) $f(3.1, 3.9) \approx 5 + \frac{3}{5}(3.1-3)$ Δx

$+ \frac{4}{5}(3.9-4)$ Δy

$= 5 + \frac{3}{50} - \frac{4}{50} = 4.98$

~~f(3.1,3.9)~~ exact: 4.98197...

§12.4 The total differential

review of 1D: $dy := f'(x)\,dx$ defining

the differential to curve $y = f(x)$

two new independent variables !!

Multivariable case: $dz := f_x(x,y)\,dx + f_y(x,y)\,dy$

total differential.

x, y still variables (!)

Compare to linear approx... Ok to think $dx = \Delta x$, $dy = \Delta y$

$\Delta z \approx dz$ shorthand notation for the linear approx.

If $\quad \Delta z \approx dz$

then $\quad \Delta z = dz + \underbrace{\varepsilon_1(dx, dy)\,dx + \varepsilon_2(dx, dy)\,dy}_{\text{error}}$

(?)

Def'n If I can write Δz in this way and if

$$\lim_{(dx, dy)\to(0,0)} \varepsilon_1(dx, dy) = 0$$

$$\lim_{(dx, dy)\to(0,0)} \varepsilon_2(dx, dy) = 0$$

then we say f is <u>differentiable</u>

Intuitive: <u>the</u> <u>linear</u> approx is the best possible linear approx.

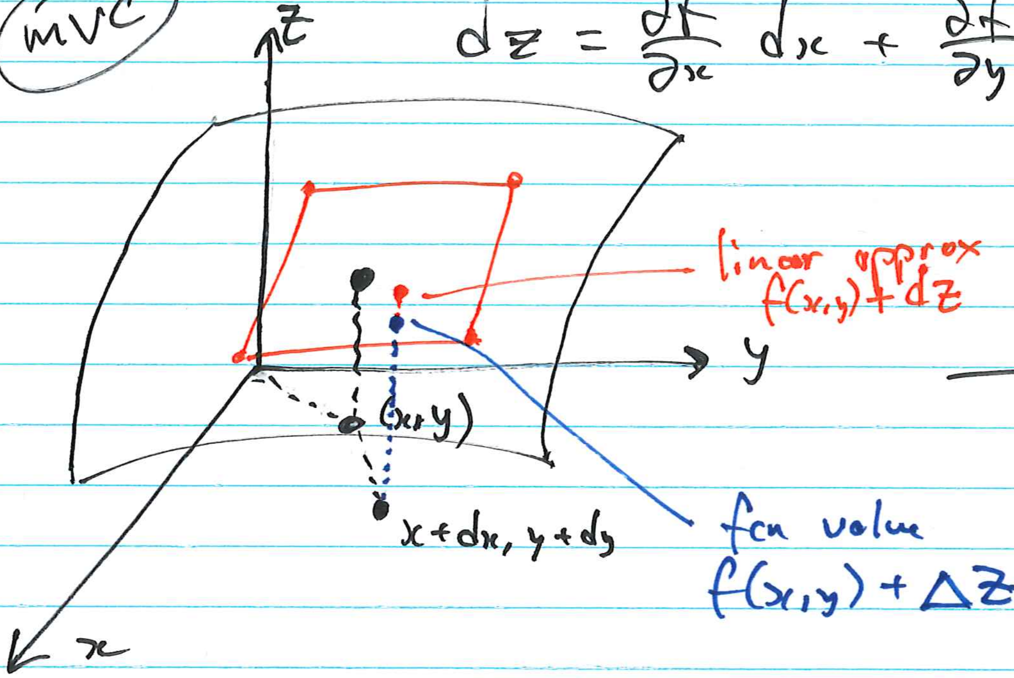
Thm If f_x and f_y exist ~~at~~ near (a,b) and they are continuous at (a,b) then f is <u>differentiable</u> at (a,b)

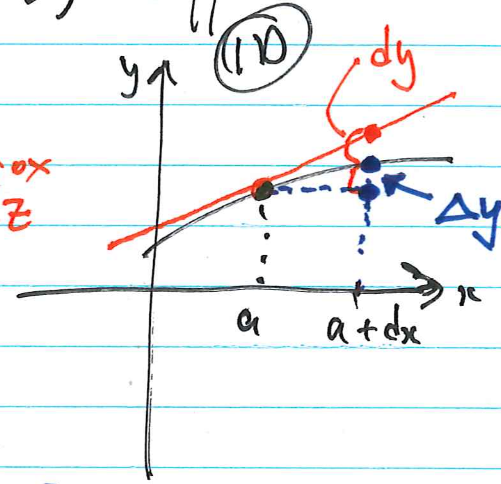
End of theory :)

Return to practical matters :)

MVC

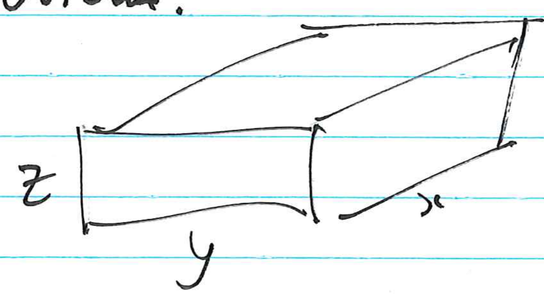
$$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \qquad dz \approx \Delta z$$





Ex Rectangular box 80 × 100 × 50 cm measured within 1mm. Estimate the max error in the volume.

Sol'n $V = xyz$



$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \frac{\partial V}{\partial z}dz$$

(the total differential)

$$= yz\,dx + xz\,dy + xy\,dz$$

tangent hyperplane??, don't want geometry!

$dx = dy = dz = 0.1 \text{ cm}$, $\quad x = 80\text{cm}$
$y = 50 \text{ cm}$
$z = 100\text{cm}$

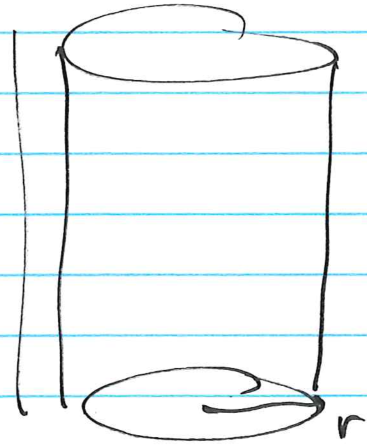
$$dV = \underset{cm}{50}\cdot\underset{cm}{100}\cdot\underset{cm}{0.1} + 80\cdot 50\cdot 0.1 + 80\cdot 100\cdot 0.1$$

$$= 500 + 400 + 800 = 1700\text{cm}^3$$

$$= 1.7L$$

Relative to V: $\dfrac{1700 \text{ cm}^3}{80\cdot 100\cdot 50 \text{ cm}^3} = 0.4\%$

<u>Ex</u> <u>Estimate</u> the amount of metal in a hollow can 10cm high, 4cm in diameter. Suppose top/bottom 0.1cm thick and that the sides are 0.05cm thick



<u>Sol'n:</u> $V = \pi r^2 h$

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh$$

$$= \underbrace{2\pi r h\, dr}_{\text{Surface area of the side}} + \underbrace{\pi r^2 dh}_{\text{surface area of the top. or bottom}}$$