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To the memory of Takuro Shintani

Let G be the group of rational points on a connected reductive group defined over the p-adic field k. Certain irreducible admissible representations σ_P of G, indexed by parabolic subgroups P of G, may be defined which are of cohomological interest. For example, when G is semi-simple and Γ is a co-compact discrete subgroup they play a crucial role in the proof of the vanishing theorem for $H^{\cdot}(\Gamma, \boldsymbol{C})$ (see [4] or Chapter XI of [2]). One of their important properties is that when G is simple, none of them are unitary except for the trivial and the Steinberg representations (indexed by G and minimal parabolics, respectively).

These representations were introduced in [4], but only much more recently, in the book [2] by Borel and Wallach, did proofs first appear. These were simpler in many ways than my original ones, especially where unitarity is concerned, since in the meantime an effective criterion for non-unitarity was found by R. Howe (see [8]). Nonetheless what I propose to do in this paper is exhibit my earlier argument. On the one hand, the technique used in it will apply in situations where Howe's criterion is not applicable and on the other it yields a more precise result. I include also some discussion of irreducibility even though it duplicates [2], since it is only a minor digression. And in an appendix I deal with some matters of cohomology.

These matters are only indirectly related to work of Shintani. However, he was interested in the representation theory of p-adic groups throughout his professional life, and I hope therefore that this paper will serve to honor his memory.

Incidentally, it might be considered the third in a series beginning with [6] and [7], which I draw on for calculations. Thanks are due to A. Borel and D. Wigner for invaluable encouragement and suggestions.

Because I am concerned with illustrating a technique, I will assume for convenience that the reductive group G is a Chevalley group, hence arising by base extension from a group scheme over Spec $(\mathcal{O})$.

I use a $\square$ to conclude proofs, but $\square\square$ to conclude long proofs with intermediate results, with the name of the result thereafter in parentheses.

1. For each parabolic subgroup $P \subseteqq G$, let π_P be the right regular representation of G on $C^\infty(P\backslash G)$, the space of all locally constant functions on $P\backslash G$, values in $\boldsymbol{C}$. Since $P\backslash G$ is compact, this is clearly admissible. When $P \subseteqq Q$ the canonical projection $P\backslash G \to Q\backslash G$ induces an embedding of π_Q into π_P. Define σ_P to be the quotient of π_P by the sum of all the π_Q with Q properly containing P.

1.1. THEOREM: *Each σ_P is irreducible, and $\sigma_P \cong \sigma_Q$ if and only if P and Q are conjugate. The representation π_P possesses a Jordan-Hölder series whose irreducible factors are the σ_Q with $Q \supseteqq P$, each with multiplicity one.*

The Q containing a given P form a finite set, and the π_Q form a filtration of π_P indexed by this finite partially ordered set. Furthermore no two parabolics containing P are conjugate. Of course if P and Q are conjugate then $\pi_P \cong \pi_Q$ and $\sigma_P \cong \sigma_Q$. Hence the substance of the theorem lies in two assertions:

(1.1.1) If σ_P and σ_Q have any composition factors in common, then P and Q are conjugate;

(1.1.2) Each σ_P is irreducible.

In fact we will need later (in the discussion of unitarity) more precise statements which have these as consequence. Unfortunately, in order to introduce them I must also introduce some more notation.

Choose a minimal parabolic subgroup $P_\phi \subseteqq G$, also arising by base extension from Spec$(\mathcal{O})$. Let $P_\phi = A_\phi N_\phi$ be a Levi decomposition. Let

Σ = roots of $\mathfrak{g}$ (Lie algebra of G) with respect to A_ϕ
Σ^+ = positive roots determined by P_ϕ
Δ = basis of Σ^+
W = Weyl group of Σ, also $N_G(A_\phi)/A_\phi$.

For each subset $\Theta \subseteqq \Delta$ let

Σ_Θ = linear span of Θ in Σ
$\Sigma_\Theta^+ = \Sigma_\Theta \cap \Sigma^+$
N_Θ = unipotent subgroup of N_ϕ whose Lie algebra is the sum of eigenspaces $\mathfrak{g}_\alpha$ $(\alpha \in \Sigma^+ - \Sigma_\Theta^+)$
$A_\Theta = \cap \ker(\alpha)$ $(\alpha \in \Theta)$
M_Θ = centralizer of A_Θ
W_Θ = Weyl group of the root system Σ_Θ.

Thus $W_\Theta \subseteqq M_\Theta$ and is generated by the elementary reflections w_α $(\alpha \in \Theta)$. The map $\Theta \longmapsto P_\Theta = M_\Theta N_\Theta$ is a bijection between subsets of Δ and parabolics containing P_ϕ. For each Θ let π_Θ, σ_Θ be what I earlier would have called π_{P_Θ}, σ_{P_Θ}.

Since every parabolic subgroup of G is conjugate to a unique P_Θ, what is to be proven is:

(1.1.3) If σ_Θ and σ_Ω have composition factors in common, then $\Theta=\Omega$;

(1.1.4) Each σ_Θ is irreducible.

For convenience in dealing with subscripts, *from now on throughout this paper let* $P=P_\phi$, $A=A_\phi$, etc. For each $\Theta\subseteqq\Delta$ let δ_Θ be the modulus character of P_Θ:

$$\delta_\Theta(m)=|\det \mathrm{Ad}_{\mathfrak{n}_\Theta}(m)| \qquad (m\in M_\Theta)$$

whose restriction to A is

$$\delta_\Theta(a)=\prod|\alpha(a)| \qquad (\alpha\in\Sigma^+-\Sigma^+_\Theta).$$

I will write δ_ϕ as just δ. Overloading the notation slightly, for $w\in W$ define δ_w to be $(w\delta^{-1/2})\delta^{1/2}$, so that for $a\in A$,

$$\delta_w(a)=\prod|\alpha(a)| \qquad (\alpha>0,\ w^{-1}\alpha<0).$$

For any character χ of M_Θ, the representation $\mathcal{I}nd(\chi|P_\Theta, G)$ of G induced by it is the right regular representation of G on the space of all locally constant $f: G\to \boldsymbol{C}$ such that $f(nmg)=\chi(m)f(g)$ for all $n\in N_\Theta$, $m\in M_\Theta$, $g\in G$. (*Note that there is no $\delta^{1/2}$-factor,* which would complicate my notation quite a bit.) Thus π_Θ is the same as $\mathcal{I}nd(1|P_\Theta, G)$. More generally, for each $t\in\boldsymbol{R}$ set

$$\pi^t_\Theta=\mathcal{I}nd(\delta^{-t}_\Theta|P_\Theta, G),$$

so that $\pi_\Theta=\pi^0_\Theta$.

When $\Theta=\phi$, I will write $\mathcal{I}nd(\chi|P, G)$ as $I(\chi)$.

Recall that the Jacquet module of any smooth G-representation (π, V) with respect to P is the A-module (π_N, V_N), where V_N is the universal N-trivial quotient of V (see [1], where this construction is called *restriction,* or chapter 3 of [5]). The assignment $V\rightsquigarrow V_N$ is an exact functor, so that if π_1 is a composition factor of π_2, $(\pi_1)_N$ is a composition factor of $(\pi_2)_N$. The Jacquet module appears in many different situations, but its most fundamental property is that it controls G-maps from V into principal series. Let $\Lambda: I(\chi)\to\boldsymbol{C}$ be the P-map from $I(\chi)$ to $\boldsymbol{C}$ with the P-structure χ, $f\mapsto f(1)$. Composition with Λ induces a linear map

$$\mathrm{Hom}_G(V, I(\chi)) \longrightarrow \mathrm{Hom}_M(V_N, \boldsymbol{C}(\chi))$$

which is in fact bijective (Frobenius reciprocity).

Recall that for $\Theta\subseteqq\Delta$ the cosets $W_\Theta\backslash W$ possess elements of smallest length. More precisely, w has minimal length in $W_\Theta w$ if and only if $w^{-1}(\Theta)>0$. Let $W(\Theta)$ be the inverses of such distinguished elements. Then $W(\Omega)\subseteqq W(\Theta)$ whenever $\Theta\subseteqq\Omega$. Let $W(\Theta, \Delta-\Theta)$ be the complement in $W(\Theta)$ of the union of the $W(\Omega)$ with Ω strictly larger than Θ: $w\in W(\Theta, \Delta-\Theta)$ when $w(\Theta)>0$, $w(\Delta-\Theta)<0$.

I call a character χ of A *regular* if the characters $(w\chi)\delta_w$ $(w\in W)$ are all distinct. This is equivalent to: the characters $w(\chi\delta^{-1/2})$ are all distinct.

1.2. LEMMA: *Let χ be a character of M_Θ whose restriction to A is regular. If $I=\mathcal{I}nd(\chi|P_\Theta, G)$ then I_N is isomorphic to*

$$\oplus(w\chi)\delta_w \qquad (w\in W(\Theta)).$$

PROOF: This sort of thing is well known, so I will just recall the proof sketchily. Start with the double coset decomposition $G=\bigcup P_\Theta w^{-1}P$ $(w\in W(\Theta))$. Filter the space I by P-stable subspaces accordingly: f lies in I_w $(w\in W(\Theta))$ if

$$\mathrm{Supp}(f)\subseteqq \bigcup P_\Theta x^{-1}P \qquad (x\geqq w).$$

Here I use the ordinary ordering on $W: y\leqq x$ if and only if PyP is contained in the closure of PxP. For $x, y\in W(\Theta)$ this means also that $P_\Theta y^{-1}P$ is contained in the closure of $P_\Theta x^{-1}P$. The filtration is thus *decreasing*: $I_x\subseteqq I_y$ when $x\geqq y$. The map

$$f\longmapsto \int_{N\cap wP_\Theta w^{-1}\backslash N} f(w^{-1}n)dn$$

(having chosen Haar measures) identifies the Jacquet module of $I_w/\sum I_x$ $(x\geqq w,\ x\neq w)$ with $(w\chi)\delta_w$. This gives by exactness a filtration of I_N also indexed by $W(\Theta)$, which yields by the regularity assumption a direct sum decomposition. □

1.3. COROLLARY: *The Jacquet module of σ_Θ is isomorphic to $\oplus\delta_w$ $(w\in W(\Theta, \Delta-\Theta))$.*

PROOF: Apply 1.2 to the representations π_Ω with $\Theta\subseteqq\Omega$. □

1.4. PROPOSITION: *If σ_Θ and σ_Ω have any composition factors in common, then $\Theta=\Omega$.*

PROOF: The fundamental theorem of principal series (Chapter 6 of [5]) asserts that if π is any non-zero composition factor of $I(\chi)$, then $\pi_N\neq 0$. The Proposition follows from 1.3, which implies that if the Jacquet modules of σ_Θ and σ_Ω have any factors in common, then $\Theta=\Omega$. □

If χ is a character of M_Θ, then upon restriction it becomes a character of A. Therefore there exists a canonical embedding τ of $\mathcal{I}nd(\chi|P_\Theta, G)$ into $I(\chi)$. If χ is regular, then as a special case of 1.2 (with $\Theta=\phi$) one has $I(\chi)_N\cong\oplus(w\chi)\delta_w$ $(w\in W)$, and by Frobenius reciprocity corresponding to the projections onto the summands one has G-intertwining operators

$$T_w: I(\chi)\longrightarrow I((w\chi)\delta_w).$$

This map is determined by the condition that if the support of f lies in

$\bigcup Px^{-1}P$ $(x \geqq w)$ then

$$T_w f(1) = \int_{N \cap wNw^{-1} \backslash N} f(w^{-1}n)dn\,. \tag{1.4}$$

On the other hand, one also has the intertwining operator $\tau_w : \mathcal{I}nd\,(\chi | P_\Theta, G) \to I((w\chi)\delta_w)$, specified this time by the condition

$$\tau_w f(1) = \int_{N \cap wP_\Theta w^{-1} \backslash N} f(w^{-1}n)dn$$

for suitable f. But because $w \in W(\Theta)$, $N \cap wP_\Theta w^{-1} = N \cap wNw^{-1}$, so that this diagram commutes:

$$\begin{array}{ccc} & \mathcal{I}nd\,(\chi | P_\Theta, G) & \\ {}^{\tau=\tau_1}\swarrow & & \searrow^{\tau_w} \\ I(\chi) & \longrightarrow & I((w\chi)\delta_w) \end{array}$$

Now suppose in particular that $w = w_\Theta = w_\iota w_{\iota,\Theta}$, where w_ι is the largest element in W and $w_{\iota,\Theta}$ is that in W_Θ. Then w_Θ lies in $W(\Theta)$, and indeed $w_\Theta(\Theta) = w_\iota(-\Theta) = \bar{\Theta}$, the *conjugate* of Θ in Δ. The set $\{\alpha > 0 | w\alpha < 0\}$ is in this case $\Sigma^+ - \Sigma^+_\Theta$, while $\{\alpha > 0 | w^{-1}\alpha < 0\}$ is $\Sigma^+ - \Sigma^+_{\bar{\Theta}}$. Therefore $\delta_{w_\Theta} = \delta_{\bar{\Theta}}$ and $N \cap w_\Theta N w_\Theta^{-1} \backslash N \cong N_{\bar{\Theta}}$. The map

$$f \longmapsto \int_{N \cap w_\Theta N w_\Theta^{-1} \backslash N} f(w_\Theta^{-1}n)dn$$

giving rise to τ_w is the same as

$$f \longmapsto \int_{N_{\bar{\Theta}}} f(w_\Theta^{-1}n)dn\,,$$

which gives rise to a G-map T^χ_Θ fitting into this commutative diagram:

$$\begin{array}{ccc} \mathcal{I}nd\,(\chi | P_\Theta, G) & \xrightarrow{T^\chi_\Theta} & \mathcal{I}nd\,((w_\Theta\chi)\delta_{\bar{\Theta}} | P_{\bar{\Theta}}, G) \\ \downarrow & & \downarrow \\ I(\chi) & \xrightarrow{T_{w_\Theta}} & I((w_\Theta\chi)\delta_{\bar{\Theta}}) \end{array}$$

where both vertical maps are canonical embeddings. When $\chi = \delta_{\bar{\Theta}}^{-t}$, I will write T^χ_Θ as $T^t_\Theta : \pi^t_\Theta \to \pi^{-t-1}_{\bar{\Theta}}$.

1.5. PROPOSITION: *When* $t \geqq 0$ *or* $t \leqq -1$, $\delta_{\bar{\Theta}}^{-t}$ *is a regular character of* A. *Furthermore*:

(a) *when* $t > 0$, T^t_Θ *is an isomorphism*;
(b) *when* $t = 0$, $T^0_\Theta : \pi_\Theta \to \pi^{-1}_{\bar{\Theta}}$ *induces an isomorphism of* σ_Θ *with the image.*

For the proof we need two preliminary basic results.

If χ is for the moment any unramified character of A then $I(\chi)$, as is well known, possesses a one-dimensional subspace of elements fixed by the compact open subgroup $K=G(\mathcal{O})$. More precisely, because of the Iwasawa decomposition $G=PK$ there exists a unique $\varphi_K=\varphi_{K,\chi}$ in $I(\chi)$ with $\varphi_K(1)=1$, φ_K fixed by K. If χ is regular then $T_w: I(\chi)\to I((w\chi)\delta_w)$ is defined, and it must satisfy

(1.6.1) $$T_w(\varphi_K)=c_w(\chi)\varphi_K$$

for some scalar $c_w(\chi)$. Of course T_w is only strictly defined after some choice of Haar measure for N and certain subgroups, but for some suitable choice this scalar has been calculated (see for example [6]):

(1.6.2) $$c_w(\chi)=\prod c_\alpha(\chi) \qquad (\alpha>0,\ w\alpha<0)$$

where

(1.6.3) $$c_\alpha(\chi)=(1-q^{-1}\chi\delta^{-1/2}(a_\alpha))(1-\chi\delta^{-1/2}(a_\alpha))^{-1}.$$

In this formula a_α is the image of a generator of $\mathfrak{p}$ under the multiplicative co-root $\alpha_*: k^\times\to G$. Recall that α_* is characterized by the condition $\gamma(\alpha_*(x))=x^{n(\gamma,\alpha)}$ (in the notation of p. 145 in [3]). In transferring the formula from [6], one must keep in mind that the induced representations there are normalized by a $\delta^{1/2}$-factor.

1.6. LEMMA: *Suppose χ is any regular character of A. Then $T_w: I(\chi)\to I((w\chi)\delta_w)$ is an isomorphism if and only if $c_w(\chi)c_{w^{-1}}((w\chi)\delta_w)\neq 0$.*

PROOF: Consider the composition of

$$T_w: I(\chi)\longrightarrow I((w\chi)\delta_w)$$

with

$$T_{w^{-1}}: I((w\chi)\delta_w)\longrightarrow I(\chi).$$

Because χ is regular, $I(\chi)$ possesses only the scalar multiplications as G-endomorphisms, so this composition must be a scalar. However, its effect on φ_K is multiplication by $c_w(\chi)c_{w^{-1}}((w\chi)\delta_w)=$(say) C. If $C\neq 0$, the $T_{w^{-1}}$ and T_w are essentially inverse to each other, since the same reasoning applies to $T_w\circ T_{w^{-1}}$ as well. If $C=0$ and T_w is an isomorphism, then $T_{w^{-1}}$ is identically 0, a contradiction to the defining equation (1.4). □

1.7. LEMMA: (a) *For any $\Theta\subseteq\Delta$,*

$$\delta_\Theta(a_\alpha)\begin{cases}<1 & \alpha\in\Sigma^+-\Sigma^+_\Theta\\ =1 & \alpha\in\Sigma^+_\Theta.\end{cases}$$

(b) *For $\Theta=\phi$, $\delta(a_\alpha)\leqq q^{-2}$ for all $\alpha>0$ and $\delta(a_\alpha)=q^{-2}$ if and only if $\alpha\in\Delta$.*

PROOF: Part (b) is a translation of a standard calculation (p. 168 of [3]).

For part (a): the character δ_Θ is the same as $\delta_{w_{\bar\Theta}^{-1}}$. The inequality of (a) becomes

$$\delta(a_\alpha)<(w_{\Theta}^{-1}\delta)(a_\alpha)=\delta(a_{w_\Theta\alpha}) .$$

But as a consequence of (b), $\delta(a_\gamma)<1$ for $\gamma>0$, $\delta(a_\gamma)>1$ for $\gamma<0$. Apply this to $\gamma=\alpha$, $\gamma=w_\Theta\alpha$.

If $\alpha\in\bar\Theta$, then $w_\Theta\alpha\in\bar\Theta\subseteqq\Delta$ so by (b)

$$\delta(a_\alpha)=q^{-2}=(w_\Theta^{-1}\delta)(a_\alpha)$$

and hence $\delta_\Theta(a_\alpha)=1$ in this case. But for any $\gamma\in\Sigma_\Theta^+$, a_γ is a monomial in the a_α $(\alpha\in\Theta)$, so that $\delta_\Theta(a_\gamma)=1$ as well. □

Now to begin the proof of 1.5. From 1.7 one deduces that for all $t\geqq0$ and $\alpha>0$,

$$\delta_\Theta^{-t}(a_\alpha)\delta^{-1/2}(a_\alpha)\geqq\delta^{-1/2}(a_\alpha)>1 .$$

If χ is any character of A, it is regular if and only if $w(\chi\delta^{-1/2})=\chi\delta^{-1/2}$ only for $w=1$. But if $\chi\delta^{-1/2}(a_\alpha)>1$ for all $\alpha>0$, this is clearly all right. Therefore δ_Θ^{-t} is regular for $t\geqq0$. Since its transform under w_Θ is $(w_\Theta\delta_\Theta^{-t})\delta_{w_\Theta}=\delta_{\bar\Theta}^{t+1}$, this is true also for δ_Θ^{-t} with $t\leqq-1$.

In particular, T_Θ^t is always defined for $t\geqq0$ or $t\leqq-1$, and in fact we have this commutative diagram:

(1.7)
$$\begin{array}{ccc} \pi_\Theta^t & \xrightarrow{\;T_\Theta^t\;} & \pi_{\bar\Theta}^{-1-t} \\ \downarrow & & \downarrow \\ I(\delta_\Theta^{-t}) & \xrightarrow{\;T_{w_\Theta}\;} & I(\delta_{\bar\Theta}^{t+1}) . \end{array}$$

(a special case of 1.5.2) for all such t. It can be easily seen that in order to prove that T_Θ^t is an isomorphism, it suffices to prove that T_{w_Θ} is, since one also has the diagram

$$\begin{array}{ccc} \pi_{\bar\Theta}^{-1-t} & \xrightarrow{\;T_{\bar\Theta}^{-1-t}\;} & \pi_\Theta^t \\ \downarrow & & \downarrow \\ I(\delta_{\bar\Theta}^{t+1}) & \xrightarrow{\;T_{w_{\bar\Theta}}\;} & I(\delta_\Theta^{-t}) . \end{array}$$

Therefore by 1.6 one has to check that if $\chi=\delta_\Theta^{-t}$ with $t>0$ and $w=w_\Theta$, that

$c_w(\chi)c_{w^{-1}}((w\chi)\delta_w)\neq 0$, or equivalently (again interchanging t and $-t-1$, Θ and $\bar{\Theta}$) that for $t>0$ or $t<-1$, $c_w(\chi)\neq 0$. According to (1.6.2) and (1.6.3), this is equivalent to $\chi\delta^{-1/2}(a_\alpha)\neq q$ for every $\alpha>0$ with $w_\Theta\alpha<0$; in other words for $\alpha\in\Sigma^+ -\Sigma^+_\Theta$. But from 1.7 it follows that for such α and $\chi=\delta_\Theta^{-t}$, $t>0$, $\chi(a_\alpha)>1$ while $\delta^{1/2}(a_\alpha)q\leqq 1$. As for $\chi=\delta_\Theta^{-t}$ $(t<-1)$,

$$\chi\delta^{-1/2}(a_\alpha)<\delta_\Theta\delta^{-1/2}(a_\alpha)=w_\Theta^{-1}\delta^{-1/2}(a_\alpha)=\delta^{-1/2}(a_{w_\Theta(\alpha)})\leqq q^{-1}$$

since $\delta_\Theta=(w_\Theta^{-1}\delta^{-1/2})\delta^{1/2}$. This concludes the proof of 1.5 (a).

Now for 1.5 (b). The map which by Frobenius reciprocity gives rise to T^0_Θ is the projection onto the factor $\delta_{\bar{\Theta}}=\delta_{w_\Theta}$ of the Jacquet module of π_Θ. If Ω strictly contains Θ, then by 1.2 (with $\chi=1$) this projection is null on π_Ω, which therefore lies in the kernel of T^0_Θ. Hence T^0_Θ factors through the canonical projection $\pi_\Theta\rightarrow\sigma_\Theta$. It remains to be shown that the induced map from σ_Θ into $I(\delta_{\bar{\Theta}})$ is an injection. By the fundamental theorem on the principal series, it suffices to show that the Jacquet module of the kernel is null, or indeed that the Jacquet module of the image of σ_Θ under T^0_Θ contains each δ_w ($w\in W(\Theta, \Delta-\Theta)$) as a quotient. For this, by Frobenius reciprocity, it will suffice to show that all the $I(\delta_w)$ with $w\in W(\Theta, \Delta-\Theta)$ are isomorphic to $I(\delta_{w_\Theta})$. By 1.6 this amounts to proving (1) for $w\in W(\Theta, \Delta-\Theta)$ and $x=ww_\Theta^{-1}$, $c_x(\delta_{\bar{\Theta}})\neq 0$; (2) for $w\in W(\Theta, \Delta-\Theta)$ and $x=w_\Theta w^{-1}$, $c_x(\delta_w)\neq 0$. By (1.6.2), (1.6.3) these amount to (1′) for $\alpha>0$ with $ww_\Theta^{-1}\alpha<0$, $\delta_{\bar{\Theta}}(a_\alpha)\neq\delta^{1/2}(a_\alpha)q$; (2′) for $\alpha>0$ with $w_\Theta w^{-1}\alpha<0$, $\delta_w(a_\alpha)\neq\delta^{1/2}(a_\alpha)q$.

Since $\delta_{\bar{\Theta}}=(w_\Theta\delta^{-1/2})\delta^{1/2}$, the first comes to

$$\delta^{-1/2}(a_{w_\Theta^{-1}\alpha})\neq q \tag{1.8.1}$$

for $\alpha>0$ with $ww_\Theta^{-1}\alpha<0$. But by 1.7 (b) if this does not hold then $w_\Theta^{-1}\alpha\in\Delta$. Since $w_\Theta^{-1}(\bar{\Theta})=\Theta$, while $w_\Theta^{-1}(\Sigma^+-\Sigma^+_{\bar{\Theta}})<0$, $w_\Theta^{-1}\alpha\in\Delta$ and $\alpha>0$ only when $\alpha\in\bar{\Theta}$. But then $w_\Theta^{-1}\alpha\in\Theta$ and $ww_\Theta^{-1}\alpha>0$. Hence (1.8.1) is always true.

The second similarly amounts to

$$\delta^{-1/2}(a_{w^{-1}\alpha})\neq q \tag{1.8.2}$$

when $\alpha>0$, $w_\Theta w^{-1}\alpha<0$. Again, if (1.8.2) does not hold them $w^{-1}\alpha\in\Delta$. But if $w_\Theta w^{-1}\alpha<0$ and $w^{-1}\alpha\in\Delta$, then $w^{-1}\alpha\in\Delta-\Theta$ and $\alpha>0$ lies in $w(\Delta-\Theta)$, a contradiction to the condition $w(\Delta-\Theta)<0$. □□ (Proposition 1.5)

The same argument gives also the proof of the irreducibility of σ_Θ as well as:

1.8. PROPOSITION: *For $t>0$, π^t_Θ is irreducible.*

As I have remarked in the introduction, this proof is taken from that in [2]; the basic idea is ascribed there to Silberger. It is also used very elegantly by Rodier in [9]. □□ (Theorem 1.1)

2. We now come to the main point.

2.1. THEOREM: *Assume G to be simple. Then σ_Θ is unitary if and only if $\Theta=\phi$ or Δ.*

The proof will be long. At least one part is easy: if $\Theta=\Delta$ then σ_Θ is the trivial representation while if $\Theta=\phi$ then σ_Θ is the Steinberg representation. In the second case its Jacquet module is δ_ϕ, so that by a standard criterion it is square-integrable (see [5], for example). In both cases σ_Θ is unitary.

The proof of 2.1 in [2] relies essentially on Jacquet modules (see pp. 340–343). If $\Theta\neq\phi$ then $\Delta-\Theta\neq\Delta$ and $A_{\Delta-\Theta}$ is not the trivial subgroup of A. The set $W(\Theta, \Delta-\Theta)$ contains $w=w_{l,\Delta-\Theta}$, which commutes with $A_{\Delta-\Theta}$, so that δ_w is trivial on $A_{\Delta-\Theta}$. Because of the relationship between Jacquet modules and the asymptotic behaviour of matrix coefficients this implies that the matrix coefficients of σ_Θ remain constant, asymptotically, in the direction of $A_{\Delta-\Theta}$, so that in particular they do not vanish at infinity. But then by Howe's criterion σ_Θ cannot be unitary unless it is the trivial representation of G.

The same relationship also implies that if V is any admissible representation of G and $\tilde{V}$ its contragredient then the Jacquet module of V with respect to P and that of $\tilde{V}$ with respect to the opposite P^- are canonically contragredient. Since P^- is conjugate to P by means of w_l, this implies that if V is isomorphic to $\tilde{V}$ then V_N must be isomorphic to the contragredient of $w_l V_N w_l^{-1}$. If this is applied to $V=\sigma_\Theta$, using Corollary 1.3, and considering that σ_Θ is defined over $\boldsymbol{R}$, one sees that *σ_Θ cannot be isomorphic to its conjugate contragredient, much less be unitary, unless $\Theta=\bar{\Theta}$. We will assume this from now on.* Now the contragredient of π_Θ^t is isomorphic always to π_Θ^{-t-1}; the pairing is given by

$$\langle f_1, f_2\rangle=\int_{P_\Theta\backslash G} f_1(x)f_2(x)dx$$

where since $f_1\in\mathcal{I}nd(\delta_\Theta^{-t}|P_\Theta, G)$ and $f_2\in\mathcal{I}nd(\delta_\Theta^{t+1}|P_\Theta, G)$ the product lies in $\mathcal{I}nd(\delta_\Theta|P_\Theta, G)$, so the integral makes sense. By Theorem 1.5, then, T_Θ^t for $t>0$ induces an isomorphism of π_Θ^t with its contragredient and T_Θ^0 induces the isomorphism of σ_Θ with its contragredient (since now $\bar{\Theta}=\Theta$): in other words, the formula

$$(f_1, f_2)=\langle T^t f_1, f_2\rangle \tag{2.1}$$

gives the G-invariant pairings of π_Θ^t $(t>0)$ or σ_Θ $(t=0)$ with themselves.

The pairing defined by (2.1) is of course not Hermitian, but that is not important. *From now on we will assume all representations and maps to be real,* which is certainly possible as one may verify by reviewing all the definitions.

2.2. LEMMA: *The pairing defined by Equation* (2.1) *is symmetric.*

In other words: we have the map

$$T=T^t: \pi_\Theta^t \longrightarrow \pi_\Theta^{-t-1} \cong \tilde{\pi}_\Theta^t$$

hence also its contragredient $\tilde{T}$, also from π_Θ^t to π_Θ^{-t-1}. The claim is that these are the same. Because of the regularity of δ_Θ^{-t}, $\tilde{T}$ must at least be a scalar multiple of T. But simply because $T(\varphi_K)$ is a constant times φ_K, this multiple is just T itself. □

At this point we must introduce more notation. Recall that G and the minimal parabolic subgroup P arise by base extension from schemes over Spec($\mathcal{O}$). Subsets $\Theta \subseteqq \Delta$ parametrize also subschemes of $G(\mathcal{O})$ containing $P(\mathcal{O})$. Let $K=K_\Delta$ be the maximal compact subgroup $G(\mathcal{O})$ of $\mathcal{O}$-valued points on G, and for each $\Theta \subseteqq \Delta$ define

$$K_\Theta = \text{inverse image in } K \text{ of } P_\Theta(\mathcal{O}/\mathfrak{p}).$$

In particular, K is an Iwahori subgroup of G. One has the decompositions

$$K = \bigcup K_\Theta w^{-1} K_\phi \qquad (w \in W(\Theta))$$
$$G = \bigcup P_\Theta w^{-1} K_\phi \qquad (w \in W(\Theta)).$$

Consequently if $V=\mathcal{I}nd\,(\chi | P_\Theta, G)$, where χ is any character of M_Θ whose restriction to A is unramified, then restriction of functions to K gives an isomorphism of V^{K_ϕ} with $\boldsymbol{R}[K_\Theta \backslash K/K_\phi]$ ($\boldsymbol{R}$-valued functions on the finite double coset space), which from now on I will call $\mathcal{V}_\Theta$ (this is because $G=PK$). Since $K_\Theta \subseteqq K_\Omega$ whenever $\Theta \subseteqq \Omega$, there are canonical embedding $\mathcal{V}_\Theta \subseteqq \mathcal{V}_\Omega$. There is a natural choice of a positive definite inner product on $\mathcal{V}_\Theta$ also, namely integration over K. This assigns to the characteristic function of the double coset $K_\Theta x^{-1} K_\phi$ the length (meas $(K_\Theta x^{-1} K_\phi))^{1/2} = (q^{l(x)} \sum q^{l(y)})^{1/2}$ $(y \in W_\Theta)$ (here I assume meas K_ϕ to be 1), and in fact these characteristic functions form an orthogonal basis with respect to the inner product $f_1 \cdot f_2$. The subspace of σ_Θ of vectors fixed by K_ϕ can be identified with $\mathcal{V}_\Theta / \sum \mathcal{V}_\Omega$ $(\Theta \subseteqq \Omega,\ \Theta \neq \Omega)$, or with the orthogonal complement in $\mathcal{U}_\Theta$ of the sum $\sum \mathcal{V}_\Omega$. Since T_Θ^0 annihilates the π_Ω with $\Theta \subseteqq \Omega$, $\Theta \neq \Omega$, the sum $\sum \mathcal{V}_\Omega$ lies in the radical of the inner product on $\mathcal{V}_\Theta$ induced by it. In fact it is exactly the radical, by Proposition 1.5. In addition, 1.5 implies that T_Θ^t has radical 0 for $t>0$.

From now on by T_Θ^t *I will always mean its restriction to the subspace of* K_ϕ*-fixed vectors in* π_Θ^t, *which I will identify with* $\mathcal{V}_\Theta$.

Integrating over $P_\Theta \backslash G$ is the same as integrating over K (after suitable normalization). The spaces π_Θ^t for all t may be identified with each other as K-spaces, so the map T_Θ^t may be considered as one from $\mathcal{V}_\Theta$ to itself. The pairings $(f_1, f_2) = \langle T_\Theta^t f_1, f_2 \rangle$ and $T_\Theta^t f_1 \cdot f_2$ are, therefore, up to a positive scalar, the same (this for all t). Hence T_Θ^t is also a symmetric operator with respect to the inner product $f_1 \cdot f_2$, and in particular $\mathcal{V}_\Theta$ is stable under T_Θ^0, which is invertible on it by remarks I just made.

The following result is, as we will see in a moment, a refinement of Theorem 2.1:

2.3. THEOREM: *The signature of T_Θ^0 on $\mathcal{V}_\Theta$ is the same as that of $L(w_\iota)$* (left multiplication by w_ι) *on* $\boldsymbol{R}[W_\Theta \backslash W]/\sum \boldsymbol{R}[W_\Omega \backslash W]$ $(\Theta \subseteqq \Omega,\ \Theta \neq \Omega)$.

This makes sense since $w_\iota W_\Omega w_\iota^{-1} = W_{\bar{\Omega}}$ and $\Theta = \bar{\Theta}$.

Before I begin the proof of Theorem 2.3, let me first show how it implies Theorem 2.1. Incidentally, by the signature of T_Θ^0 I mean of course the dimension of its $\pm$ eigenspaces, and similarly for $L(w_\iota)$.

Since $W_\Omega \cap W_\Xi = W_{\Omega \cap \Xi}$, the space $\boldsymbol{R}[W_\Theta \backslash W]/\sum \boldsymbol{R}[W_\Omega \backslash W]$ is, in the Grothendieck group of $\{1, w_\iota\}$, just

$$(-1)^{|\Theta|} \sum_{\Theta \subseteqq \Omega} (-1)^{|\Omega|} \boldsymbol{R}[W_\Omega \backslash W]\,.$$

(Of course $L(w_\iota)$ interchanges $\boldsymbol{R}[W_\Omega \backslash W]$ and $\boldsymbol{R}[W_{\bar{\Omega}} \backslash W]$, so that this does not make strict good sense, but the meaning should be clear.)

2.4. LEMMA: *If $\Omega = \bar{\Omega} \neq \Delta$ then $L(w_\iota)$ has no fixed points in $W_\Omega \backslash W$.*

PROOF: For $w \in W$, $w_\iota w \in W_\Omega w$ if and only if $w_\iota \in W_\Omega$, or if and only if $\Omega = \Delta$. □

Therefore $L(w_\iota)$ acting on each $\boldsymbol{R}[W_\Omega \backslash W]$ has ± 1 with equal multiplicity unless $\Omega = \Delta$, assuming $\Omega = \bar{\Omega}$. When $\Omega \neq \bar{\Omega}$ the same is clearly true of $\boldsymbol{R}[W_\Omega \backslash W] + \boldsymbol{R}[W_{\bar{\Omega}} \backslash W]$. Thus the signature of $L(w_\iota)$ on $\boldsymbol{R}[W_\Theta \backslash W]/\sum \boldsymbol{R}[W_\Omega \backslash W]$ is either $(m, m+1)$ or $(m+1, m)$ if $2m+1 =$ the dimension of the space. In particular, the signature cannot be definite unless $m=0$, and thus Theorem 2.3 implies Theorem 2.1.

The basic idea of the proof of Theorem 2.3 is to find a path of quadratic forms, all non-degenerate on the space $\mathcal{V}_\Theta$, from T_Θ^0 to $L(w_\iota)$ (making the identification $W_\Omega w \leftrightarrow K_\Omega w K_\phi$ between the spaces on which the two operators act). This does not seem to be possible, at least in a natural way, but a modification of this idea does work, as we will see.

To begin I will introduce two new operators which I call E and S acting on the space $\mathcal{V}_\phi$ (which contains all $\mathcal{V}_\Theta$). *The operator E takes $K_\phi w K_\phi$ to $K_\phi \bar{w} K_\phi$*, where $w \mapsto \bar{w}$ is the involution $w \mapsto w_\iota w w_\iota^{-1}$. Since $w_\iota(-\alpha) \in \Delta$ whenever $\alpha \in \Delta$, this permutes the elementary reflections in W. The map E takes each $\mathcal{V}_\Omega$ to $\mathcal{V}_{\bar{\Omega}}$, hence $\mathcal{V}_\Theta$ to itself.

To define S, I recall that $\mathcal{V}_\phi$ possesses the structure of an algebra with convolution as the product: it is the *Hecke algebra* $\mathcal{H}(K, K_\phi)$ of K with respect to K_ϕ. It may also be defined in terms of generators and relations: let $C(w)=K_\phi w K_\phi$ for every $w\in W$. Then of course these form a basis of $\mathcal{H}(K, K_\phi)$ over $\boldsymbol{C}$. The relations are:

$$\prod C(w_i)=C(\prod w_i) \tag{2.2.1}$$

whenever the length of $\prod w_i$ is $\sum l(w_i)$;

$$C(w_1)C(w_2)C(w_1)\cdots=C(w_2)C(w_1)C(w_2)\cdots \tag{2.2.2}$$

where each side possesses r terms, if r is the order of w_1w_2 in W, and w_1, w_2 are elementary reflections in W;

$$(C(w)+1)(C(w)-q)=0 \tag{2.2.3}$$

if w is an elementary reflection.

The operator S is multiplication on the right by $C(w_l)=K_\phi w_l K_\phi$. Since it commutes with multiplication on the left, it takes each $\mathcal{V}_\Omega$ to itself.

2.4. PROPOSITION: (a) *The involution E is an automorphism of the algebra $\mathcal{H}(K, K_\phi)$. It is symmetric with respect to the inner product $f_1\cdot f_2$.* (b) *The operator S is also symmetric with respect to the inner product. Both S and E are invertible.*

PROOF: Only the symmetricity and the invertibility of S require comment. The second follows from relations (2.2.1) and (2.2.3): (2.2.1) says S is the product of right multiplication by the $C(w_i)$ if $w_l=w_1\cdots w_n$ is a reduced expression for w_l as a product of elementary reflections. Equation (2.2.3) implies that each $C(w_i)$ is invertible.

The symmetricity of S is a consequence of the more general proposition that in any smooth unitary representation of a p-adic group G the adjoint of the function $f\in C(G)$ is $f(x^{-1})$. This is because the metric on $\mathcal{V}_\phi$ I have defined is the restriction of a K-invariant metric, and because $w_l^{-1}=w_l$. □

2.5. PROPOSITION: *For any $t\geqq 0$, the operators T^t_Θ, E, and S all commute.*

PROOF: For T^t_Θ and S this is immediate, since T^t_Θ is the restriction of a G-map, and the identification of $\mathcal{I}nd\,(\chi|P_\Theta, G)^{K_\phi}$ with $\mathcal{V}_\Theta$ is a K-isomorphism.

The case of S and E is also clear since $\overline{w}_l=w_l$ and E is an automorphism of algebras.

Only the case of E and the T^t_Θ is not simple. The proof I give seems to me a little too technical to be pleasant, but I have found no other so short. First we need (now for this computational proof, but also later on for a more serious reason):

2.6. LEMMA: *Let χ be any unramified character of A, $\alpha\in\Delta$. Then the intertwining operator T_{w_α}: $I(\chi)\rightarrow I((w_\alpha\chi)\delta_{w_\alpha})$ amounts to left multiplication in $\mathcal{H}(K, K_\phi)$ by*

$$(c_\alpha(\chi)-1)+q^{-1}C(w_\alpha)$$

on the subspace $\mathcal{V}_\phi$ of vectors fixed by K_ϕ.

This is essentially Theorem 3.4 of [6]. Incidentally, since T_{w_α} is an $\mathcal{H}(K, K_\phi)$-morphism ($\mathcal{H}$ acting on the right on $\mathcal{V}_\phi$), it amounts to left multiplication by something in $\mathcal{H}(K, K_\phi)$ and in order to calculate by what one must see what effect it has, say, on K_ϕ. □

Now express w_Θ (which is, recall, $w_l w_{l,\Theta}$) as a product of elementary reflections

$$w_\Theta=w_{\alpha_n}\cdots w_{\alpha_1} \tag{2.6.1}$$

where n is the length of w_Θ. The operator T_Θ^t is the restriction of $T_{w_\Theta}: I(\chi)\rightarrow I((w_\Theta\chi)\delta_\Theta)$ to the subspace π_Θ^t, where $\chi=\delta_\Theta^{-t}$, so one may apply the Lemma to calculate it. Because of the way the T_w's compose, $T_{w_\Theta}=T_{\alpha_n}\cdot T_{\alpha_{n-1}}\cdot\cdots\cdot T_{\alpha_1}$ (where I have abbreviated the subscripts, replacing w_{α_i} by α_i).

To reduce congestion in notation, for the moment define

$$\varepsilon_\alpha(\tau)=(c_\alpha(\tau)-1)+q^{-1}C(w_\alpha)$$

for every $\alpha\in\Delta$ and τ an unramified character of A. Thus, according to Lemma 2.6 and the factorization above, T_{w_Θ} is left multiplication by the product

$$\varepsilon_{\alpha_n}((w_{n-1}\chi)\delta_{w_{n-1}})\cdots\varepsilon_{\alpha_1}(\chi)\,. \tag{2.6.2}$$

Therefore for every $\varphi\in\mathcal{V}_\Theta$, $E(T_\Theta^t(\varphi))$ is the product

$$E(\varepsilon_{\alpha_n}((w_{n-1}\chi)\delta_{w_{n-1}}))\cdots E(\varepsilon_{\alpha_1}(\chi))$$

applied to $E(\varphi)$, since E is an algebra automorphism. Explicitly,

$$E(\varepsilon_\alpha(\tau))=(c_\alpha(\tau)-1)+q^{-1}C(w_{\bar\alpha})\,.$$

Since $\bar\Theta=\Theta$,

$$\overline{w_\Theta}=w_\Theta=w_{\bar\alpha_n}\cdots w_{\bar\alpha_1}$$

so that T_Θ^t is also left multiplication by the product

$$\varepsilon_{\bar\alpha_n}((\overline{w_{n-1}}\chi)\delta_{\overline{w_{n-1}}})\cdots\varepsilon_{\bar\alpha_1}(\chi)\,.$$

Hence $E\circ T_\Theta^t=T_\Theta^t\circ E$ if

$$c_{\overline{\alpha_i}}((\overline{w_{i-1}}\chi)\delta_{\overline{w_{i-1}}})=c_{\alpha_i}((w_{i-1}\chi)\delta_{w_{i-1}})$$

for every i. But from the explicit expression for c_α (Equation (1.6.3)) one has that $c_\alpha(\tau)$ is some rational function $F(\tau\delta^{-1/2}(a_\alpha))$, where F does not depend on α or τ. Hence

$$c_\alpha((w\chi)\delta_w)=F(w(\chi\delta^{-1/2})(a_\alpha))$$
$$=F(\chi\delta^{-1/2}(a_{w^{-1}\alpha}))$$

and

$$c_{\bar{\alpha}}((\bar{w}\chi)\delta_{\bar{w}})=F((w_\iota w w_\iota^{-1})(\chi\delta^{-1/2})(a_{\bar{\alpha}}))\,.$$

Now χ here$=\delta_\Theta^{-t}$ so that $w_\iota\chi=\chi^{-1}$ since $w_\iota(\Theta)=-\Theta$. And $w_\iota(\delta^{1/2})=\delta^{-1/2}$, while $w_\iota\alpha=-\bar{\alpha}$. So this expression is also

$$F(w(\chi^{-1}\delta^{1/2})(a_{\bar{\alpha}}^{-1}))=F(w(\chi\delta^{-1/2})(a_\alpha)) \quad \Box\Box \text{ (Proposition 2.5).}$$

We will now begin to construct a homotopy of operators. First of all, we will let the t in T_Θ^t go to infinity. This sort of thing is probably always interesting, and in this case the answer is simple as well. According to the calculations above, after a short manipulation one sees that T_Θ^t is left multiplication by the product of the elements

$$(c_{w_{i-1}^{-1}(\alpha_i)}(\chi)-1)+q^{-1}C(w_{\alpha_i})$$

where $\chi=\delta_\Theta^{-t}$. Recall that

$$c_\gamma(\chi)=(1-q^{-1}\chi\delta^{-1/2}(a_\gamma))(1-\chi\delta^{-1/2}(a_\gamma))^{-1}\,.$$

Now the set of roots $\{w_{i-1}^{-1}(\alpha_i)\}$ is just the set $\{\gamma>C\,|\,w_\Theta\gamma<0\}=\Sigma^+-\Sigma_\Theta^+$. We know from Lemma 1.7 that as $t\to\infty$, $\delta_\Theta^{-t}(a_\gamma)\to\infty$ for $\gamma\in\Sigma^+-\Sigma_\Theta^+$, and therefore $c_\gamma(\delta_\Theta^{-t})\to q^{-1}$. We have thus arrived at:

2.7. LEMMA: *The limit of the operator T_Θ^t as $t\to\infty$ is left multiplication by the product*

$$(q^{-1}C(w_{\alpha_n})+(q^{-1}-1))\cdots(q^{-1}C(w_{\alpha_1})+(q^{-1}-1))$$

in $\mathcal{H}(K, K_\phi)$, where

$$w_\Theta=w_{\alpha_n}\cdots w_{\alpha_1}\,,$$

is a reduced expression for w_Θ as a product of elementary reflections.

It will not be needed below, but it may be helpful if I interpret this expression. It is not obvious, for example, that it depends only on w_Θ and not on the particular reduced product expression, except as a consequence of this elaborate calculation. In fact there is a direct way to see this. From the relation (2.2.3) in $\mathcal{H}(K, K_\phi)$ one deduces immediately that

$$C(w_\alpha)(q^{-1}C(w_\alpha)+(q^{-1}-1))=1\,,$$

or

$$q^{-1}C(w_\alpha)+(q^{-1}-1)=C(w_\alpha)^{-1}.$$

So the product in Lemma 2.7 is also

$$C(w_{\alpha_n})^{-1}\cdots C(w_{\alpha_1})^{-1}=(C(w_{\alpha_1})\cdots C(w_{\alpha_n}))^{-1}$$
$$=C(w_\Theta)^{-1}$$

since $w_\Theta=w_\Theta^{-1}=w_{\alpha_1}\cdots w_{\alpha_n}$.

Letting $t\to\infty$ is half of our homotopy. In the other half we let the prime power $q\to1$. It is well known that this makes sense: for any number q one may define a Hecke algebra $\mathcal{H}(q)$ with basis $C(w)$ ($w\in W$) according to the relations (2.2). It depends, of course, simply on the root system. One has in each case the operators $T_{\Theta,q}$, E_q, S_q completely analogous to the operators defined above, where $T_{\Theta,q}$ is now the operator given in Lemma 2.6. The algebra $\mathcal{H}(q)$ still possesses an inner product with respect to which the $C(w)$ are orthogonal, and $\|C(w)\|^2=q^{l(w)}$. For $q\geqq1$ the generators T_q, E_q, S_q are still mutually commuting, invertible, symmetric.

When $q=1$, $\mathcal{H}(q)$ is the group algebra of W. In this case,

$T_{\Theta,1}$=left multiplication by w_Θ
S_1=right multiplication by w_l
E_1=conjugation by w_l.

On the space $\mathcal{V}_\Theta$, since $w_\Theta=w_l w_{l,\Theta}$ and $w_{l,\Theta}\in W_\Theta$, $T_{\Theta,1}$ is also left multiplication by w_l. Hence on $\mathcal{V}_\Theta$, $E_1=T_{\Theta,1}\circ S$, or in other words *the product* $T_{\Theta,1}\cdot E_1\cdot S_1$ *is the identity.*

Now in all the variations that have been made, from $t=0$ to ∞, then from q=our prime power to 1, the product $T\cdot E\cdot S$ remains symmetric and invertible (as well as real). Therefore, *for every* $t>0$.

$$T_\Theta^t\circ E\circ S$$

is positive definite. Hence *for* $t=0$, *it is positive semi-definite, and* in fact *positive definite on* $\mathcal{V}_\Theta$. But in all the variation also, the space $\mathcal{V}_\Theta$ remained stable under the product $E\cdot S$. For $q=1$, this product is left multiplication by w_l. In conclusion, then, T_Θ^0 and this left multiplication by w_l must have the same signature on $\mathcal{V}_\Theta$. □□ (Theorem 2.3)

2.8. Remark: It has been shown above that certain representations σ are not unitary by showing that the given G-invariant inner product is not positive definite on the subspace $\sigma^{K\phi}$. Is there a converse to this? Suppose σ any subrepresentation of $I(\chi)$ with χ unramified. If a G-invariant Hermitian inner product is positive definite on $\sigma^{K\phi}$ is it everywhere positive definite?

Appendix. Smooth cohomology.

The representations σ_P are distinguished by their cohomological properties. This is covered thoroughly in Chapter X of [2], but for certain nasty technical reasons I will recall in a moment, they were forced to introduce notions from the theory of continuous cohomology, that is to say they had to deal with topologies on the vector spaces on which the groups act. Since the notion of a smooth representation involves no such topology, it should be possible in principle to avoid this, but certainly at the time [2] was written it was not at all clear how to do it. What I want to sketch below is how a recent observation of P. Blanc [10] allows one to develop a satisfactory algebraic theory.

For the moment let G be any locally profinite topological group. To emphasize the algebraic nature of what I will say, let $\boldsymbol{F}$ be any field of characteristic 0. Choose on G a left-invariant Haar measure with respect to which the measure of compact open sets is rational. Define $\mathcal{C}_G$ to be the category whose objects are smooth G-modules over $\boldsymbol{F}$ — that is to say (π, V) with V a vector space over $\boldsymbol{F}$ and $\pi: G \to \mathrm{Aut}_{\boldsymbol{F}}(V)$ a homomorphism such that the isotropy group $v \in V$ is always open — and whose morphisms are $\boldsymbol{F}$-linear G-maps.

Let $C_c^\infty(G, \boldsymbol{F})$ (which I abbreviate as $C_c^\infty(G)$ here) be the space of locally constant $\boldsymbol{F}$-valued functions on G of compact support. It becomes a ring by convolution:

$$(f*h)(x)=\int_G f(y)h(y^{-1}x)dy$$

and every smooth G-module becomes a module over this ring:

$$\pi(f)v=\int_G f(g)\pi(g)v\,dg\,.$$

The integral is really a finite sum since v is fixed by some compact open subgroup K and so is f. This gives what is in fact an equivalence of categories between (i) smooth G-modules and (ii) modules over $C_c^\infty(G)$. From one standpoint, $C_c^\infty(G)$-modules are complicated because $C_c^\infty(G)$ does not have a unit. At any rate, for each compact open subgroup K one does have a projection operator $\mathcal{P}_K = (\mathrm{meas}\, K)^{-1}\,\mathrm{char}_K$. By applying it one can see that in $\mathcal{C}_G$ the functor $V \rightsquigarrow H^0(K, V) = V^K$ is exact.

If (π, V) is a smooth G-module, its contragredient $(\tilde{\pi}, \tilde{V})$ is the contragredient representation of G on the subspace of vectors in the total linear dual of V with open isotropy subgroup. The assignment $\pi \rightsquigarrow \tilde{\pi}$ is also an exact functor.

If H is a closed subgroup of G and (σ, U) a smooth H-module the representation $\mathrm{Ind}(U|H, G)$ is the right regular representation R of G on the space of all $f: G \to U$ satisfying

(i) $f(hg)=\sigma(h)f(g)$ for all $h \in G$, $g \in G$;

(ii) there exists an open subgroup K such that $R_k f = f$ for all $k \in K$.
This representation is smooth because of (ii).

A.1. LEMMA: *The functor* $U \rightsquigarrow \mathrm{Ind}(U|H, G)$ *is exact.*

The proof is straightforward. I mention it specifically because it requires the exactness of $U \rightsquigarrow U^K$ when K is compact, which in turn relies on the fact that the modules we are concerned with are vector spaces over a field of characteristic 0. One might be tempted to work with the whole category of G-modules with open isotropy subgroups, but then it seems A.1 could fail to hold.

Let $\Lambda: \mathrm{Ind}(U|H, G) \to U$ be the H-map $f \mapsto f(1)$.

A.2. LEMMA (Frobenius reciprocity): *For any smooth G-module V, composition with* Λ *is an isomorphism*

$$\mathrm{Hom}_G(V, \mathrm{Ind}(U|H, G)) \cong \mathrm{Hom}_H(V, U).$$

The proof is standard.

A.3. COROLLARY: *If U is injective in* $\mathcal{C}_H$, $\mathrm{Ind}(U|H, G)$ *is injective in* $\mathcal{C}_G$.

In particular, if $H=1$ then $\mathrm{Ind}(U|1, G)$ is always G-injective. Now if U is a G-module then the map $u \mapsto f_u$ where

$$f_u(g) = gu$$

is a G-embedding of U into $\mathrm{Ind}(U|1, G)$. Hence $\mathcal{C}_G$ (which is clearly an abelian category) possesses sufficiently many injectives to guarantee that every object has an injective resolution. If

$$0 \longrightarrow U \longrightarrow U^0 \longrightarrow U^1 \longrightarrow \cdots$$

is such a resolution and V is any smooth G-modules the Ext-groups are defined according to the recipe

$$\mathrm{Ext}^{\cdot}_G(V, U) = \text{cohomology of the complex } \mathrm{Hom}_G(V, U^{\cdot}).$$

In particular $H^{\cdot}(G, U) = \mathrm{Ext}^{\cdot}_G(\boldsymbol{F}, U)$.

It is at this point that things become tricky. As is always true, for calculating any cohomology groups one needs forms of Shapiro's lemma and the Hochschild-Serre spectral sequence. But in the category of smooth G-modules the G-injectives are not generally H-injective, and for this reason the standard arguments break down. (In other words, there are examples of G and H where $\mathrm{Ind}(\boldsymbol{F}|1, G)$ is not H-injective.) One might hope that G-injectives V at least satisfy $\mathrm{Ext}^m_H(U, V)=0$ for $m>0$, and I know no counterexamples. At any rate, the technical reason Borel and Wallach in [2] introduce continuous cohomology (where of course $F=\boldsymbol{R}$ or $\boldsymbol{C}$) is because then at least one can go a bit further and at least obtain Shapiro's Lemma, although with Hochschild-Serre things still remain nasty (as Wigner and I discovered after publishing [11], when Deligne pointed out a serious error in the proof of the spectral sequence there).

One way out of the mire was provided by [10]:

A.4. THEOREM (Blanc): *The G-module $C_c^\infty(G)$, given the right-regular representation of G, is projective in $\mathcal{C}_G$.*

PROOF: I will simply copy that in [10], even though it seems to me somewhat mysterious. Choose any $\varphi\in C_c^\infty(G)$ with

$$\int_G \varphi(x)dx=1 .$$

For every $g\in G$ and $f\in C_c^\infty(G)$ define

$$f_g(x)=f(x)\varphi(xg) .$$

This satisfies

(A.4.1) $$\int_G f_g dg=f .$$

(A.4.2) $$R_h f_{h^{-1}g}=(R_h f)_g .$$

Given a diagram

$$\begin{array}{ccccc} & & C_c^\infty(G) & & \\ & & \downarrow F & & \\ U & \xrightarrow{P} & V & \longrightarrow & 0 \end{array}$$

let S be an F-linear section of P and define

$$\bar{F}(f)=\int_G gSg^{-1}F(f_g)dg .$$

One applies (A.4.1) and (A.4.2) to see that this integral makes sense — i.e. is really a finite sum — and that it is a G-map which lifts F. □

Now let V be any smooth G-space. It follows from A.4 that $C_c^\infty(G)\otimes V$, with G acting only on the left factor, is projective in $\mathcal{C}_G$. The map

$$f\otimes v\longmapsto \int_G f(x^{-1})\pi(x)v\,dx$$

is a G-map from $C_c^\infty(G)\otimes V$ to V, which is surjective since for every $v\in V$ there always exists $f\in C_c^\infty(G)$ with $\pi(f)v=v$. Thus:

A.5. COROLLARY: *Every object in $\mathcal{C}_G$ has a projective resolution.*

As a consequence, we have a second way to calculate Ext-groups. Let $U_\cdot\to U$ be a projective resolution and let V be any smooth G-space. Then

$\mathrm{Ext}^{\cdot}_G(U, V)$ is the cohomology of the complex $\mathrm{Hom}_G(U., V)$. One can also define *homology* spaces, starting with $H_0(G, U)$ which is the quotient of U by the subspace spanned by the $\pi(g)u-u$ $(g\in G,\ u\in U)$ and is the maximal G-trivial quotient of U. The higher groups are defined by

$$H.(G, U)=\text{homology of } H_0(G, U.)\,.$$

A.6. LEMMA: *Let H be a closed subgroup of G.*

(a) *As an H-module*

$$C_c^\infty(G)\cong C_c^\infty(H)\otimes C_c^\infty(G/H)$$

where H acts trivially on the second factor;

(b) *If H is normal, then as G/H-modules*

$$H_0(H, C_c^\infty(G))\cong C_c^\infty(G/H)\,.$$

PROOF: Part (a) follows from the existence of a cross-section of $G\to G/H$ (Prop. 1 on page I-2 of [12]). For part (b), consider the map from $C_c^\infty(G)$ to $C_c^\infty(G/H)$ defined by

$$\mathcal{P}f(x)=\int_H f(h^{-1}x)\,dh\,.$$

Then $\mathcal{P}(R_g f)=R_g(\mathcal{P}f)$ so that in particular $\mathcal{P}(R_h f)=\mathcal{P}(f)$ for $h\in H$ so that it factors through $H_0(H, C_c^\infty(G))$. The induced map is clearly surjective, and in order to conclude it must be shown that if $\mathcal{P}f=0$ then

$$f=\sum(R_{h_i}f_i-f_i)$$

for some choice of h_i, f_i. Again using a cross-section, this reduces to the proposition that for any $f\in C_c^\infty(H)$, if its integral over H is 0 then f may be so expressed. But if $f\in C_c^\infty(H)$ and $K\subseteq H$ is compact and open, with $R_k f=f$ for all $k\in K$, then

$$\begin{aligned}f&=\sum_{H/K} f(x)R_{x^{-1}}(\text{char } K)\\ &=\sum_{H/K} f(x)[R_{x^{-1}}(\text{char } K)-(\text{char } K)]+(\text{meas } K)^{-1}\int_H f(x)\,dx\,.\\ &=\sum_{H/K} f(x)[R_{x^{-1}}(\text{char } K)-(\text{char } K)]\end{aligned}$$

if the integral is 0. □

A.7. COROLLARY: *Let H be a closed subgroup of G.*

(a) *Any G-projective in $\mathcal{C}_G$ is also H-projective in $\mathcal{C}_H$;*

(b) *If H is normal in G and U is G-projective in $\mathcal{C}_G$ then $H_0(H, U)$ is G/H-projective in $\mathcal{C}_{G/H}$.*

There are several applications. The first is a version of Shapiro's Lemma.

A.8. PROPOSITION: *Let H be a closed subgroup of G, U a smooth H-module, V a smooth G-module. Then composition with Λ: $\mathrm{Ind}(U|H, G) \to U$ induces an isomorphism*

$$\mathrm{Ext}_G^{\cdot}(V, \mathrm{Ind}(U|H, G)) \cong \mathrm{Ext}_H^{\cdot}(V, U).$$

PROOF: Let $V. \to V$ be a projective resolution of V. Then the left-hand side is the cohomology of

$$\mathrm{Hom}_G(V., \mathrm{Ind}(U|H, G))$$

which by Frobenius reciprocity is the same as

$$\mathrm{Hom}_H(V., U).$$

But by A.7 the $V.$ are H-projective, so this is also $\mathrm{Ext}_H^{\cdot}(V, U)$. □

Another application is a variant of Hochschild-Serre.

A.9. PROPOSITION: *Suppose H to be closed and normal in G, U and V smooth G-modules, with H acting trivially on V. Then there is a spectral sequence*

$$E_2^{p,q} \cong \mathrm{Ext}_{G/H}^{p}(H_q(H, U), V) \Rightarrow \mathrm{Ext}_G^{p+q}(U, V).$$

PROOF: Consider the functors $F: U \rightsquigarrow H_0(H, U)$ from $\mathcal{C}_G$ to $\mathcal{C}_{G/H}$ and $E: U \rightsquigarrow \mathrm{Hom}_{G/H}(U, V)$ from $\mathcal{C}_{G/H}$ to $\mathcal{C}_1$. The composite $E \circ F$ is $\mathrm{Hom}_G(U, V)$ since H acts trivially on V. The first is covariant, the second contravariant. The first takes projectives to projectives by A.7 (b), and by A.7 (a) its derived functors are H-homology. A standard argument gives the spectral sequence. □

There are a few more tricks in this business which are useful to know.

A.10. PROPOSITION: *Let V be a smooth G-module, $\tilde{V}$ its smooth dual. Then $H^{\cdot}(G, \tilde{V})$ is canonically isomorphic to the $\boldsymbol{F}$-linear dual of $H.(G, V)$.*

A.11. PROPOSITION: *Suppose that V is an admissible G-module: for each open subgroup K, the space V^K has finite dimension. Then for any smooth G-module U, $\mathrm{Ext}_G^{\cdot}(U, V)$ is canonically isomorphic to $\mathrm{Ext}_G^{\cdot}(\tilde{V}, \tilde{U})$.*

Both proofs are left as an exercise. For A.11 the main point is that V is admissible if and only if it is the contragredient of $\tilde{V}$.

Now return to the notation of the main part of the paper: G reductive and p-adic, etc. If V is a smooth G-module and $P=MN$ is any subgroup then the Jacquet module V_N is nothing other than $H_0(N, V)$. Combining A.8 and A.9 one sees:

A.12. THEOREM: *Let P be parabolic subgroup of G with Levi decomposition $P=MN$. If (σ, U) is a smooth representation of P trivial on N and (π, V) is a smooth G-representation, then there is a canonical isomorphism*

$$\mathrm{Ext}^{\cdot}_{G}(V, \mathrm{Ind}(U|P, G)) \cong \mathrm{Ext}^{\cdot}_{M}(V_N, U).$$

From this point on, the arguments in [2] can be followed practically word-for-word, occasionally using A.10 or A.11, to yield the main result:

A.13. THEOREM: *Let G be semi-simple. For any parabolic P in G*

$$H^{\cdot}(G, \sigma_P) = \begin{cases} C & \textit{in dimension}=\textit{parabolic rank of } P \\ 0 & \textit{otherwise.} \end{cases}$$

If (π, V) is any irreducible admissible representation of G and $H^{\cdot}(G, V)\neq 0$, then $V\cong\sigma_P$ for some P.

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