

# Extended automorphic forms on the upper half plane

W. Casselman

**Abstract:** A variant of Hadamard's notion of *partie finie* is applied to the theory of automorphic functions on arithmetic quotients of the upper half-plane. As a consequence, conceptually simple proofs of the volume formula and the Maass-Selberg relations are given. This technique interprets Zagier's idea of renormalization (Jour. Fac. Sci. Univ. Tokyo **28** (1982), 415–437) so that it can be generalized easily to higher rank quotients (which is not done in this paper, however).

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### Introduction

Formally,

$$\int_0^\infty x^s dx = \int_0^1 x^s dx + \int_1^\infty x^s dx.$$

The first integral on the right converges for  $\operatorname{Re}(s) > -1$  and is then equal to  $1/(s+1)$ . The second converges for  $\operatorname{Re}(s) < -1$  and is then equal to  $-1/(s+1)$ . Hence, defining the integral by analytic continuation, we can write (at least if  $s \neq -1$ )

$$\int_0^\infty x^s dx = 0.$$

This paper will show how to justify this calculation, and similar ones where the domain of integration is replaced by arithmetic quotients of the upper half plane and the integrand by certain automorphic forms. It will also explain by examples why being able to justify such calculations is useful. The first example will be a new calculation of the area of such a quotient, and the second will be a new derivation of the Maass-Selberg formula for the inner product of truncated Eisenstein series. These examples may not seem very interesting, but I hope they will offer evidence of the potential of the idea. I will show in a subsequent paper that all of the notions in this paper can be extended to the case of arithmetic groups of arbitrary rank, taking into account the Arthur-Langlands theory of truncation. The examples here generalize easily to give respectively the volume formula of Langlands (1966a) and Lai (1980) and the inner product formula of Langlands (1966b) and Arthur (1980) (see also Labesse-Langlands (1983) and Arthur (1982)).

When I described some of the results of this paper to Jacquet several years ago, he pointed out similarities with the ideas of Zagier (1982), and the current formulation has been strongly influenced by that paper. It provides a more automatic method of justifying Zagier's calculations, and in such a way that extension to groups of higher rank is straightforward. It is worthy of remark that one of Zagier's topics is the Selberg trace formula, which is not covered here at all. I would like to think that eventually a serious connection may be found to it.

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It is also perhaps worth remarking that the techniques of this paper extend with mild modification to reductive groups over local fields. The main application to local representation theory is an inner product formula for truncated matrix coefficients, generalizing the formulas of Waldspurger (1989) and Arthur (1991) and it turns out that this can be used to give a direct construction of the Plancherel measure.

## 1. The multiplicative group

The simplest reductive arithmetic quotient is the multiplicative group of positive real numbers. Define the *Schwartz space*  $\mathcal{S}(0, \infty)$  to be the space of all functions in the Schwartz space of  $\mathbb{R}$  which vanish identically on  $(-\infty, 0)$ . In other words a smooth function  $f$  on the open interval  $(0, \infty)$  lies in this space if and only if it and all the derivatives  $d^n f/dx^n$  vanish at infinity and at 0 more rapidly than any power of  $x$ . This is equivalent to the same condition on the derivatives  $D^n f$  where  $D$  is the multiplicatively invariant derivative  $xd/dx$ .

In this paper I define the *extended Schwartz space*  $\mathcal{S}[0, \infty)$  to be that of all smooth functions  $f$  on  $(0, \infty)$  obtained by restricting to  $(0, \infty)$  smooth functions on all of  $\mathbb{R}$  which lie in the Schwartz space of  $\mathbb{R}$  and whose Taylor series at 0 comprises just a constant term. *This definition is the crux of this paper.* It may seem at first somewhat arbitrary, but in the context of arithmetic quotients seems to balance nicely several ultimate requirements. In another context, that of representations of  $SL_2(\mathbb{R})$ , the correct definition would be simply the restrictions to  $[0, \infty)$  of all functions in the Schwartz space of  $\mathbb{R}$ . The difference between the two contexts is essentially the difference between the asymptotic behaviour of automorphic forms and of matrix coefficients—the second case is more complicated. In any event, the basic idea in all cases is a variant of Hadamard’s definition of the *partie finie*.

I will define the spaces of *tempered distributions*  $A(0, \infty)$  and *extended distributions*  $A[0, \infty)$  to be the respective duals. The multiplicative group  $\mathbb{R}^{\text{pos}}$  acts on these spaces by the right regular representation.

The canonical inclusion gives a short exact sequence of  $\mathbb{R}^{\text{pos}}$ -modules

$$0 \rightarrow \mathcal{S}(0, \infty) \rightarrow \mathcal{S}[0, \infty) \rightarrow \mathbb{C} \rightarrow 0$$

where the third term is the trivial representation, and dually

$$0 \rightarrow \mathbb{C} \rightarrow A[0, \infty) \rightarrow A(0, \infty) \rightarrow 0. \tag{1.1}$$

Let  $\mathbb{C}[s]$  be the one-dimensional module on which  $\mathbb{R}^{\text{pos}}$  acts by the character  $x^s$ , and on which the differential operator  $D$  acts as multiplication by  $s$ . The integration formula

$$\langle x^s, f \rangle = \int_0^\infty f(x)x^s dx/x$$

corresponds to an  $\mathbb{R}^{\text{pos}}$ -injection of  $\mathbb{C}[s]$  into  $A(0, \infty)$ . As a functional as well as a function it thus satisfies the equation  $Dx^s = sx^s$ .

**1.1. Proposition.** *For  $s \neq 0$  there exists a unique eigenvector of  $D$  in  $A[0, \infty)$  extending the functional  $x^s$  on  $\mathcal{S}(0, \infty)$ .*

I will call the extension  $\mathfrak{e}(x^s)$ . Zagier calls it the *renormalization* of  $x^s$ .

*Proof.* Any short exact sequence of  $C[D]$ -modules

$$0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$$

gives a long exact sequence

$$\begin{aligned} 0 \rightarrow \text{Ker}_A(D - s) \rightarrow \text{Ker}_B(D - s) \rightarrow \text{Ker}_C(D - s) \\ \rightarrow A/(D - s)A \rightarrow B/(D - s)B \rightarrow C/(D - s)C \rightarrow 0. \end{aligned}$$

Hence if  $D - s$  is invertible on  $C$  with itself, inclusion induces an isomorphism of the  $\mathbb{C}[D]$ -modules  $A/(D - s)A$  and  $B/(D - s)B$ .

Apply this argument to the sequence (1.1). If  $s$  is not 0 then  $D - s$  is an isomorphism on  $\mathbb{C}$ , so that restriction induces an isomorphism between the subspaces of elements of  $A[0, \infty)$  and  $A(0, \infty)$  annihilated by  $D - s$ .  $\square$

Another way of formulating the main point of the argument is to say that the *spectrum* of the quotient  $\mathcal{S}[0, \infty)/\mathcal{S}(0, \infty)$  (with respect to the operator  $D$ ) is exactly  $\{0\}$ .

More explicitly, since the derivative of any  $f$  in  $\mathcal{S}[0, \infty)$  lies in  $\mathcal{S}(0, \infty)$ , for every  $s \neq 0$  integration by parts suggests the definition

$$\langle \mathfrak{e}(x^s), f \rangle = -\frac{1}{s} \langle x^{s+1}, df/dx \rangle$$

This definition of the extended functional  $\mathfrak{e}(x^s)$  (a very simple case of Hadamard's *partie finie*) is easily seen to agree with the previous one. It shows that the definition of the extension may be characterized as by analytic continuation.

More generally, if  $M$  is any module over the polynomial algebra  $\mathbb{C}[D]$  on which multiplication by  $D$  is bijective (say on which  $D - s$  acts nilpotently for some  $s \neq 0$ ), then the proof of Proposition 1.1 shows that any  $D$ -covariant map from  $M$  to  $A(0, \infty)$  possesses a unique lifting to  $A[0, \infty)$ . Corresponding to the product formula

$$\begin{aligned} Dx^s \log^n x &= sx^s \log^n x + nx^s \log^{n-1} x \\ x^s \log^n x &= \frac{1}{s} Dx^s \log^n x - \frac{n}{s} x^s \log^{n-1} x \end{aligned}$$

or (again) integration by parts, we define the extension of the distribution associated to  $x^s \log^n x$  by induction on  $n$ :

$$\langle \epsilon(x^s \log^n x), f \rangle = -\frac{1}{s} \int_0^\infty (df/dx) x^s \log^n x dx - \frac{n}{s} \langle \epsilon(x^s \log^{n-1} x), f \rangle .$$

For any  $s$  in  $\mathbb{C}$  define  $A_s(0, \infty)$  to be the subspace of distributions of the form  $x^s P(\log x)$  where  $P$  is a polynomial. Then in summary:

**1.2. Proposition.** *For  $s \neq 0$  every element in  $A_s(0, \infty)$  possesses a canonical lifting to  $A[0, \infty)$ . More precisely, if  $P$  is a polynomial of degree  $n$  then  $\epsilon(x^s P(\log x))$  varies meromorphically with  $s$ , holomorphically in the region  $\mathbb{C} - \{0\}$ , and with a pole of order  $n$  at 0. In particular the residue of  $x^s$  at 0 is  $\delta_0$ .*

We are not quite to the point where we can integrate  $x^s$  over all of  $(0, \infty)$ , but close. Define now third and fourth Schwartz spaces  $\mathcal{S}(0, \infty]$  and  $\mathcal{S}[0, \infty]$  to be respectively the smooth functions  $f$  on  $(0, \infty)$  such that  $f(1/x)$  lies in  $\mathcal{S}[0, \infty)$ ; the sums of functions in  $\mathcal{S}[0, \infty)$  and  $\mathcal{S}(0, \infty]$ . Define  $A(0, \infty]$ ,  $A[0, \infty]$  to be their duals. Since the constant functions are elements of  $\mathcal{S}[0, \infty]$ , any element of  $A[0, \infty]$  can in effect be integrated over the whole interval  $(0, \infty)$ . All that remains is to show:

**1.3. Proposition.** *For each  $s \neq 0$  every element  $\Phi$  of  $A(0, \infty)$  possesses a canonical extension  $\epsilon(\Phi)$  to  $A[0, \infty]$ . This functional varies meromorphically with  $s$  and has a single pole at 0.*

*Proof.* Not much different from that of Proposition 1.1. Explicitly one can define the extension locally near 0 and  $\infty$ , by writing an element of  $\mathcal{S}[0, \infty]$  as a sum of two parts.  $\blacksquare$

If  $\delta_\infty$  is the functional on  $\mathcal{S}[0, \infty]$  which takes  $f$  to  $f(\infty)$ :

**1.4. Proposition.** *The residue of the extension of  $x^s$  at  $s = 0$  is equal to  $\delta_0 - \delta_\infty$ .*

*Proof.* I leave this as an exercise.  $\blacksquare$

**1.5. Corollary.** *For  $s \neq 0$ ,*

$$\langle \epsilon(x^s \log^n x), 1 \rangle \left( \text{formally } \int_0^\infty x^s \log^n x dx/x \right) = 0.$$

*Proof.* Look at the case  $n = 1$ : by definition of the dual representation

$$\begin{aligned}\langle Dx^s, 1 \rangle &= s \langle x^s, 1 \rangle \\ &= \langle x^s, D1 \rangle = 0. \quad \blacksquare\end{aligned}$$

It is useful to keep in mind that multiplication makes the space  $A[0, \infty]$  into a module over  $\mathcal{S}[0, \infty]$ :

$$\langle f\Phi, \varphi \rangle = \langle \Phi, f\varphi \rangle.$$

Similarly of course if  $s, t, s+t$  are all non-zero then the product of any two elements  $F_t$  and  $F_s$  in  $A_s$  and  $A_t$ , which lies in  $A_{s+t}$ , as well as  $F_s$  and  $F_t$ , have canonical extensions. The product rule for differentiation holds in all these cases. Since  $\langle D(F_s \cdot F_t), 1 \rangle = 0$ :

**1.6. Proposition.** *For any two functions  $F_s$  and  $F_t$  in  $A_s[0, \infty)$  and  $A_t[0, \infty)$  (with all of  $s, t, s+t$  not 0) we have*

$$\langle DF_s \cdot F_t, 1 \rangle = - \langle F_s \cdot DF_t, 1 \rangle,$$

and more generally

$$\langle LF_s \cdot F_t, 1 \rangle = \langle F_s \cdot L^*F_t, 1 \rangle$$

for any polynomial  $L$  in  $D$ , where  $L^*$  is the differential operator adjoint to  $L$ .

It is also useful to keep in mind that if  $\chi$  is the characteristic function of a half-line  $(0, T]$  or  $[T, \infty)$ , then  $\chi(x)x^s$  can be interpreted canonically as an element of  $A[0, \infty]$ : write  $f$  in  $\mathcal{S}[0, \infty]$  as a sum of elements  $f_1, f_2$  in  $\mathcal{S}[0, \infty]$ , one with support on the half-line, the other eventually identically vanishing on the half-line. Then the definition

$$\langle \mathbf{e}(\chi(x)x^s), f \rangle := \langle \mathbf{e}(x^s), f_1 \rangle + \langle \chi(x)x^s, f_2 \rangle$$

is independent of the particular choice of the  $f_i$ . One could also define the functionals by means of integration by parts:

$$\int_0^T x^s f(x) dx/x = \frac{T^s f(T)}{s} - \frac{1}{s} \int_0^T (df/dx)x^s dx.$$

Of course

$$\int_0^T x^s dx/x = T^s/s, \quad \int_T^\infty x^s dx = -T^s/s$$

for all  $s$  where  $\mathbf{e}(x^s)$  is defined.

## 2. The upper half plane

Let  $\Gamma$  be a discrete subgroup of  $\mathrm{SL}_2(\mathbb{R})$  such that (a)  $\Gamma \backslash \mathfrak{H}$  has finite volume and (b)  $\Gamma$  has one cusp. The second condition is for convenience. Let  $P$  be the subgroup of upper triangular matrices in  $\mathrm{SL}_2(\mathbb{R})$ ,  $N$  its subgroup of unipotent matrices. For further convenience I will assume that the cusp is  $\infty$  and that  $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$  generates  $\Gamma \cap N$ . The best known example would be  $\mathrm{SL}_2(\mathbb{Z})$ .

Choose on  $\mathfrak{H}$  the  $\mathrm{SL}_2(\mathbb{R})$ -invariant metric

$$(dx^2 + dy^2)/y^2$$

with associated volume form and Laplacian

$$\omega = \frac{dx dy}{y^2}, \quad \Delta = y^2(\partial^2/\partial x^2 + \partial^2/\partial y^2).$$

I recall that the Schwartz space  $\mathcal{S}(\Gamma \backslash \mathfrak{H})$  is the space of smooth functions  $f$  on  $\Gamma \backslash \mathfrak{H}$  such that  $f$  and all its derivatives  $\partial^n f / \partial^m x \partial^{n-m} y$  decrease more rapidly than any power of  $y$  as  $y$  goes to infinity. Define the *extended Schwartz space*  $\mathcal{S}_{\mathrm{Ext}}(\Gamma \backslash \mathfrak{H})$  of the quotient to be the space of all smooth functions  $f$  on  $\Gamma \backslash \mathfrak{H}$  which may be expressed as the sum of a function in  $\mathcal{S}(\Gamma \backslash \mathfrak{H})$  and a function of  $y$  alone which lies in  $\mathcal{S}(0, \infty]$ . Define  $A(\Gamma \backslash \mathfrak{H})$ ,  $A_{\mathrm{Ext}}(\Gamma \backslash \mathfrak{H})$  to be the respective dual spaces.

We have a short exact sequence of  $\Delta$ -spaces

$$0 \rightarrow \mathcal{S}(\Gamma \backslash \mathfrak{H}) \rightarrow \mathcal{S}_{\mathrm{Ext}}(\Gamma \backslash \mathfrak{H}) \rightarrow \mathbb{C} \rightarrow 0.$$

Of course since functions in  $\mathcal{S}_{\mathrm{Ext}}(\Gamma \backslash \mathfrak{H})$  are bounded, there is a canonical embedding of  $L^1(\Gamma \backslash \mathfrak{H})$ , and of its subspace  $L^2(\Gamma \backslash \mathfrak{H})$ , into  $A_{\mathrm{Ext}}(\Gamma \backslash \mathfrak{H})$ .

The space  $A(\Gamma \backslash \mathfrak{H})$  contains the space  $A_{\mathrm{umg}}(\Gamma \backslash \mathfrak{H})$  of functions  $f$  on  $\Gamma \backslash \mathfrak{H}$  such that for some integer  $n$  all the  $\Delta^m f$  are of order  $y^n$  at infinity (such functions are of *uniform moderate growth*). It is well known (see Casselman (1984) for an awkward attempt at a direct proof) that the asymptotic behaviour of  $f$  at infinity is under this assumption controlled by that of its constant term, defined as the function  $f_0(y)$  where

$$f_0(y) = \int_0^1 f(x + iy) dx.$$

The exact statement is that for  $y \gg 0$  the function  $f(x + iy) - f_0(y)$  has order of decrease greater than any inverse power of  $y$ . The space  $A_{\mathrm{umg}}(\Gamma \backslash \mathfrak{H})$  contains as a subspace that of automorphic

forms, which have in addition to the growth condition the requirement that they be annihilated by some ideal in  $\mathbb{C}[\Delta]$  of finite codimension.

For any finite subset  $S \subseteq \mathbb{C}$  define  $A^S(\Gamma \backslash \mathfrak{H})$  to be the space of functions in  $A_{\text{umg}}(\Gamma \backslash \mathfrak{H})$  whose constant term is a sum of terms of the form  $y^s P(\log y)$  for  $s \in S$ ,  $P$  a polynomial. If  $\Phi$  is any element of  $A^S(\Gamma \backslash \mathfrak{H})$  where  $S$  does not contain 1, then we may extend  $\Phi$  to an element  $\mathfrak{e}(\Phi)$  of  $A_{\text{Ext}}(\Gamma \backslash \mathfrak{H})$  in the following way: given  $f$  in  $\mathcal{S}_{\text{Ext}}(\Gamma \backslash \mathfrak{H})$ , let  $f_\infty$  be the function  $\chi(y)f_0(y)$  made  $\Gamma$ -invariant, where  $f_0(y)$  is the constant term of  $f$ , and  $\chi(y)$  is a suitable smooth cut-off function, identically 1 for  $y$  large, identically 0 for  $y$  a little less large. Then  $f - f_\infty$  lies in the Schwartz space and we set

$$\langle \mathfrak{e}(\Phi), f \rangle = \langle \Phi, f - f_\infty \rangle_{\Gamma \backslash \mathfrak{H}} + \langle \mathfrak{e}(\varphi), f_\infty / y \rangle_{(0, \infty]}$$

where  $\varphi$  is the constant term of  $\Phi$ . The factor  $1/y$  arises because of the difference between the measures  $dy/y$  used in the last section and  $dy/y^2$  used in this one. This definition turns out not to depend on the choice of cut-off  $\chi(y)$ . It is canonical in any of several ways:

**2.1. Proposition.** *This manner of extension is compatible with  $\Delta$ :*

- (a) For any  $\Phi$  in  $A^S(\Gamma \backslash \mathfrak{H})$  where  $S$  does not contain 1,  $\Delta \mathfrak{e}(\Phi) = \mathfrak{e}(\Delta \Phi)$ .
- (b) For any two functions  $\Phi_1$  and  $\Phi_2$  in  $A^{S_1}(\Gamma \backslash \mathfrak{H})$  and  $A^{S_2}(\Gamma \backslash \mathfrak{H})$ , where  $S_1 + S_2$  does not contain 1, Green's formula holds:

$$\langle \Delta \Phi_1 \cdot \Phi_2, 1 \rangle = \langle \Phi_1 \cdot \Delta \Phi_2, 1 \rangle.$$

These follow from the definition and the corresponding results in one dimension.

Recall that the *Eisenstein series*  $E_s$  for  $s$  in  $\mathbb{C}$  with  $\text{Re}(s) > 1$  is defined by the convergent series

$$E_s(z) = \sum_{\Gamma \cap P \backslash \Gamma} y(\gamma z)^s.$$

If  $\Gamma = \text{SL}_2(\mathbb{Z})$  then this is the Maass function

$$\sum_{(c,d)=1, c>0} \frac{y^s}{|cz + d|^{2s}}.$$

The function  $E_s$  is real-analytic on  $\Gamma \backslash \mathfrak{H}$ , and an eigenfunction of  $\Delta$  with eigenvalue  $s(s-1)$ . It continues meromorphically as a function of the parameter  $s$  to all of  $\mathbb{C}$ . For all  $s$  where defined satisfies the equation

$$\Delta E_s = s(1-s)E_s;$$

and near infinity it is the sum of a Schwartz function and its *constant term*

$$y^s + c(s)y^{1-s}$$

where  $c(s)$  is a meromorphic scalar function on  $\mathbb{C}$ . Furthermore  $E_s$  satisfies the functional equation

$$E_s = c(s)E_{1-s}.$$

If  $\Gamma$  is  $\mathrm{SL}_2(\mathbb{Z})$  then

$$c(s) = \frac{\xi(2s-1)}{\xi(2s)} \quad (\xi(s) = \pi^{s/2}\Gamma(s/2)\zeta(s)).$$

**2.2. Corollary.** *Any  $E_s$  where  $E_s$  is defined and  $s \neq 0, 1$  has a unique extension which remains an eigenvector of  $\Delta$ .*

The poles of  $E_s$  in the right hand plane  $\mathrm{Re}(s) \geq \frac{1}{2}$  are known to lie in the interval  $(\frac{1}{2}, 1]$ . The residue at 1 is a constant function (since it is harmonic and asymptotic to a constant). It is in fact the constant

$$c^*(1) = \mathrm{Residue}_{s=1} c(s).$$

If  $\Gamma$  is  $\mathrm{SL}_2(\mathbb{Z})$  this is just  $1/\xi(2)$ . This residue is of course also the residue of  $E(s)$  considered as a tempered distribution on  $\Gamma \backslash \mathfrak{H}$ . The extended distribution  $\mathfrak{e}(E_s)$  has also a simple pole, but its residue is slightly more complicated. Note that at least since constants are square-integrable, they may be canonically identified with extended distributions.

**2.3. Proposition.** *The residue of  $\mathfrak{e}(E_s)$  at  $s = 1$  is  $c^*(1) - \delta_\infty$ .*

Since constants lie in  $\mathcal{S}_{\mathrm{Ext}}(\Gamma \backslash \mathfrak{H})$ , any extended distribution may be paired with constants, or in other words integrated over  $\Gamma \backslash \mathfrak{H}$ . Since the extension of  $E_s$  is still an eigenfunction for  $\Delta$ , an argument entirely similar to that for the proof of Proposition 1.4 implies:

**2.4. Lemma.** *For non-integral  $s$  where  $E_s$  is defined*

$$\langle E_s, 1 \rangle \left( \text{formally } \int_{\Gamma \backslash \mathfrak{H}} E_s \frac{dx dy}{y^2} \right) = 0.$$

Letting  $s = 1$  and applying Proposition 2.3:

**2.5. Proposition.** *The volume of the quotient is*

$$\int_{\Gamma \backslash \mathfrak{H}} \frac{dx dy}{y^2} = \frac{1}{c^*(1)}.$$

The product of two Eisenstein series  $E_s$  and  $E_t$  has as asymptotic expansion near infinity the sum of four terms whose exponents are  $s + t$ ,  $2 - s - t$ ,  $1 - s + t$ ,  $1 - t + s$ . As long as  $s$  and  $t$  are not singular parameters and as long as none of these exponents is 1, the product  $E_s E_t$  can be extended canonically, hence integrated over all of the quotient. From Proposition 2.1 and the usual argument for the orthogonality of eigenvectors:

**2.6. Proposition.** *Whenever the product  $E_s E_t$  may be canonically extended*

$$\langle E_s E_t, 1 \rangle \left( \text{formally } \int_{\Gamma \backslash \mathfrak{H}} E_s E_t \frac{dx dy}{y^2} \right) = 0.$$

Let  $T$  be a large positive number. Define the corresponding *truncation*  $\Lambda^T(F)$  of any function  $F$  on  $\Gamma \backslash \mathfrak{H}$  to be just  $F$  in the region  $y \leq T$  of the fundamental domain, but  $F - F_0$  in the region  $y > T$ . Let  $C^T(F)$  be the remainder, equal to 0 inside and  $F_0$  outside. The decomposition

$$F = \Lambda^T(F) + C^T(F)$$

is orthogonal in practically any sense. If  $F$  is an automorphic form, then its truncation is rapidly decreasing near infinity. In particular the truncation of  $E_s$  will be square-integrable. Since truncation is an orthogonal projection, it may be therefore defined on the spaces of tempered distributions and extended tempered distributions by duality.

**2.7. Corollary.** (The Maass-Selberg formula) *For any values of  $s$  and  $t$*

$$\langle \Lambda^T(E_s), \Lambda^T(E_t) \rangle = \frac{T^{s+t-1} - c(s)c(t)T^{1-s-t}}{s+t-1} + \frac{c(t)T^{s-t} - c(s)T^{t-s}}{s-t}.$$

*Proof.* By Corollary 2.2, for generic  $s$  and  $t$  the product of  $E_s$  and  $E_t$  may be integrated over  $\Gamma \backslash \mathfrak{H}$ , and by Proposition 2.6 the integral is 0. But expressing  $E_s$  as  $\Lambda^T(E_s) + C^T(E_s)$  we then have

$$\langle \Lambda^T(E_s), \Lambda^T(E_t) \rangle + \langle C^T(E_s), C^T(E_t) \rangle = 0,$$

where the second term must be interpreted in the light of the remarks at the end of the previous section, that is to say by analytic continuation. Thus in the same spirit

$$\langle \Lambda^T(E_s), \Lambda^T(E_t) \rangle = \int_0^T (y^s + c(s)y^{1-s})(y^t + c(t)y^{1-t}) \frac{dy}{y^2}$$

which gives the Corollary.  $\square$

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