Analysis on SL(2)

Representations of $\mathrm{SL}_2(\mathbf{R})$

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This essay will explain what is needed about representations of $\mathrm{SL}_2(\mathbb{R})$ in the elementary parts of the theory of automorphic forms. On the whole, I have taken the most straightforward approach, even though the techniques used are definitely not valid for other groups.

This essay is about representations. It says nothing about what might be called invariant harmonic analysis on G. It does not discuss orbital integrals at all, and says very little about characters. Nor does it prove anything deep about the relationship between representations of G and those of its Lie algebra, or about versions of Fourier transforms. These topics involve too much analysis to be dealt with in this algebraic essay. I shall deal with those matters elsewhere.

Unless specified otherwise in this essay:

$$\begin{aligned} G &= \mathrm{SL}_2(\mathbb{R}) \\ K &= \text{ the maximal compact subgroup } \mathrm{SO}_2 \text{ of } G \\ A &= \text{ the subgroup of diagonal matrices} \\ N &= \text{ subgroup of unipotent upper triangular matrices} \\ P &= \text{ subgroup of upper triangular matrices} \\ &= AN \\ \mathfrak{g}_{\mathbb{R}} &= \text{Lie algebra } \mathfrak{sl}_2(\mathbb{R}) \\ \mathfrak{g}_{\mathbb{C}} &= \mathbb{C} \otimes_{\mathbb{R}} \mathfrak{g}_{\mathbb{R}} \,. \end{aligned}$$

Most of the time, it won't matter whether I am referring to the real or complex Lie algebra, and I'll skip the subscript.

Much of this essay is unfinished. There might well be some annoying errors, for which I apologize in advance. Constructive complaints will be welcome. There are also places where much have yet to be filled in, and these will be marked by one of two familiar signs:

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Introduction

Suppose for the moment G to be an arbitrary semi-simple group defined over $\mathbb{Q}$. One of the most interesting questions in number theory is, *what irreducible representations of $G(\mathbb{A})$ occur discretely in the right regular representation of $G(\mathbb{A})$ on $\mathrm{L}^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$?* A closely related question can be made more concrete. Suppose K to be a compact open subgroup of $G(\mathbb{A})$, and $\Gamma = G(\mathbb{Q}) \cap K$. *What irreducible representations of $G(\mathbb{R})$ occur discretely in $\mathrm{L}^2(\Gamma\backslash G(\mathbb{R}))$? With what multiplicity?* And for applying the trace formula to them, we also ask, *What are their characters (which are distributions)?*

These questions are extremely difficult, and they have motivated much investigation into harmonic analysis of reductive groups defined over local fields. Representations of semi-simple (and reductive) groups over $\mathbb{R}$ have been studied since about 1945, and much has been learned about them, but much remains to be done. The literature is vast, and it is difficult to know where to begin in order to get any idea of what it is all about.

There are a number of technical issues that complicate matters. The first is that one immediately switches away from representations of G to certain representations of its Lie algebra. This is justified by some relatively deep theorems in analysis, but after an initial period this is easily absorbed. The representations of the Lie algebra are in fact representations simultaneously of $\mathfrak{g}$ and a maximal compact subgroup K of G. These are assumed to be compatible in an obvious sense. The dependence on K is weak, since all choices are conjugate in G. The important condition on these representations is that the restriction to K be a direct sum of irreducible representations, each with finite multiplicity. Such representations of $(\mathfrak{g}, K)$ are called Harish-Chandra modules. In view of the original questions posed above, it is good to know that the classification of irreducible unitary representations of G is equivalent to that irreducible unitarizable, Harish-Chandra modules.

The classification of unitarizable Harish-Chandra modules has not yet been carried out for all G, and the classification that does exist is somewhat messy. The basic idea for the classification goes back to the origins of the subject—first classify all irreducible Harish-Chandra modules, and then decide which are unitarizable. The classification of all irreducible Harish-Chandra modules asked for here has been known for a long time. One useful fact is that every irreducible Harish-Chandra modules may be embedded in one induced from a finite-dimensional representations of a minimal parabolic subgroup. This gives a great deal of information, and in particular often allows one to detect unitarizability.

This technique gives a great deal of information, but some important things require some other approach. The notable exceptions are those representations of G that occur discretely in $\mathrm{L}^2(G)$, which make up the discrete series. These are generally dealt with on their own, and then one can look also at representations induced from discrete series representations of parabolic subgroups. Along with some technical adjustments, these are the components of Langlands' classification of all irreducible Harish-Chandra modules. One of the adjustments is that one must say something about embeddings of arbitrary Harish-Chandra modules into $C^\infty(G)$, and say something about the asymptotic behaviour of functions in the image.

Obtaining character formulas of irreducible Harish-Chandra modules is another important problem. It is apparently very difficult, and involves the phenomenon of endoscopy, according to which harmonic analysis on G is tied to harmonic analysis on certain subgroups of G called endoscopic.

The traditional approaches mentioned so far have not been entirely successful. A more uniform approach is based on D-modules on flag varieties, but even here there seems to be much unfinished business.

This essay is almost entirely concerned with just the particular group $G = \mathrm{SL}_2(\mathbb{R})$. Although this is a relatively simple group, it is instructive to see that nearly all interesting phenomena mentioned alreday appear here. I shall touch, eventually, on all themes mentioned above, but in the present version a few are left out.

The subject of unitary representations of non-compact semi-simple groups begin with [Bargmann:1947]. Bargmann's initial work classifying representations of $\mathrm{SL}_2(\mathbb{R})$ already made the step from G to $(\mathfrak{g}, K)$, if not rigourously. Even now it is worthwhile to include an exposition of Bargmann's paper, because it is almost the only case where results can be obtained without introducing sophisticated tools. But another reason for looking at $\mathrm{SL}_2(\mathbb{R})$ closely is that the sophisticated tools one does need eventually are for $\mathrm{SL}_2(\mathbb{R})$ relatively simple, and it is valuable to see them in that simple form. That is my primary goal in this essay.

In Part I I shall essentially follow Bargmann's calculations to classify all irreducible Harish-Chandra modules over $(\mathfrak{sl}_2, \mathrm{SO}(2))$. The unitarizable ones among these will be classified. The techniques used here will be relatively simple calculations in the Lie algebra $\mathfrak{sl}_2$.

Bargmann's techniques become impossible for groups other than $\mathrm{SL}_2(\mathbb{R})$. One needs in general a way to classify representations that does not require such explicit computation. There are several ways to do this. One is by means of induction from parabolic subgroups of G, and this is what I'll discuss in Part II (but just for $\mathrm{SL}_2(\mathbb{R})$). These representations are now called principal series.

It happens, even for arbitrary reductive groups, that every irreducible Harish-Chandra module can be embedded in one induced from a finite-dimensional representation of a minimal parabolic subgroup. This is important, but does not answer many interesting questions. Some groups, including $\mathrm{SL}_2(\mathbb{R})$ possess certain irreducible unitary representations said to lie in the discrete series. They occur discretely in the representation of G on $\mathrm{L}^2(G)$, and they require new methods to understand them. This is done for $\mathrm{SL}_2(\mathbb{R})$ in Part III,

For $\mathrm{SL}_2(\mathbb{R})$ the discrete series representations can be realized in a particularly simple way in terms of holomorphic functions. This is not true for all semi-simple groups, and it is necessary sooner or later to understand how arbitrary representations can be embedded in the space of smooth functions on the group. This is in terms of matrix coefficients, explained at the end of Part III.

In order to understand the role of representations in the theory of automorphic forms, it is necessary to understand Langlands' classification of representations, and this is explained in Part IV.

Eventually I'll include in this essay how D-nodules on the complex projective line can be used to explain many phenomena that appear otherwise mysterious, but I have not done so yet. Other topics I've not yet included are the relationship between the asymptotic behaviour of matrix coefficients and embeddings into principal series, and the relationship between representations of $\mathrm{SL}_2(\mathbb{R})$ and Whittaker functions.

The best standard reference for this material seems to be [Knapp:1986] (particularly Chapter II, but also scattered references throughout), although many have found enlightenment elsewhere. A very different approach is Chapter I of [Vogan:1981]. Many books give an account of Bargmann's results, but for most of the rest I believe the account here is new.

Part I. Bargmann's classification

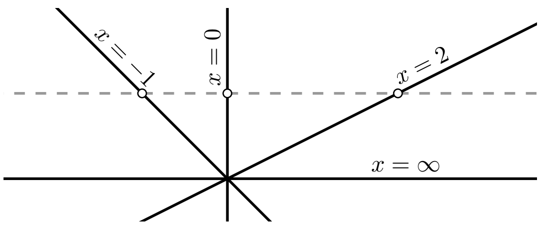
1. Representations of the group and of its Lie algebra

What strikes many on first sight of the theory of representations of $G = \mathrm{SL}_2(\mathbb{R})$ is that it is very rarely looks at representations of G! Instead, one works almost exclusively with certain representations of its Lie algebra. And yet ... the theory is ultimately about representations of G. In this first section I'll try to explain this paradox, and summarize the consequences of the substitution of Lie algebra for Lie group.

I want first to say something first about why one is really interested in representations of G. The main applications of representation theory of groups like G are to the theory of automorphic forms. These are functions of a certain kind on arithmetic quotients $\Gamma\backslash G$ of finite volume, for example when $\Gamma = \mathrm{SL}_2(\mathbb{Z})$. The group G acts on this quotient on the right, and the corresponding representation of G on $\mathrm{L}^2(\Gamma\backslash G)$ is unitary. *Which unitary representations of G occur as discrete summands of this representation?* There is a conjectural if partial answer to this question, at least for congruence groups, which is an analogue for the real prime of Ramanujan's conjecture about the coefficients of holomorphic forms. It was Deligne's proof of the Weil conjectures that gave at the same time a proof of Ramanujan conjecture, and from this relationship the real analogue acquires immediately a certain cachet.

As for why one winds up looking at representations of $\mathfrak{g}$, this should not be unexpected. After all, even the classification of finite-dimensional representations of G comes down to the classification of representations of $\mathfrak{g}$, which are far easier to work with. For example, one might contrast the action of the unipotent upper triangular matrices of G on $S^n\mathbb{C}^2$ with that of its Lie algebra. What might be surprising here is that the representations of $\mathfrak{g}$ one looks at are not usually at the same time representations of G. It is this that I want to explain.

I'll begin with an example that should at least motivate the somewhat technical aspects of the shift from G to $\mathfrak{g}$. The projective space $\mathbb{P}_{\mathbb{R}} = \mathbb{P}^1(\mathbb{R})$ is by definition the space of all lines in $\mathbb{R}^2$. The group G acts on $\mathbb{R}^2$ by linear transformations, and it takes lines to lines, so it acts on $\mathbb{P}_{\mathbb{R}}$ as well. There is a standard way to assign coordinates on $\mathbb{P}_{\mathbb{R}}$ by thinking of this space as $\mathbb{R}$ together with a point at ∞. Formally, every line but one passes through a unique point $(x, 1)$ in $\mathbb{R}^2$, and this is assigned coordinate x. The exceptional line is parallel to the x-axis, and is assigned coordinate ∞.



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In terms of this coordinate system $\mathrm{SL}_2(\mathbb{R})$ acts by fractional linear transformations:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : x \longmapsto (ax+b)/cx+d\,,$$

as long as we interpret $x/0$ as ∞. This is because

$$\begin{aligned}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} z \\ 1 \end{bmatrix} &= \begin{bmatrix} az+b \\ cz+d \end{bmatrix} \\ &= (cz+d)\begin{bmatrix} \dfrac{az+b}{cz+d} \\ 1 \end{bmatrix}.\end{aligned}$$

The isotropy subgroup fixing ∞ is P, and $\mathbb{P}_{\mathbb{R}}$ may be identified with the quotient G/P. Since $K \cap P = \{\pm I\}$ and K acts transitively on $\mathbb{P}_{\mathbb{R}}$, as a K-space $\mathbb{P}_{\mathbb{R}}$ may be identified with $\{\pm 1\}\backslash K$. The action of G on $\mathbb{P}_{\mathbb{R}}$ gives rise to a representation of G on functions on $\mathbb{P}_{\mathbb{R}}$: $L_g f(x) = f(g^{-1} \cdot x)$. (The inverse is necessary here in order to have $L_{g_1 g_2} = L_{g_1} L_{g_2}$, as you can check.) But there are in fact several spaces of functions available—for example real analytic functions, smooth functions, continuous functions, functions that are square-integrable when restricted to K, or the continuous duals of any of these infinite-dimensional vector spaces. We can make life a little simpler by restricting ourselves to smooth representations (ones that are stable with respect to derivation by elements of the Lie algebra $\mathfrak{g}$), in which case the space of continuous functions, for example, is ruled out. But even with this restriction there are several spaces at hand.

But here is the main point: *All of these representations of G should be considered more or less the same*, at least for most purposes. The representation of K on $\mathbb{P}_{\mathbb{R}}$ is as multiplication on $K/\{\pm 1\}$, among the functions on this space are those which transform by the characters ε^{2n} of K where

$$\varepsilon\colon \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \longmapsto (c + is) .$$

The way to make the essential equivalence of all these spaces precise is to *replace all of them by the subspace of functions which when restricted to K are a finite sum of eigenfunctions*. This subspace is the same for all of these different function spaces, and in some sense should be considered the 'essential' representation. However, it has what seems to be at first one disability—it is *not* a representation of G, since g takes eigenvectors for K to eigenvectors for the conjugate gKg^{-1}, which is not generally the same. To make up for this, it *is* stable with respect to the Lie algebra $\mathfrak{g}_{\mathbb{R}} = \mathfrak{sl}_2(\mathbb{R})$, which acts by differentiation.

The representation of $\mathrm{SL}_2(\mathbb{R})$ on spaces of functions on $\mathbb{P}_{\mathbb{R}}$ is a model for other representations and even other reductive groups G. I recall that a continuous representation (π, V) of a locally compact group G on a topological vector space V is a homomorphism π from G to $\mathrm{Aut}(V)$ such that the associated map $G \times V \to V$ is continuous. The TVS V is always assumed in this essay to be locally convex, Hausdorff, and quasi-complete. This last condition guarantees that if f is a continuous function of compact support on G with values in V then the integral

$$\int_G f(g)\, dg$$

is well defined. One basic fact about this integral is contained in the convex hull of the image of f, scaled by the measure of its support. If f is in $C_c^\infty(G)$, one can hence define the operator-valued integral

$$\pi(f) = \int_G f(g)\pi(g)\, dg .$$

If (π, V) is a continuous representation of a maximal compact subgroup K, let $V_{(K)}$ be the subspace of vectors that are contained in a K-stable subspace of finite dimension. These are called the **K-finite vectors** of the representation. If G is $\mathrm{SL}_2(\mathbb{R})$, it is known that any finite-dimensional space on which K acts continuously will be a direct sum of one-dimensional subspaces on each of which K acts by a character, so $V_{(K)}$ is particularly simple. There are a number of somewhat subtle things about this construction.

(1) *If (π, V) is any continuous representation of K, the subspace $V_{(K)}$ is dense in V.*

This is a basic fact about representations of a compact group, and a consequence of the Stone-Weierstrass Theorem.

A continuous representation of G on a complex vector space V is called **admissible** if $V_{(K)}$ is a direct sum of characters of K, each occurring with *finite multiplicity*.

(2) *If (π, V) is an admissible representation of G, the vectors in $V_{(K)}$ are smooth, and $V_{(K)}$ is stable under the Lie algebra $\mathfrak{g}$.*

Suppose (σ, U) to be an irreducible representation of K, ξ the corresponding projection operator in $C^\infty(K)$. The σ-component V_σ of V is the subspace of vectors fixed by $\pi(\xi)$. If (f_n) is chosen to be a Dirac sequence, then $\pi(f_n)v \to v$ for all v in V. The operators $\pi(\xi\pi(f)\pi(\xi)$ are therefore dense in the finite-dimensional space $\mathrm{End}(V_\sigma)$, hence make up the whole of it. But $\xi * f * \xi$ is also smooth and of compact support. Hence any vector v in V_σ may therefore be expressed as $\pi(f)$ for some f in $C_c^{infty}(G)$.

As for the second assertion, if v lies in V_σ then $\pi(\mathfrak{g})f$ lies in the direct sum of spaces V_τ as τ ranges over the irreducible components of $\mathfrak{g} \otimes \sigma$.

A **unitary** representation of G is one on a Hilbert space whose norm is G-invariant. One of the principal goals of representation theory is to classify representations that occur as discrete summands of arithmetic quotients. This is an extremely difficult task. But these representations are unitary, and classifying unitary representations of G is a first step towards carrying it out.

(3) *Any irreducible unitary representation of G is admissible.*

This is the most difficult of these claims, requiring serious analysis. It is usually skipped over in expositions of representation theory, probably because it is difficult and also because it is not often needed in practice. The most accessible reference seems to be §4.5 (on 'large' compact subgroups) of [Warner:1970].

In fact, admissible representations are ubiquitous. At any rate, we obtain from an admissible representation of G a representation of $\mathfrak{g}$ and K satisfying the following conditions:

(a) as a representation of K it is a direct sum of smooth irreducible representations (of finite dimension), each with finite multiplicity;
(b) the representations of $\mathfrak{k}$ associated to that as subalgebra of $\mathfrak{g}$ and that as Lie algebra of K are the same;
(c) for k in K, x in $\mathfrak{g}$

$$\pi(k)\pi(X)\pi^{-1}(k) = \pi(\mathrm{Ad}(k)X)\,.$$

Any representation of the pair $(\mathfrak{g}, K)$ satisfying these is called **admissible**.

I'll make some more remarks on this definition. It is plainly natural that we require that $\mathfrak{g}$ act. But why K? There are several reasons. (•) What we are really interested in are representations of G, or more precisely representations of $\mathfrak{g}$ that come from representations of G. But a representation of $\mathfrak{g}$ can only determine the representation of the connected component of G. The group K meets all components of G, and fixes this problem. For groups like $\mathrm{SL}_2(\mathbb{R})$, which are connected, this problem does not occur, but for the group $\mathrm{PGL}_2(\mathbb{R})$, with two components, it does. (•) One point of having K act is to distinguish $\mathrm{SL}_2(\mathbb{R})$ from other groups with the same Lie algebra. For example, $\mathrm{PGL}_2(\mathbb{R})$ has the same Lie algebra as $\mathrm{SL}_2(\mathbb{R})$, but the standard representation of its Lie algebra on $\mathbb{R}^2$ does not come from one of $\mathrm{PGL}_2(\mathbb{R})$. Requiring that K acts picks out a unique group in the isogeny class, since K contains the centre of G. (•) Any continuous representation of K, such as the restriction to K of a continuous representation of G, decomposes in some sense into a direct sum of irreducible finite-dimensional representations. This is not true of the Lie algebra $\mathfrak{k}$. So this requirement picks out from several possibilities the class of representations we want.

Admissible representations of $(\mathfrak{g}, K)$ are not generally semi-simple. For example, the space V of smooth functions on $\mathbb{P}_\mathbb{R}$ contains the constant functions, but they are not a G-stable summand of V.

If (π, V) is a smooth representation of G assigned a G-invariant Hermitian inner product, the corresponding equations of invariance for the Lie algebra are that

$$Xu \bullet v = -u \bullet Xv$$

for X in $\mathfrak{g}_\mathbb{R}$, or

$$Xu \bullet v = -u \bullet \overline{X}v$$

for X in the complex Lie algebra $\mathfrak{g}_\mathbb{C}$. An admissible $(\mathfrak{g}_\mathbb{C}, K)$-module is said to be unitary if there exists a positive definite Hermitian metric satisfying this condition. In some sense, unitary representations are by far the most important. At present, the major justification of representation theory is in its applications to automorphic forms, and the most interesting ones encountered are unitary.

Here is some more evidence that the definition of an admissible representation of $(\mathfrak{g}_{\mathbb{C}}, K)$ is reasonable:

(a) admissible representations of $(\mathfrak{g}, K)$ that are finite-dimensional are in fact representations of G;
(b) if the continuous representation (π, V) of G is admissible, the map taking each $(\mathfrak{g}, K)$-stable subspace $U \subseteq V_{(K)}$ to its closure in V is a bijection between $(\mathfrak{g}_{\mathbb{C}}, K)$-stable subspaces of $V_{(K)}$ and closed G-stable subspaces of V;
(c) every admissible representation of $(\mathfrak{g}, K)$ is $V_{(K)}$ for some continuous representation (π, V) of G;
(d) a unitary admissible representation of $(\mathfrak{g}_{\mathbb{C}}, K)$ is $V_{(K)}$ for some unitary representation of G;
(e) there is an exact functor from admissible representations of $(\mathfrak{g}, K)$ to those of G.

Item (a) is classical.

For (b), refer to Théorème 3.17 of [Borel:1972].

I am not sure what a good reference for (c) is. For irreducible representations, it is a consequence of a theorem of Harish-Chandra that every irreducible admissible $(\mathfrak{g}, L)$-module is a subquotient of a principal series representation. This also follows from the main result of [Beilinson-Bernstein:1982].

The first proof of (d) was probably by Harish-Chnadra, but I am not sure exactly where.

For (e), refer to [Casselman:1989] or [Bernstein-Krötz:2010].

Most of the rest of this essay will be concerned only with admissible representations (π, V) of $(\mathfrak{g}, K)$, although it wil be good to keep in mind that the motivation for studying them is that they arise from representations of G.

Curiously, although nowadays representations of $\mathrm{SL}_2(\mathbb{R})$ are seen mostly in the theory of automorphic forms, the subject was introduced by the physicist Valentine Bargmann, in pretty much the form we see here. I do not know what physics problem led to his investigation.

2. The Lie algebra

The Lie algebra of $G_{\mathbb{R}}$ is the vector space $\mathfrak{sl}_2(\mathbb{R})$ of all matrices in $M_2(\mathbb{R})$ with trace 0. It has Lie bracket $[X, Y] = XY - YX$. There are two useful bases of the complexified algebra $\mathfrak{g}_{\mathbb{C}}$, mirroring a duality that exists throughout representation theory.

The split basis. The simplest basis is this:

$$\begin{aligned} h &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \\ \nu_+ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \nu_- &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \end{aligned}$$

with defining relations

$$\begin{aligned} [h, \nu_\pm] &= \pm 2\, \nu_\pm \\ [\nu_+, \nu_-] &= h\,. \end{aligned}$$

If θ is the Cartan involution $X \mapsto -{}^tX$ then $\nu_+^\theta = -\nu_-$. For this reason, sometimes in the literature $-\nu_-$ is used as part of a standard basis instead of ν_-.

The compact basis. This is a basis which is useful when we want to do calculations involving K. The first element of the new basis is

$$\kappa = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which spans the Lie algebra $\mathfrak{k}$ of K. The rest of the new basis is to be made up of eigenvectors of K. The group K is compact, and its characters are all complex, not real, so in order to decompose the adjoint action of $\mathfrak{k}$ on $\mathfrak{g}$ we so we must extend the real Lie algebra to the complex one. The $\mathfrak{k}_{\mathbb{R}}$-stable space complementary to $\mathfrak{k}_{\mathbb{R}}$ in $\mathfrak{g}_{\mathbb{R}}$ is that of symmetric matrices

$$\begin{bmatrix} a & b \\ b & -a \end{bmatrix} .$$

and its complexification decomposes into the sum of two conjugate eigenspaces for κ spanned by

$$\begin{aligned} x_+ &= \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \\ x_- &= \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \end{aligned}$$

with relations

$$\begin{aligned} [\kappa, x_\pm] &= \pm 2i\, x_\pm \\ [x_+, x_-] &= -4i\,\kappa\,. \end{aligned}$$

The Casimir operator. The centre of the universal enveloping algebra of $\mathfrak{sl}_2$ is the polynomial algebra generated by the Casimir operator

$$\Omega = \frac{h^2}{4} + \frac{\nu_+\nu_-}{2} + \frac{\nu_-\nu_+}{2}$$

which has alternate expressions

$$\begin{aligned} \Omega = &\quad \frac{h^2}{4} - \frac{h}{2} + \nu_+\nu_- \\ = &\quad \frac{h^2}{4} + \frac{h}{2} + \nu_-\nu_+ \\ = &-\frac{\kappa^2}{4} + \frac{x_-x_+}{8} + \frac{x_+x_-}{8} \\ = &-\frac{\kappa^2}{4} - \frac{\kappa i}{2} + \frac{x_-x_+}{4} \\ = &-\frac{\kappa^2}{4} + \frac{\kappa i}{2} + \frac{x_+x_-}{4}\,. \end{aligned}$$

There is good reason to expect any semi-simple Lie algebra to have such an element in the centre of its enveloping algebra, but for now I'll just point out that one can prove by explicit computation that Ω commutes with all of $\mathfrak{g}$ and is invariant even under the adjoint action of G. (I refer to the Appendix for more information.)

3. Classification of irreducible representations

Suppose (π, V) to be an irreducible admissible representation of $(\mathfrak{g}, K)$. Following Bargmann, I shall list all possibilities.

Step 1. I recall that V by definition is an algebraic direct sum of one-dimensional spaces on which K acts by some character ε^n. For a given n, the whole eigenspace for ε^n is finite-dimensional.

- *If v is an eigenvector for K with eigencharacter ε^n then $\pi(\kappa)v = niv$ and $\pi(\kappa)\pi(x_\pm) = (n \pm 2)i\pi(x_\pm)$.*

Proof. Since

$$\pi(\kappa)v = \frac{d}{dt}\bigg|_{t=0} e^{int}v = niv$$

and

$$\pi(\kappa)\pi(x_\pm)v = \pi(x_\pm)\pi(\kappa)v + \pi([\kappa, x_\pm])v\,.$$

Step 2. The Casimir Ω commutes with K and therefore preserves the finite-dimensional eigenspaces of K. It must possess in each one at least one eigenvector. If $v \neq 0$ and $\pi(\Omega)v = \gamma v$ then since Ω commutes with $\mathfrak{g}$ the space of all eigenvectors for $\pi(\Omega)$ with eigenvalue γ is stable under $\mathfrak{g}$ and K. Since π is irreducible:

- *The Casimir operator acts as the scalar γ on all of V.*

Step 3. Suppose $v_n \neq 0$ to be an eigenvector for K, so that $\pi(\kappa)v_n = niv_n$. For $k > 0$ set

$$\begin{aligned} v_{n+2k} &= \pi(x_+^k)v_n \\ v_{n-2k} &= \pi(x_-^k)v_n \end{aligned}$$

so that $\pi(\kappa)v_m = mi\, v_m$ for all m.

Note that v_m is defined only for m of the same parity as n.

- *The space V is that spanned by all the v_m.*

Proof. Since π is irreducible, it suffices to prove that the space spanned by the v_m is stable under $\mathfrak{g}$.

Suppose that $\pi(\Omega) = \gamma I$. Since each v_m is an eigenvector for κ, I must show that V is stable under $x_\pm$. Let $V_{\leq n}$ be the space spanned by the v_m with $m \leq n$, and similarly for $V_{\geq n}$. Since $\pi(x_+)V_{\geq n} \subseteq V_{>n}$ and $\pi(x_-)V_{\leq n} \subseteq V_{<n}$, the claim will follow from these two others: (i) $\pi(x_+)V_{<n} \subseteq V_{\leq n}$; (ii) $\pi(x_-)V_{>n} \subseteq V_{\geq n}$.

Recall that

$$\begin{aligned} x_+x_- &= 4\Omega + \kappa^2 - 2\kappa i \\ x_-x_+ &= 4\Omega + \kappa^2 + 2\kappa i\,. \end{aligned}$$

Then:

$$\begin{aligned} v_{n-2k-2} &= \pi(x_-)v_{n-2k} \\ \pi(x_+)v_{n-2k-2} &= \pi(x_+x_-)v_{n-2k} \\ &= (4\gamma - m^2 + 2m)v_{n-2k} \\ v_{n+2k+2} &= \pi(x_+)v_{n+2k} \\ \pi(x_-)v_{n+2k+2} &= \pi(x_+x_-)v_{n+2k} \\ &= (4\gamma - m^2 - 2m)v_{n+2k}\,. \end{aligned}$$

This concludes the proof.

Step 4. The choice of v_n at the beginning of our argument was arbitrary. As a consequence of the previous step:

- Whenever $v_\ell \neq 0$ and $\ell \leq m$, say $m = ell + 2k$, then v_m is a scalar multiple of $\pi(x_+^k)v_\ell$,

Step 5. As a consequence of the step two back, the dimension of the eigenspace for each ε^n is at most one. It may very well be zero—that is to say, some of the v_n might be 0.

- *The set of m with $v_m \neq 0$— up to a factor i the spectrum of $\pi(\kappa)$—is an interval, possibly infinite, in the subset of $\mathbb{Z}$ of some given parity.*

I'll call such an interval a *parity interval*.

It follows from the previous step that if $\ell \le m = \ell + 2k$ and both $v_\ell \neq 0$ and $v_m \neq 0$ that v_m is a non-zero multiple of $\pi(x_+^k v_\ell$. Hence none of the $\pi(x_+^j)v_\ell$ for $0 \le j \le k$ vanish.

Step 6. If $v_m \neq 0$ and $\pi(x_-)v_m = 0$ then $\gamma = m^2/4 - m/2$. This is because

$$\pi(x_+x_-)v_m = (4\gamma - m^2 + 2m)v_m \, .$$

Conversely, suppose $\gamma = m^2/4 - m/2$. If $v = \pi(x_-)v_m$, then $\pi(x_+)\pi(x_-)v_m = \pi(x_+)v = 0$. The subspace spanned by the $\pi(x_-^k)v$ is $\mathfrak{g}$-stable, but because π is irreducible it must be trivial. So $v = 0$. Similarly for $\pi(x_+)v_m$.

- *Suppose $v_m \neq 0$. Then $\pi(x_-)v_m = 0$ if and only if $\gamma = m^2/4 - m/2$, and $\pi(x_+)v_m = 0$ if and only if $\gamma = m^2/4 + m/2$.*

Step 7. There now exist four possibilities:

(a) *The spectrum of κ is a finite parity interval.*

In this case, choose $v_n \neq 0$ with $\pi(x_+)v_n = 0$. Then $\gamma = n^2/4 - n/2$ and V is spanned by the v_{n-2k}. These must eventually be 0, so for some m we have $v_m \neq 0$ with $\pi(x_-)v_m = 0$. But then also $\gamma = m^2/4 + m/2$, which implies that $m = -n$. Therefore $n \ge 0$, the space V has dimension $n+1$, and it is spanned by v_n, v_{n-2}, ..., v_{-n}. I call this representation FD_n.

(b) *The spectrum of κ is some parity subinterval* $(-\infty, n]$. That it is to say, some $v_n \neq 0$ is annihilated by x_+ and V is the infinite-dimensional span of the $\pi(x_-^k)v_n$.

In particular, none of these is annihilated by x_-.

Say $v_n \neq 0$ and $\pi(x_+)v_n = 0$. Here $\gamma = n^2/4 - n/2$. If $n \ge 0$ then v_{-n} would be annihilated by x_-, so $n < 0$. The space V is spanned by the non-zero vectors v_{n-2k} with $k \ge 0$. I call this representation DS_n^-, for reasons that will appear later.

(c) *The spectrum of κ is some parity subinterval* $[n, \infty)$. That is to say, some $v_m \neq 0$ is annihilated by x_- but none is annihilated by x_+.

Say $v_n \neq 0$ and $\pi(x_-)v_n = 0$. Here $\gamma = n^2/4 - n/2$. If $n \le 0$ then v_{-n} would be annihilated by x_+, so $n > 0$. The space V is spanned by the non-zero vectors v_{n+2k} with $k \ge 0$. I call this representation DS_n^+.

(d) *The spectrum of κ is some parity subinterval* $(-\infty, \infty)$. That is to say, no $v_m \neq 0$ is annihilated by either x_+ or x_-.

In this case $v_n \neq 0$ for all n. We cannot have $\gamma = ((m+1)^2 - 1)/4$ for any m of the same parity of the K-weights occurring. Furthermore, $\pi(x_+)$ is both injective and surjective. We therefore choose a new basis (v_m) such that

$$\pi(x_+)v_m = v_{m+2}$$

for all m. And then

$$\begin{aligned}\pi(x_-)v_m &= \pi(x_-x_+)v_{m-2} \\ &= 4\gamma - (m-2)^2 - 2(m-2))v_{m-2} \\ &= (4\gamma - m^2 - 2m)v_{m-2} \, .\end{aligned}$$

I call this representation $\mathrm{PS}_{\gamma,n}$. The isomorphism class depends only on the parity of n.

Step 8. I have shown that any irreducible admissible representation of $(\mathfrak{g}, K)$ falls into one of the types listed above. This is a statement of uniqueness. There now remain several more questions to be answered. *Existence—do we in fact get in every case a representation of* $(\mathfrak{g}, K)$ *with the given parameters? Do these representations come from representations of* G*? Which of them are unitary?*

As for existence, it can be shown directly, without much trouble, that in every case the implicit formulas above define a representation of $(\mathfrak{g}, K)$, but I'll show this in a later section by a different method. I'll answer the second question at the same time. I'll answer the third in a moment.

4. Duals

Suppose (π, V) to be an admissible $(\mathfrak{g}, K)$-module. Its **admissible dual** is the subspace of K-finite vectors in the linear dual $\widehat{V}$ of V. The group K acts on it so as to make the pairing K-invariant:

$$\langle \pi(k)\widehat{v}, \pi(k)v\rangle = \langle \widehat{v}, v\rangle, \quad \langle \pi(k)\widehat{v}, v\rangle = \langle \widehat{v}, \pi(k^{-1})v\rangle \,,$$

and the Lie algebra $\mathfrak{g}$ acts on it according to the specification that matches G-invariance if G were to act:

$$\langle \pi(X)\widehat{v}, v\rangle = -\langle \widehat{v}, \pi(X)v\rangle \,.$$

If K acts as the character ε on a space U, it acts as ε^{-1} on its dual. So if the representation of K on V is the sum of characters ε^k, then that on $\widehat{V}$ is the sum of the ε^{-k}. Thus we can read off immediately that the dual of FD_n is FD_n (this is because the longest element in its Weyl group happens to take a weight to its negative), and the dual of $\mathrm{DS}_n^{\pm}$ is $\mathrm{DS}_n^{\mp}$. What about $\mathrm{PS}_{\gamma,n}$?

[casimir-dual] **4.1. Lemma.** *If* $\pi(\Omega) = \gamma \cdot I$ *then* $\widehat{\pi}(\Omega) = \gamma \cdot I$.

Proof. Just use the formula

$$\Omega = \Omega = \frac{h^2}{4} + \frac{\nu_+\nu_-}{2} + \frac{\nu_-\nu_+}{2}$$

and the definition of the dual representation. ∎

So $\mathrm{PS}_{\gamma,n}$ is isomorphic to its dual.

The **Hermitian dual** of a (complex) vector space V is the linear space of all conjugate-linear functions on V, the maps

$$f \colon V \to \mathbb{C}$$

such that $f(cv) = \overline{c} f(v)$. If K acts on V as the sum of ε^k, on its Hermitian dual it also acts like that—in effect, because K is compact, one can impose on V a K-invariant Hermitian norm. I'll write Hermitian pairings as $u \bullet v$. For this the action of $\mathfrak{g}_{\mathbb{C}}$ satisfies

$$\pi(X)u \bullet v = -u \bullet \pi(\overline{X})v \,.$$

The Hermitian dual of FD_n is itself, the Hermitian dual of $\mathrm{DS}_n^{\pm}$ is itself, and the Hermitian dual of $\mathrm{PS}_{\gamma,n}$ is $\mathrm{PS}_{\overline{\gamma},n}$. A unitary representation is by definition isomorphic to its Hermitian dual, so a necessary condition that $\mathrm{PS}_{\gamma,n}$ be unitary is that γ be real. It is not sufficient, since the Hermitian form this guarantees might not be positive definite.

5. Unitarity

Which of the representations above are unitary? I recall that (π, V) is isomorphic to its Hermitian dual if if and only if there exists on V an Hermitian form which is $\mathfrak{g}_\mathbb{R}$-invariant. For $\mathrm{SL}_2(\mathbb{R})$ this translates to the conditions

$$\pi(\kappa)u \bullet u = -u \bullet \pi(\kappa)u$$
$$\pi(x_+)u \bullet u = -u \bullet \pi(x_-)u$$

for all u in V. It is unitary if this form is positive definite:

$$u \bullet u > 0 \text{ unless } u = 0\,.$$

It suffices to construct the Hermitian form on eigenvectors of κ. We know from the previous section that π is its own Hermitian dual for all FD_n, $\mathrm{DS}_n^{\pm}$, and for $\mathrm{PS}_{\gamma,n}$ when γ is real. I'll summarize without proofs what happens in the first three cases: (1) FD_n is unitary if and only if $n = 0$ (the trivial representation). (2–3) Given a lowest weight vector v_n of DS_n^+, there exists a unique invariant Hermitian norm on DS_n^+ such that $v_n \bullet v_n = 1$. Hence DS_n^+ is always unitary. Similarly DS_n^-. I leave verification of these claims as an exercise. The representations $\mathrm{PS}_{\gamma,n}$ are more interesting. Let's first define the Hermitian form when γ is real.

The necessary and sufficient condition for the construction of the Hermitian form $v \bullet v$ is that

$$\pi(x_+)v_m \bullet v_{m+2} = -v_m \bullet \pi(x_-)v_{m+2}$$

for all m. This translates to

$$v_{m+2} \bullet v_{m+2} = -(4\gamma - m^2 - 2m)v_m \bullet v_m\,.$$

So we see explicitly that 4γ must be real. But if the form is to be unitary, we require in addition that

$$4\gamma < (m+1)^2 - 1$$

for all m, or $\gamma < -1/4$ in the case of odd m and $\gamma < 0$ in the case of even m. This conclusion seems a bit arbitrary, but we shall see later some of the reasons behind it.

6. *Appendix.* The Casimir element

The **Killing form** of a Lie algebra is the inner product

$$K(X,Y) = \mathrm{trace}\,(\mathrm{ad}_X \mathrm{ad}_Y)\,.$$

If G is a Lie group, the Killing form of its Lie algebra is G-invariant. By definition, it is non-degenerate for semi-simple Lie algebras. For $\mathfrak{g} = \mathfrak{sl}_2$ with basis h, $e_\pm$ the matrix of the Killing form is

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & -4 \\ 0 & -4 & 0 \end{bmatrix}.$$

The Killing form gives an isomorphism of $\mathfrak{g}$ with its linear dual $\widehat{\mathfrak{g}}$, and thus of $\widehat{\mathfrak{g}} \otimes \widehat{\mathfrak{g}}$ and $\widehat{\mathfrak{g}} \otimes \mathfrak{g}$, which may be identified with $\mathrm{End}(\mathfrak{g})$. The Casimir element Ω is the image of the Killing form itself. Explicitly

$$\Omega = (1/2) \sum X_i X_i^\vee$$

with the sum over a basis X_i and the corresponding dual basis elements $X_i^\vee$.

It lies in the centre of $U(\mathfrak{g})$ precisely because the Killing form is $\mathfrak{g}$-invariant. For $\mathfrak{sl}_2$ it is $(1/4)h^2 - (1/2)e_+e_- - (1/2)e_-e_+$, which may be manipulated to give other expressions.

The group G acts on the upper half plane $\mathcal{H}$ ny non-Euclidean isometries, and in this action the Casimir becomes the Laplacian.

Part II. The principal series

7. Vector bundles

Do the representations of $(\mathfrak{g}, K)$ that we have constructed come from representations of G?

In the introduction I discussed the representation of G on $C^\infty(\mathbb{P})$. It is just one in an analytic family. The others are on spaces of sections of real analytic line bundles. This is such a common notion that I shall explain it here, at least in so far as it concerns us. There are two versions—one real analytic, the other complex analytic. Both are important in representation theory although the role of the bundle itself, as opposed to its space of sections, is often hidden by the mechanism of induction of representations.

Suppose G to be a Lie group, H a closed subgroup. For the moment, I'll take $G = \mathrm{SO}(3)$, $H = \mathrm{SO}(2)$. The group G acts by rotations on the two-sphere $S = \mathbb{S}^2$ in $\mathbb{R}^3$, and H can be identified with the subgroup of elements of G fixing the north pole $P = (0, 0, 1)$. Elements of H amount to rotations in the (x, y) plane around the z-axis. The group G acts transitively on S, so the map taking g to $g(P)$ identifies G/H with S.

The group G also acts on tangent vectors on the sphere—the element g takes a vector v at a point x to a tangent vector $g_*(v)$ at the point $g(x)$. For example, H just rotates tangent vectors at P. This behaves very nicely with respect to multiplication on the group:

$$g_*h_*(v) = g_*(h_*(v)) \,.$$

Let T be the tangent space at P. If v is a tangent at $g(x)$, then $g_*^{-1}(v)$ is a tangent vector at P. If we are given a smoothly varying vector field on S—i.e. a tangent vector v_x at every point of the sphere, varying smoothly with x—then we get a smooth function Θ from G to T by setting

$$\Theta(g) = g_*^{-1}(v_{g(P)}) \,.$$

Recall that $H = \mathrm{SO}(2)$ fixes P and therefore acts by rotation on T. This action is just h_*, which takes T to itself. The function $\Theta(g)$ satisfies the equation

$$\begin{aligned}\Theta(gh) &= h_*^{-1}g_*^{-1}v_{gh(P)} \\ &= h_*^{-1}g_*^{-1}v_{g(P)} \\ &= h_*^{-1}g_*^{-1}v_{g(P)} \\ &= h_*^{-1}(\Theta(g)) \,.\end{aligned}$$

Conversely, any smooth function $\Theta\colon G \to T$ satisfying this condition gives rise to a smoothly varying vector field on S.

The space T_S of all tangent vectors on the sphere is a **vector bundle** on it. To be precise, every tangent vector is a pair (x, v) where x is on S and v is a tangent vector at x. The map taking (x, v) to x is a projection onto S, and the inverse image is the tangent space at x. There is no canonical way to choose coordinates on S, or to identify the tangent space at one point with that at another. But if we choose a coordinate system in the neighbourhood of a point x we may identify the local tangent spaces with copies of $\mathbb{R}^2$. Therefore locally in the neighbourhood of every point the tangent bundle is a product of an open subset of S with $\mathbb{R}^2$. These are the defining properties of a **vector bundle**.

> *A vector bundle of dimension n over a smooth manifold M is a smooth manifold B together with a projection $\pi\colon B \to M$, satisfying the condition that locally on M the space B and the projection π possess the structure of a product of a neighbourhood U and $\mathbb{R}^n$, together with projection onto the first factor.*

A **section** of the bundle is a function $s\colon M \to B$ such that $\pi(s(x)) = x$. For one example, smooth functions on a manifold are sections of the trivial bundle whose fibre at every point is just $\mathbb{C}$. For another, a vector field on S amounts to a section of its tangent bundle. A vector bundle is said to be trivial if it is isomorphic to a product of M and some fibre. One reason vector bundles are interesting is that they can be highly non-trivial. For example, the tangent bundle over S is definitely not trivial because every vector field on S vanishes somewhere, whereas if were trivial there would be lots of non-vanishing fields. In other words, vector bundles can be *topologically* interesting.

Now let $H \subseteq G$ be arbitrary. If (σ, U) is a smooth representation of H, we can define the **associated vector bundle** on $M = G/H$ to be the space B of all pairs (g, u) is $G \times U$ modulo the equivalence relation $(gh, \sigma^{-1}(h)u) \sim (g, u)$. This maps onto G/H and the fibre at the coset H is isomorphic to U. The space of all sections of B over M is then isomorphic to the space

$$\Gamma(M, B) = \{f \colon G \to U \mid f(gh) = \sigma^{-1}(h)f(g) \text{ for all } g \in G, h \in H\}.$$

This is itself a vector space, together with a natural representation of G:

$$L_g s(x) = s(g^{-1}(x)).$$

This representation is that **induced** by σ from H to G. In most situations it is not really necessary to consider the vector bundle itself, but just its space of sections. In some, however, knowing about the vector bundle is useful.

If G and H are complex groups and (σ, U) is a complex-analytic representation of H, one can define a complex structure on the vector bundle. In this case the space of holomorphic sections may be identified with the space of all complex analytic maps f from G to U satisfying the equations $f(gh) = \sigma^{-1}(h)f(g)$.

If the group G acts transitively on M, B is said to be a **homogeneous** bundle if G acts on it compatibly with the action on M. If x is a point of M, the isotropy subgroup G_x acts on the fibre U_x at x. The bundle B is then isomorphic to the bundle associated to G_x and the representation on U_x.

One example of topological interest will occur in the next section. Let sgn be the character of A taking a to $\mathrm{sgn}(a)$. This lifts to a character of P, and therefore one can construct the associated bundle on $\mathbb{P} = P\backslash G$. Now $\mathbb{P}$ is a circle, and this vector bundle is topologically a Möbius strip with this circle running down its middle.

One may define vector bundles solely in terms of the **sheaf** of their local sections. This turns out to be a surprisingly valuable idea, but I postpone explaining more about it until it will be immediately useful.

In the next section I will follow a different convention than the one I follow here. Here, the group acts on spaces on the left, so if G acts transitively with isotropy group G_x the space is isomorphic to G/G_x. In the next section, it will act on the right, so the space is now $G_x\backslash G$. Sections of induced vector bundle then becomes functions $F \colon G \to U$ such that $F(hg) = \sigma(h)F(g)$. Topologists generally use the convention I follow in this section, but analysts the one in the next section. Neither group is irrational—the intent of both is to avoid cumbersome notation. Topologists are interested in actions of groups on spaces, analysts in actions on functions.

One important line bundle on a manifold is that of **one-densities**. If the manifold is orientable, these are the same as differential forms of degree equal to the dimension of the manifold, but otherwise not. The point of one-densities is that one may integrate them canonically. (Look at the appendix to see more about one-densities on homogeneous spaces.)

8. The principal series

In this essay, a **character** of any locally compact group H is a continuous homomorphism from H to $\mathbb{C}^\times$.

[chars-of-Rx] **8.1. Lemma.** *Any character of $\mathbb{R}^\times_{>0}$ is of the form $x \mapsto x^s$ for some unique s in $\mathbb{C}$.*

Proof. Working locally and applying logarithms, this reduces to the claim that any continuous additive function f from $\mathbb{R}$ to iself is linear. For this, say $\alpha = f(1)$. It is easy to see that $f(m/n) = (m/n)\alpha$ for all integers $m, n \neq 0$, from the claim follows by continuity. ∎

The multiplicative characters of A are therefore all of the form

$$\chi = \chi_s\,\mathrm{sgn}^n\colon \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \longmapsto |t|^s \mathrm{sgn}^n(t)$$

with the dependence on n only through its parity, even or odd. Each of these lifts to a character of P, taking $p = an$ to $\chi(a)$. Since every ν in N is a commutator in P, every one of its characters arises in this way. In particular, we have the **modulus character**

$$\delta = \delta_P\colon\ p \longmapsto \big|\det\nolimits_{\mathrm{Ad}_{\mathfrak{n}}}(p)\big|, \qquad \begin{bmatrix} t & x \\ 0 & 1/t \end{bmatrix} \longmapsto t^2 .$$

This plays a role in duality for $G = \mathrm{SL}_2(\mathbb{R})$. There is no G-invariant measure on $P\backslash G$, but instead the things one can integrate are one-densities, which are for this P and this G the same as one-forms on $P\backslash G$. The space of smooth one-densities on $P\backslash G$ may be identified with the space of functions

$$\big\{ f\colon\ G \longrightarrow \mathbb{C} \,\big|\, f(pg) = \delta(p) f(g) \text{ for all } p \in P, g \in G \big\} .$$

This is not a canonical identification, but it is unique up to a scalar multiplication. It is common to realize this integration in a simple manner.

[iwasawa-factorization] **8.2. Lemma.** *Every g in G can be factored uniquely as $g = nak$ with n in N, a in $|A|$, k in K.*

Here by $|A|$ I mean the connected component of A.

Proof. This is probably familiar, but I'll need the explicit factorization eventually. The key to deriving it is to let G act on the right on $\mathbb{P}^1(\mathbb{R})$ considered as non-zero row vectors modulo non-zero scalars. The group P is the isotropy group of the line $\langle\langle 0, 1\rangle\rangle$, so it must be shown that K acts transitively. The final result is:

[iwasawa-f] **(8.3)**
$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1/r & (ac+bd)/r \\ 0 & r \end{bmatrix} \begin{bmatrix} d/r & -c/r \\ c/r & d/r \end{bmatrix} \quad \big(r = \sqrt{c^2+d^2}\big) . \qquad ∎$$

Given this, we may choose integration on $P\backslash G$ to be integration over K:

$$\int_{P\backslash G} f(x)\, dx = \int_K f(k)\, dk .$$

For any character χ of P define the smooth representation induced by it to be the right regular representation of G on

$$\mathrm{Ind}^\infty(\chi) = \mathrm{Ind}^\infty(\chi\,|\,P, G) = \{ f \in C^\infty(G) \,|\, f(pg) = \chi\delta^{1/2}(p) f(g) \text{ for all } p \in P, g \in G\} .$$

This is, as you will see by comparing this definition with one in the previous section, the space of sections of a certain homogeneous line bundle over $\mathbb{P}$. Whether it is topologically trivial or not depends on the parity of the character χ.

For $\chi = \delta^{-1/2}$ we get the space $C^\infty(P\backslash G)$ of smooth functions on $P\backslash G \cong \mathbb{P}_\mathbb{R}$, and for $\chi = \delta^{1/2}$ we get the space of smooth one-densities on $P\backslash G$, the linear dual of the first. If f lies in $\mathrm{Ind}^\infty(\chi)$ and φ in $\mathrm{Ind}^\infty(\chi^{-1})$ the product will lie in $\mathrm{Ind}^\infty(\delta_P^{1/2})$ and the pairing

$$\langle f, \varphi\rangle = \int_{P\backslash G} f(x)\varphi(x)\,dx$$

determines an isomorphism of one space, at least formally, with part of the dual of the other. In particular, if $|\chi| = 1$ so $\chi^{-1} = \overline{\chi}$ the induced representation is unitary—i.e. possesses a G-invariant positive definite Hermitian form. As we'll see in a moment, on these the Casimir acts as a real number in the range $(-\infty, -1/4]$. This accounts for some of the results in a previous section about unitary representations, but not all.

Because $G = PK$ and $K \cap P = \{\pm 1\}$, restriction to K induces a K-isomorphism of $\mathrm{Ind}^\infty(\chi)$ with the space of smooth functions f on K such that $f(-k) = \chi(-1)f(k)$. The subspace $\mathrm{Ind}(\chi)$ spanned by eigenfunctions of K is thus a direct sum $\oplus\,\varepsilon^{2k}$ if $n = 0$ and $\oplus\,\varepsilon^{2k+1}$ if $n = 1$. The functions

$$\varepsilon_s^n(g) = \chi_s \mathrm{sgn}^n \delta^{1/2}(p)\varepsilon^n(k) \quad (g = pk)$$

form a basis of the K-finite functions in $\mathrm{Ind}(\chi_s\mathrm{sgn}^n)$. In terms of this basis, how does $(\mathfrak{g}, K)$ act?

[PSC] 8.4. Proposition. *We have*

$$\begin{aligned} R_\kappa\,\varepsilon_s^n &= ni\,\varepsilon_{s,n} \\ R_{x_+}\,\varepsilon_s^n &= (s+1+n)\,\varepsilon_s^{n+2} \\ R_{x_-}\,\varepsilon_s^n &= (s+1-n)\,\varepsilon_s^{n+2}\,. \end{aligned}$$

Proof. The basic formula in this sort of computation is the equation

$$[R_X f](g) = [L_{-gXg^{-1}}f](g)\,. \qquad \textbf{[L-R] (8.5)}$$

This follows from the trivial observation that

$$\frac{f(g\cdot\exp(tX)) - 1}{t} = \frac{f(\exp(t[\mathrm{Ad}(g)](X)\cdot g) - f(g)}{t} \quad (\text{informally, } g.X = gXg^{-1}\cdot g)\,.$$

In our case $f = \varepsilon_s^n$ is the only K-eigenfunction f in the representation for ε^n with $f(1) = 1$, and we know that $\pi(x_+)$ changes weights by $2i$, so we just have to evaluate $R_{x_+}\varepsilon_s^n$ at 1. Since

$$x_+ = \alpha - \kappa i - 2i\nu_+$$

we have

$$[R_{x_+}\varepsilon_s^n](1) = \chi_s\mathrm{sgn}^n(\alpha) + 1 - i(ni) = (s+1+n)\,.$$

The formula for x_- is similar. ∎

[casimir-ps] 8.6. Proposition. *The Casimir element acts on* $\mathrm{Ind}(\chi_s\mathrm{sgn}^n)$ *as* $(s^2-1)/4$.

Proof. This is a corollary of the previous result, but is better seen more directly. Again, we just have to evaluate $(R_\Omega f)(1)$ for f in $\mathrm{Ind}(\chi_s\mathrm{sgn}^n)$:

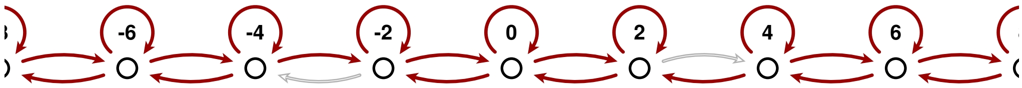
$$(R_\Omega f)(1) = (L_h^2/4 - L_h/2 + L_{\nu_+}R_{\nu_-})f(1) = (s+1)^2/4 - (s+1)/2 = s^2/4 - 1/4\,. \ ∎$$

These formulas have consequences for irreducibility. Let's look at one example, the representation $\pi = \mathrm{Ind}(\chi_s\mathrm{sgn}^n)$ with $n = 0$, $s = -3$. We have in this case

$$\begin{aligned} R_{x_+}\,\varepsilon_s^{2k} &= (-2+2k)\,\varepsilon_s^{2k+2} \\ R_{x_-}\,\varepsilon_s^{2k} &= (-2-2k)\,\varepsilon_s^{2k-2} \end{aligned}$$

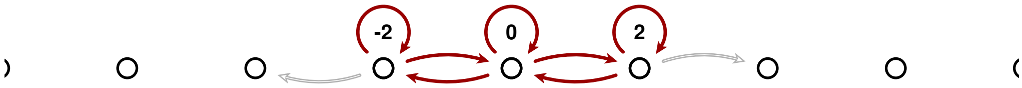
We can make a labeled graph out of these data: one node μ_k for each even integer $2k$, an edge from μ_k to μ_{k+1} when x_+ does not annihilate ε_s^{2k}, and one from μ_k to μ_{k-1} when x_- does not annihilate ε_s^{2k}. The only edges missing are for x_+ when $n = 1$ and for x_- for $n = -1$.

The graph we get is this:



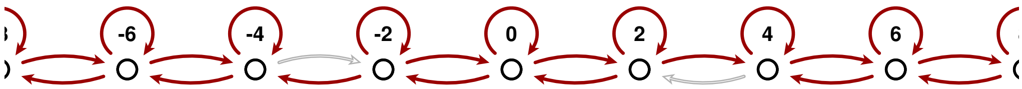
images/ps1.eps

You can read off from this graph that the subgraph



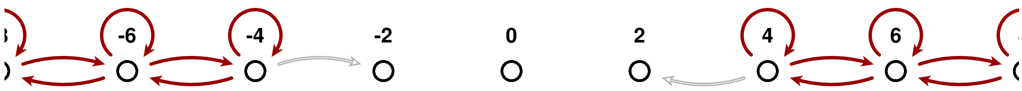
images/ps2.eps

represents an irreducible representation of dimension 3 embedded in π. The dual of π is $\widetilde{\pi} = \mathrm{Ind}(\chi_{3,0})$, whose graph looks like this:



images/ps3.eps

You can read off from this graph that the subgraph



images/ps4.eps

represents the sum of two irreducible representations contained in $\widetilde{\pi}$, each of infinite dimension. We have similar pictures whenever s is an integer of the same parity as $n-1$. In all other cases, the graph corresponding to $\mathrm{Ind}(\chi_s \mathrm{sgn}^n)$ connects all nodes in both directions. This leads to:

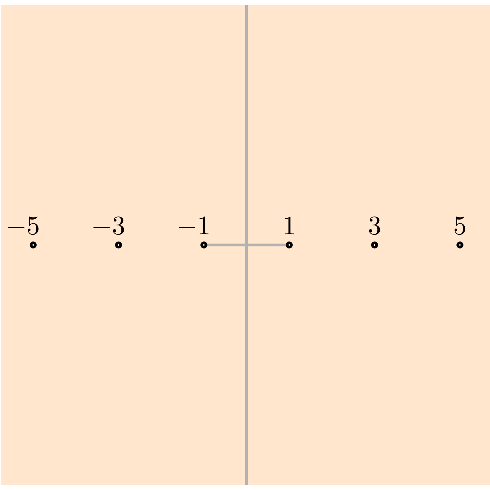
[irr-ps] **8.7. Proposition.** *The representation* $\mathrm{Ind}(\chi_s \mathrm{sgn}^n)$ *is irreducible except when* s *is an integer* m *of the same parity as* $n - 1$*. In the exceptional cases:*

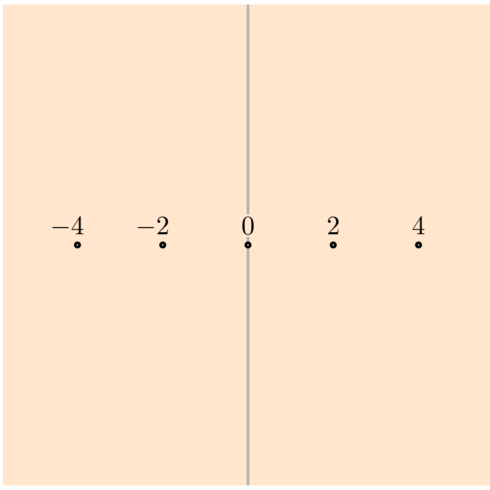
(a) *if* $s = -m \le -1$*, then* $\mathrm{Ind}(\chi_s \mathrm{sgn}^{m-1})$ *contains the unique irreducible representation of dimension* m*. The quotient is the direct sum of two infinite dimensional representation,* D^-_{m+1} *with weights* $-(m+1) - 2k$ *for* $k \ge 0$ *and* D^+_{m+1} *of weights* $m + 1 + 2k$ *with* $k \ge 1$*;*

(b) *if* $s = 0$ *and* $n = 1$ *then* $\mathrm{Ind}(\chi_s \mathrm{sgn}^n)$ *itself is the direct sum of two infinite dimensional representation,* D^-_1 *with weights* $-2k - 1$ *for* $k \ge 0$ *and* D^+_1 *of weights* $2k + 1$ *with* $k \ge 0$*;*

(c) *for* $s = m$ *with* $m \ge 1$ *we have the decompositions dual to these.*

In these diagrams the points of irreducibility and the unitary parameters are shown:





9. The Gamma function

In the next section I shall try to make clearer why $\mathrm{Ind}(\chi_s \mathrm{sgn}^n)$ and $\mathrm{Ind}(\chi_{-s} \mathrm{sgn}^n)$ are generically isomorphic representations of $(\mathfrak{g}_{\mathbb{C}}, K)$. For this, we'll need to know some explicit integration formulas involving the Gamma function. In this section I'll review what will be needed. My principal reference for this brief account is Chapter VIII of [Schwartz:1965].

The Gamma function interpolates the factorial function $n!$ on integers to a meromorphic function on all of $\mathbb{C}$. The definition of $\Gamma(z)$ for $\mathrm{RE}(z) > 0$ is

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \,.$$

Integration by parts shows that

$$\Gamma(z+1) = z\Gamma(z)\,.$$

It is easy to calculate $\Gamma(1) = 1$, and then from this you can see that $\Gamma(n+1) = n!$ for all positive integers n. This functional equation also allows us to extend the definition to all of $\mathbb{C}$ with simple poles at the non-positive integers:

$$\Gamma(z) = \frac{\Gamma(z+1)}{z} = \frac{\Gamma(z+2)}{z(z+1)} = \frac{\Gamma(z+3)}{z(z+1)(z+2)} = \cdots$$

Many definite integrals can be evaluated in terms of $\Gamma(z)$. For one thing, a change of variables $t = s^2$ in the definition of $\Gamma(z)$ gives

[z2-gamma] **(9.1)**
$$\Gamma(z) = 2\int_0^\infty e^{-s^2} s^{2z-1}\, ds \,.$$

nect, define the Bets function

$$B(u,v) = 2\int_0^{\pi/2} \cos^{2u-1}(\theta)\, \sin^{2v-1}(\theta)\, d\theta \,.$$

[beta-formula] **9.2. Proposition.** *We have*

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}\,.$$

♡ [z2-gamma] *Proof.* Start with (9.1). Moving to two dimensions and switching to polar coordinates:

$$\begin{aligned}\Gamma(u)\Gamma(v) &= 4\int\int_{s\geq 0, t\geq 0} e^{-s^2-t^2} s^{2u-1} t^{2v-1}\, ds\, dt \\ &= 4\int\int_{r\geq 0, 0\leq\theta\leq\pi/2} e^{-r^2} r^{2(u+v)-1} \cos^{2u-1}\theta\, \sin^{2v-1}\theta\, dr\, d\theta \\ &= 4\int_{r\geq 0} e^{-r^2} r^{2(u+v)-1}\, dr \int_0^{\pi/2} \cos^{2u-1}(\theta)\, \sin^{2v-1}(\theta)\, d\theta \\ &= \Gamma(u+v)B(u,v)\,.\end{aligned}$$

[gamma-talphabeta] **9.3. Corollary.** *We have*

$$\int_0^\infty \frac{t^\alpha}{(1+t^2)^\beta}\, dt = \frac{1}{2}\, \frac{\Gamma\Big(\dfrac{\alpha+1}{2}\Big)\Gamma\Big(\beta-\dfrac{\alpha+1}{2}\Big)}{\Gamma(\beta)}$$

Proof. In the formula for the Beta function, change variables to $t = \tan(\theta)$ to get

$$\begin{aligned}\theta &= \arctan(t) \\ d\theta &= dt/(1+t^2) \\ \cos(\theta) &= 1/\sqrt{1+t^2} \\ \sin(\theta) &= t/\sqrt{1+t^2}\end{aligned}$$

leading to

$$\int_0^\infty \frac{t^\alpha}{(1+t^2)^\beta}\, dr = \frac{1}{2}\, \frac{\Gamma\Big(\dfrac{\alpha+1}{2}\Big)\Gamma\Big(\beta-\dfrac{\alpha+1}{2}\Big)}{\Gamma(\beta)}\,.$$ 0

In particular

$$\Gamma^2(1/2) = \int_{-\infty}^\infty \frac{dt}{1+t^2} = \pi, \quad \Gamma(1/2) = \sqrt{\pi}\,.$$

We shall require later a generalization of this, which I learned from [Garrett:2009].

Define

$$I_{\alpha,\beta} = \int_{\mathbb{R}} (1+ix)^{-\alpha}(1-ix)^{-\beta}\, dx\,.$$

Here I take $z^s = e^{s\log z}$ throughout the region $z \notin (-\infty, 0]$. The integral converges for $\mathrm{RE}(\alpha+\beta) > 1$ and is analytic in that region.

[iab] **9.4. Proposition.** *For* $\mathrm{RE}(\alpha+\beta) > 1$

$$I_{\alpha,\beta} = \frac{2^{2-\alpha-\beta}\,\pi\,\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}\,.$$

Proof. It suffices to prove this when $\alpha > 1$, $\beta > 1$ are both real. But when α and β are real, this integral is

$$\int_{\mathbb{R}} a(x)\overline{b(x)}\, dx \quad \text{with} \quad a(x) = (1+ix)^{-\alpha},\ b(x) = (1+ix)^{-\beta}\,.$$

What Garrett points out is that this can be evaluated by the Plancherel formula

$$\int_{\mathbb{R}} a(x)\overline{b(x)}\,dx = \int_{\mathbb{R}} \widehat{a}(y)\overline{\widehat{b}(y)}\,dx\,,$$

where $\widehat{a}$, $\widehat{b}$ are Fourier transforms. To make things definite, I should say that the Fourier transform here is defined by the formula

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x)e^{-2\pi ixy}\,dx\,,$$

so that the Fourier transform of $f(x) = (1+ix)^{-\alpha}$ is

$$\int_{\mathbb{R}} (1+ix)^{-\alpha}e^{-2\pi ixy}\,dx\,.$$

I do not see how to calculate this directly without some trouble, so I follow Garrett by grabbing the other end of the string.

We start by recalling the definition

$$\Gamma(s) = \int_0^\infty e^{-t}t^s\,\frac{dt}{t}\,.$$

Setting $t = yz$ for $z > 0$ this becomes

$$\Gamma(s)z^{-s} = \int_0^\infty e^{-yz}y^s\,\frac{dy}{y}\,.$$

Both sides of this equation are analytic for $\mathrm{RE}(z) > 0$, so it remains valid in the right hand complex plane, and in particular for $z = 1 + 2\pi ix$:

$$\Gamma(s)(1+2\pi ix)^{-s} = \int_0^\infty e^{-y(1+2\pi ix)}y^s\,\frac{dy}{y} = \int_0^\infty e^{-2\pi ixy}e^{-y}y^{s-1}\,dy\,.$$

This can be interpreted as saying that $(1+2\pi ix)^{-s}$ is the Fourier transform of the function

$$f(y) = \begin{cases} 0 & \text{if } y < 0 \\ e^{-y}y^{s-1}/\Gamma(s) & \text{otherwise.} \end{cases}$$

Therefore by the Plancherel formula

$$\begin{aligned}
\int_{\mathbb{R}} (1+ix)^{-\alpha}(1-ix)^{-\beta}\,dx &= 2\pi\int_{\mathbb{R}} (1+2\pi ix)^{-\alpha}(1-2\pi ix)^{-\beta}\,dx \\
&= \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty e^{-y}y^{\alpha-1}\cdot e^{-y}y^{\beta-1}\,dy \\
&= \frac{2\pi}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty e^{-2y}y^{\alpha+\beta-1}\,\frac{dy}{y} \\
&= \frac{2^{2-\alpha-\beta}\,\pi}{\Gamma(\alpha)\Gamma(\beta)}\int_0^\infty e^{-y}y^{\alpha+\beta-1}\,\frac{dy}{y} \\
&= \frac{2^{2-\alpha-\beta}\,\pi\,\Gamma(\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(\beta)}\,.
\end{aligned}$$

10. Frobenius reciprocity and its consequences

For generic s the two representations $\mathrm{Ind}(\chi_s \mathrm{sgn}^m)$ and $\mathrm{Ind}(\chi_{-s} \mathrm{sgn}^m)$ are isomorphic. This section will explain this apparent accident in a way that perhaps makes it clear that a similar phenomenon ought to arise also for other Lie groups.

Suppose $H \subseteq G$ to be finite groups. If (σ, U) is a representation of H, then the representation induced by σ from H to G is the right regular representation of G on the space

$$\mathrm{Ind}(\sigma \,|\, H, G) = \{f\colon\ G \longrightarrow U \,|\, f(hg) = \sigma(h) f(g) \text{ for all } h \in H, g \in G\}\,.$$

The statement of Frobenius reciprocity in this situation is that a given irreducible representation π of G occurs as a constituent of $\mathrm{Ind}(\sigma)$ as often as σ occurs in the restriction of π to H. But the best way to say this is more precise. Let Λ_1 be the map taking f in $\mathrm{Ind}(\sigma)$ to $f(1)$. It is an H-equivariant map from $\mathrm{Ind}(\sigma)$ to U. If (π, V) is a representation of G and F a G-equivariant map from V to $\mathrm{Ind}(\sigma)$, then the composition $\Lambda_1 \cdot F$ is an H-equivariant map from V to U. This gives us a map

$$\mathrm{Hom}_G\big(\pi, \mathrm{Ind}(\sigma)\big) \longrightarrow \mathrm{Hom}_H(\pi, \sigma)\,.$$

Frobenius reciprocity asserts that this is an isomorphism. The proof simply specifies the inverse—to F on the right hand side we associate the map on the left taking v to $g \mapsto F(\pi(g)v)$.

Something similar holds for the principal series of $(\mathfrak{g}, K)$, but because the group itself doesn't act on this space one has to be careful. First of all

$$\Lambda_1\colon\ \mathrm{Ind}(\chi \,|\, P, G) \longrightarrow \mathbb{C}, \quad f \longmapsto f(1)$$

is $(\mathfrak{p}, K \cap P)$-equivariant. If (π, V) is any admissible $(\mathfrak{g}, K)$-module and $F\colon V \to \mathrm{Ind}(\chi)$ a $(\mathfrak{g}, K)$-equivariant map, then the composition $\Lambda_1 \circ F$ is a $(\mathfrak{p}, K \cap P)$-equivariant map onto $\mathbb{C}_{\chi\delta^{1/2}}$.

[frobenius] **10.1. Proposition.** (Frobenius reciprocity) *For an admissible* $(\mathfrak{g}, K)$ *representation* (π, V)*, composition with* Λ_1 *induces a bijection of* $\mathrm{Hom}_{(\mathfrak{g},K)}(V, \mathrm{Ind}(\chi))$ *with* $\mathrm{Hom}_{(\mathfrak{p},K\cap P)}(V, \mathbb{C}_{\chi\delta^{1/2}})$.

Proof. The proof for the case of finite groups can't be used without some modification since the group G doesn't necessarily act on V. But K does, and $G = PK$, so we associate to F in $\mathrm{Hom}_{(\mathfrak{p},K\cap P)}(V, \mathbb{C}_{\chi\delta^{1/2}})$ the map from V to $\mathrm{Ind}(\chi)$ taking v to the function

$$F_v(pk) = \chi\delta^{1/2}(p) F(\pi(k)v)\,.$$

I leave it as an exercise to verify that this is inverse to composition with Λ_1. $\square$

We are now going to be interested in finding maps from $\mathrm{Ind}(\chi)$ to some other $\mathrm{Ind}(\rho)$. Any $\mathfrak{p}$-map to $\mathbb{C}_{\chi\delta^{1/2}}$ is in particular $\mathfrak{n}$-trivial. Of course we have one already, talking f to $f(1)$. But there is another natural candidate for an $\mathfrak{n}$-trivial map from $\mathrm{Ind}(\chi)$ to the complex numbers, one that is suggested by the Bruhat decomposition. It tells us that $P\backslash G$ has two right N-orbits, P itself and the double coset PwN. We have seen already the $\mathfrak{n}$-trivial functional Λ_1. The new $\mathfrak{n}$-trivial map is given formally by

[intertwining-integral] **(10.2)**
$$\langle \Lambda_w, f \rangle = \int_N f(wn)\, dn\,.$$

[4B] **10.3. Proposition.** *The integral for* Λ_w *converges absolutely for any* f *in* $\mathrm{Ind}^\infty(\chi_s \mathrm{sgn}^n)$ *if* $\mathrm{RE}(s) > 0$*, and defines a continuous linear functional such that* $\Lambda_w(R_p f) = \chi_{-s,n}(p)\delta^{1/2}(p)\Lambda_w(f)$.

Proof. We can apply the Iwasawa decomposition $G = PK$ explicitly to see that

$$\begin{aligned}\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} &= \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix} \\ &= \begin{bmatrix} 1 & -x/x^2+1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/r & 0 \\ 0 & r \end{bmatrix}\left(\begin{matrix} x/r & -1/r \\ 1/r & x/r \end{matrix}\right] \quad \left(r = \sqrt{1+x^2}\right) .\end{aligned}$$

Suppose f in $\mathrm{Ind}(\chi_s \mathrm{sgn}^n)$ with $\sup_K |f(k)| = M$. We then have

$$\left|\int_N f(wn)\,dn\right| = \left|\int_N f(wnw^{-1}\,w)\,dn\right| \le M\int_{-\infty}^{\infty} \left|(1+x^2)^{-(s+1)2}\right|dx$$

which is asymptotic to $|x|^{-s-1}$ as $|x| \to \infty$ and converges for $\mathrm{RE}(s) > 0$. Also, with $\chi = \chi_s \mathrm{sgn}^n$

$$\begin{aligned}\int_N R_{na} f(wu)\,du &= \int_N f(wu\,na)\,du \\ &= \int_N f(wua)\,du \\ &= \int_N f(wa\,a^{-1}na)\,dn \\ &= \delta(a) f(wan)\,dn \\ &= \delta(a) f(waw^{-1}\,wn)\,dn \\ &= \delta(a)\chi(a^{-1})\delta^{1/2}(a^{-1})\Lambda_w(f) \\ &= \delta^{1/2}(a)\chi^{-1}(a)\Lambda_w(f) . \qquad \blacksquare\end{aligned}$$

The map

$$T_w\colon\ f \longrightarrow [g \mapsto \Lambda_w(R_g f)]$$

is then a continuous G-equivariant map from $\mathrm{Ind}^\infty(\chi_s \mathrm{sgn}^n)$ to $\mathrm{Ind}^\infty(\chi_{-s,n})$.

This map takes ε^n_s to some $c_{s,n}\varepsilon^n_{-s}$. *What is the factor* $c_{s,n}$*?* This is the same as $\Lambda_w(\varepsilon^n_s)$, so we get

[csn] **(10.4)**
$$c_{s,n} = \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{x^2+1}}\right)^{s+1} \left(\frac{x+i}{\sqrt{x^2+1}}\right)^n dx .$$

For $n = 0$ rhis gives

$$\begin{aligned}\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{x^2+1}}\right)^{s+1} dx &= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)} \\ &= \frac{\zeta_{\mathbb{R}}(s)}{\zeta_{\mathbb{R}}(s+1)}\end{aligned}$$

where I write here

$$\zeta_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$$

as the factor of the Riemann ζ function contributed by the real place of $\mathbb{Q}$. It is not an accident that this factor appears in this form—a similar factor appears in the theory of induced representations of $\mathfrak{p}$-adic SL_2, and globally these all contribute to a 'constant term' $\xi(s)/\xi(s+1)$ in the theory of Eisenstein series.

There is something puzzling about this formula. The poles of $c_{s,0}$ are at $s = 0, -2, -4, \ldots$ But there is nothing special about the principal series representations at these points. There is one interesting question that arises in connection with these poles. The poles are simple, and the residue of the intertwining operator at one of these is again an intertwining operator, also from $\mathrm{Ind}(\chi_{s,n})$ to $\mathrm{Ind}(\chi_{s,n}^{-1})$. *What is that residue?* It must correspond to an $\mathfrak{n}$-invariant linear functional on $\mathrm{Ind}^\infty(\chi_{s,n})$. It is easy to see that in some sense to be explained later that the only $\mathfrak{n}$-invariant linear functional with support on PwN is Λ_w, so it must have support on P. It will be analogous to one of the derivatives of the Dirac delta. These matters will all be explained when I discuss the Bruhat filtration of principal series representations.

For $n = \pm 1$ we get

$$\begin{aligned}\int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{x^2+1}}\right)^{s+1} \left(\frac{x \pm i}{\sqrt{x^2+1}}\right) dx &= \pm i \cdot \int_{-\infty}^{\infty} \left(\frac{1}{\sqrt{x^2+1}}\right)^{s+2} dx \\ &= \pm i \cdot \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+2}{2}\right)} \\ &= \pm i \cdot \frac{\zeta_{\mathbb{R}}(s+1)}{\zeta_{\mathbb{R}}(s+2)}\,.\end{aligned}$$

We now know $c_{s,0}$ and $c_{s,\pm1}$. The others can be calculated by a simple recursion, since

$$T_w \pi(x_+)\varepsilon_{s,n} = \pi(x_+)T_w\varepsilon_{s,n}, \quad (s+1+n)c_{s,n+2} = (-s+1+n)c_{s,n}\,,$$

leading first to

$$\begin{aligned}c_{s,n+2} &= \frac{-s+1+n}{s+1+n} \cdot c_{s,n} \\ &= -\frac{s-(n+1)}{s+(n+1)} \cdot c_{s,n}\end{aligned}$$

and then upon inverting:

$$c_{s,n-2} = -\frac{s+(n-1)}{s-(n-1)} \cdot c_{s,n}\,.$$

Finally:

$$\begin{aligned}c_{s,2n} &= (-1)^n c_{n,0} \cdot \prod_{k=1}^{n} \frac{s-(2k+1)}{s+(2k+1)} \\ c_{s,-2n} &= (-1)^n c_{n,0} \cdot \prod_{k=1}^{n} \frac{s+(2k+1)}{s-(2k+1)}\,.\end{aligned}$$

Something similar holds for n odd. These intertwining operators give us explicit isomorphisms between generic principal series $\mathrm{Ind}(\chi_s \mathrm{sgn}^n)$ and $\mathrm{Ind}(\chi_{-s}\mathrm{sgn}^n)$.

♡ **[csn]** This product formula for $c_{s,n}$ is simple enough, but I prefer something more elegant. The formula (10.4) can be rewritten as

$$c_{s,n} = \int_{\mathbb{R}} (1+ix)^{-(s+1-n)/2}(1-ix)^{-(s+1+n)/2}\,dx\,.$$

♡ **[iab]** If we set $\alpha = (s+1-n)/2$ and $\beta = -(s+1+n)/2$ in Proposition 9.4, we see that this is

(10.5) **[new-csn]**
$$c_{s,n} = \frac{2^{1-s}\pi\Gamma(s)}{\Gamma\left(\frac{s+1+n}{2}\right)\Gamma\left(\frac{s+1-n}{2}\right)}\,.$$

One curious thing about this formula is that it is not immediately apparent that it agrees with the formula
♡ **[csn]** (10.4) for $c_{s,0}$. In effect, I have proved the Legendre duplication formula

$$2(2\pi)^{-s}\Gamma(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right) \cdot \pi^{-(1+s)/2}\Gamma\left(\frac{s+1}{2}\right) = \zeta_{\mathbb{R}}(s)\zeta_{\mathbb{R}}(1+s)\,.$$

The left hand side here is called, with reason, $\zeta_{\mathbb{C}}(s)$. This is perhaps one the simplest examples of the Hasse-Davenport equation.

11. Intertwining operators and the smooth principal series

[intertwining-integral] Proposition 10.3 tells us that the intertwining operator defined by (10.2) converge in a right-hand half plane, for all functions in the smooth principal series. The explicit formulas in the previous section tell us that they continue meromorphically on the K-finite principal series. In this section I'll show in two ways that the meromorphic continuation is good on all of the smooth principal series.

The first proof is very simple. We know that

$$c_{s,n+2} = -\frac{s-(n+1)}{s+(n+1)} c_{s,n} \,,$$

and as a consequence $c_{s,n}$ is of moderate growth in n, more or less uniformly in s. Restriction to K is an isomorphism of $\mathrm{Ind}^\infty(\chi_{s,n})$ with the space of smooth functions f on K such that $f(\pm k) = (-1)^n f(k)$, which is a fixed vector space independent of s. Let ι_s be this isomorphism. Thus for f on K

$$[\iota_s f](bk) = \chi_{s,n}\delta^{1/2}(b) f(k)$$

identically. The smooth vectors in $\mathrm{Ind}(\chi_s \mathrm{sgn}^n)$ may be expanded in Fourier series when restricted to K, which implies that the Fourier coefficients decrease rapidly with n, which implies what we want.

Now for the second argument. The space $\mathbb{P} = \mathbb{P}^1(\mathbb{R})$ is the union of two copies of $\mathbb{R}$, one the lines $\langle\langle x, 1\rangle\rangle$, the other the $\langle\langle 1, x\rangle\rangle$. Fix a function ϕ on $\mathbb{P}$ with compact support on the second, and lift it back to a function on G with support on PN°. The $1-\varphi$ has support on PwN. For every function f on K define $f_{1,s} = \varphi f_s$, $f_{w,s} = (1-\varphi) f_s$, so that $f_s = f_{1,s} + f_{w,s}$. The restriction of the first to N° has compact support, and the restriction of the second to wN. Both vary analytically with s.

Recall that

$$\langle \lambda_w, f\rangle = \int_{\mathbb{R}} f(wn_x)\, dx \qquad \left(n_x = \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \right) .$$

This converges without difficulty for $f = f_{w,s}$, since the integrand has compact support on $\mathbb{R}$. It varies analytically with s. What about for $f = f_{1,s}$?

Suppose for the moment that f is an arbitrary smooth function of compact support on N°. It becomes a function in $\mathrm{Ind}^\infty(\chi_{s,n})$ by the specification

$$f(bn^\circ) = \chi_{s,n}\delta^{1/2}(b) f(n^\circ) \,.$$

The reason this defines a smooth function on G is that f has compact support on N°. To see what $\langle \Lambda_w, f\rangle$ is, we must evaluate $f(wn_x)$—i.e. factor wn_x as bn°:

$$\begin{aligned} wn_x &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & x \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -1 \\ 1 & x \end{bmatrix} \\ &= \begin{bmatrix} 1/x & -1 \\ 0 & x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1/x & 1 \end{bmatrix} . \end{aligned}$$

Thus

$$\langle \Lambda_w, f\rangle = \int_{\mathbb{R}} |x|^{-(1+s)} \mathrm{sgn}^n(x) f(1/x)\, dx \,.$$

This makes sense and converges for $\mathrm{RE}(s) > 0$ because $f(1/x) = 0$ for x small, and is bounded. We can write the integral as

$$\int_0^\infty x^{-(1+s)} f(1/x)\, dx + (-1)^n \int_0^\infty x^{-(1+s)} f(-1/x)\, dx$$

and then make a change of variables $y = 1/x$, giving

$$\int_0^\infty y^{s-1} f(y)\, dy + (-1)^n \int_0^\infty y^{s-1} f(-y)\, dy\,.$$

In this last integral, $f(y)$ is smooth of compact support. In effect, we are reduced to understanding integration against $y^{s-1}\,dy$ as a distribution on $[0, \infty)$. The integral converges for $\mathrm{RE}(s) > 0$. We can now integrate by parts:

$$\int_0^\infty y^{s-1} f(y)\, dy = \frac{-1}{s} \int_0^\infty y^s f'(y)\, dy\,,$$

to analytically continue for $\mathrm{RE}(s) > -1$. Continuing, we see that the integral may be defined meromorphically over all of $\mathbb{C}$ with possible poles in $-\mathbb{N}$. The actual location of poles for Λ_w itself depends on n.

12. The complementary series

We have seen that the imaginary line $\mathrm{RE}(s) = 0$ parametrizes unitary representations. In addition the trivial representation is unitary, and later on we'll see the holomorphic discrete series and—implicitly—the anti-holomorphic discrete series. There are, however, some more unitary representations to be found. These are the principal series $\mathrm{Ind}(\chi_{s,0})$ with s in the interval $(-1, 1)$, the so-called **complementary series** whose Casimir operator acts by a scalar in the range $(-1/4, 0)$.

The intertwining operator $T_{s,0}$ is multiplication by

$$c_{s,0} = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{s}{2}\right)}{\Gamma\left(\frac{s+1}{2}\right)}$$

which as a pole of order 1 at $s = 0$. The other constants are determined by the rule

$$\begin{aligned} c_{s,n+2} &= \frac{-s+1+n}{s+1+n} \cdot c_{s,n} \\ c_{s,n-2} &= \frac{-s+1-n}{s+1-n} \cdot c_{s,n}\,. \end{aligned}$$

These both imply that the **normalized** operator $T_w/c_{s,0}$ has limit the identity operator as $s \to 0$. Furthermore, it is an isomorphism for all s in $(-1, 1)$. But in this interval the representation $\mathrm{Ind}(\chi_{-s,0})$ is the Hermitian as well as linear dual of $\mathrm{Ind}(\chi_{s,0})$, so T_w determines in effect an Hermitian inner product on $\mathrm{Ind}(\chi_{s,0})$. It does not change signature anywhere in the interval and since it is positive definite for $s = 0$ it is positive definite throughout. Therefore

[complementary] **12.1. Proposition.** *For all s in $(-1, 1)$ the representation $\mathrm{Ind}(\chi_{s,0})$ is unitary.*

Of course we have only recovered in a different form what we deduced in an earlier section about unitary admissible modules over $(\mathfrak{g}, K)$.

13. The Bruhat filtration

14. Verma modules

15. The Jacquet module

For all of these topics refer to [Casselman:2011].

16. Whittaker models

The principal reference here is [Casselman-Hecht-Miličić:1998], which discusses Whittaker models for $\mathrm{SL}_2(\mathbb{R}$ in an elementary fashion in the introduction. There are also a few remarks about this topic in [Casselman:2011].

17. *Appendix.* Invariant integrals on quotients

If G is a locally compact group, let δ_G be its modulus character. This measures the discrepancy between left- and right-invariant measures, so that if d_ℓ is a left-invariant measure on G then $d_\ell\, gg_* = \delta_G^{-1}(g_*) d_\ell\, g)$ or, equivalently, $d_\ell\, gxg^{-1} = \delta_G(g)\, d_\ell x$. Now let $H \subseteq G$ be a closed subgroup. The quotient $H\backslash G$ will not in general possess a G-invariant measure, but this failure is easily dealt with.

Recall that a measure of compact support on a locally compact space is a continuous linear function on the space $C(G)$ of continuous $\mathbb{C}$-valued functions. Define $\Omega_c(H\backslash G)$ to be the space of continuous measures of compact support on $H\backslash G$. When G is a Lie group, elements of this space are also called one-densities on $H\backslash G$, and if $H\backslash G$ is oriented this is the same as the space of continuous forms of highest degree and of compact support. The assertion above means that $\Omega_c(H\backslash G)$ is not generally, as a G-space, isomorphic to $C_c(H\backslash G)$. But there exists a useful analogue. If χ is a character of H, define

$$\begin{aligned} Ind_c(\chi) &= Ind_c(\chi | H, G) \\ &= \big\{ f \in C(G) \text{ of compact support modulo } H \,\big|\, f(hg) = \chi(h) f(g) \text{ for all } h \in H, g \in G \big\}\,. \end{aligned}$$

If dh is a Haar measure on H, χ is a character of H, and $f \in C_c(G)$ then the integral

$$F(g) = \int_H \chi(h) f(hg)\, dh$$

will satisfy

$$\begin{aligned} F(xg) &= \int_H \chi(h) f(hxg)\, dh \\ &= \int_H \chi(yx^{-1}) f(yg)\, d\, yx^{-1} \\ &= \delta_H(x) \chi^{-1}(x) F(g) \end{aligned}$$

for all x in H, and therefore lies in $Ind_c(\delta_H \chi^{-1})$. In particular

$$\overline{f}(g) = \int_H \delta_G(h) f(hg)\, dh$$

lies in $Ind_c(\delta_H/\delta_G)$.

[G-invariant] **17.1. Proposition.** *Suppose given Haar measures dh, dg on H, G. There exists a unique G-invariant linear functional I on $Ind_c(\delta_H/\delta_G)$ such that*

$$\int_G f(g)\, dg = \langle I, \overline{f} \rangle$$

for all f in $C_c(G)$.

As a consequence, the choices of dh, dg determine an isomorphism of $Ind_c(\delta_H/\delta_G)$ with $\Omega_c(H\backslash G)$.

The proof proceeds by finding for f in $Ind_c(\delta_H/\delta_G)$ a function φ in $C_c(G)$ such that $\overline{\varphi} = f$, then verifying that

$$\int_G \varphi(g)\, dg$$

depends only on f. I refer to [Weil:1965] for details, but I shall sketch here the proof in a special case. Suppose that G is unimodular and that there exists a compact subgroup K such that $G = HK$. In particular, $H\backslash G$ is compact. This happens, for example, if G is a reductive Lie group and H is a parabolic subgroup. I claim:

[HK-invariant] **17.2. Proposition.** *Integration over K is a G-invariant integral on $Ind(\delta_H/\delta_G)$.*

Proof. Assign K total measure 1. The integral

$$f^K(g) = \int_K f(gk)\, dk$$

defines a projection of $C(G)$ onto $C(G/K)$, which possesses a G-invariant measure, unique up to scalar multiple. But $H/H\cap K = G/K$ since $G = HK$. Ht possesses an essentially unique G-invariant measure, which is therefore also a G-invariant measure. Hence

$$\int_G f(g)\, dg = \int_{H\times K} f(hk)\, dh\, dk = \int_K \overline{f}(k)\, dk$$

since

$$\overline{f}(g) = \int_H f(hg)\, dh$$

under the assumption that G is unimodular. ∎

Part III. Matrix coefficients

18. Matrix coefficients

In this part, I want to explain how the representations $\mathrm{DS}_n^{\pm}$ (for $n \ge 2$) can be embedded discretely into $\mathrm{L}^2(G)$ (and therefore earn the name 'discrete series'). I also want to explain here, albeit rather roughly, the relationship between these representations and the action of G on the complex projective line. It is this feature that explains exactly why these representations are relevant to the theory of holomorphic automorphic forms.

At this point, pretty much all we know of these representations is that they are characterized by their K-spectrum. That is not quite true, since we know they can be embedded into certain principal series representations, but in fact this also amounted just to a calculation of K-spectrum. In order to construct an embedding into $\mathrm{L}^2(G)$, we need a new way to realize these representations.

Suppose (π, V) to be any continuous admissible representation of G and $\widehat{V}$ the space of continuous linear functions on V. The group G acts on $\widehat{V}$ by the formula

$$\langle \widehat{\pi}(g)\widehat{v}, v\rangle = \langle \widehat{v}, \pi(g^{-1})v\rangle \, .$$

If v is in V and $\widehat{v}$ in $\widehat{V}$ the corresponding **matrix coefficient** is the function on G defined as

$$\Phi_{v,\hat{v}}(g) = \langle \widehat{\pi}(g)\widehat{v}, v\rangle = \langle \widehat{v}, \pi(g^{-1})v\rangle \, .$$

The terminology comes about because if π is a finite-dimensional representation on a space V with basis (e_i) and dual basis $(\widehat{e}_i)$ then $\Phi_{e_j,\hat{e}_i}(g)$ is the (i,j) entry of the matrix $\widehat{\pi}(g^{-1})$. What I am going to show is that if π is $\mathrm{DS}_n^{\pm}$ with $n \ge 2$ then its matrix coefficients lie in $\mathrm{L}^2(G)$. Since these representations are irreducible and $\mathrm{L}^2(G)$ is G-stable, it suffices to show this for a single matrix coefficient $\ne 0$.

We know that if V is any irreducible admissible representation of $(\mathfrak{g}, K)$ there exists at least one continuous representation of G for which V is the subspace of K-finite vectors. There may be several such extensions to G, but it turns out that the matrix coefficient $\Phi_{v,\hat{v}}$ does not depend on which extension is chosen. This is because (a) the matrix coefficient is an analytic function of g, so that it is determined on the connected component of G by its Taylor series at the identity; and (b) that Taylor series is completely determined by the representation of $\mathfrak{g}$. So we may refer to the matrix coefficient of $\widehat{v}$, v, even when there is no representation of G in sight.

The group G acts on both right and left on smooth functions in $C^\infty(G)$:

$$[R_g f](x) = f(xg), \quad [L_g f](x) = f(g^{-1}x) \, .$$

These are both left representations of G:

$$R_{g_1 g_2} = R_{g_1} R_{g_2}, \quad L_{g_1 g_2} = L_{g_1} L_{g_2} \, .$$

Associated to these are right and left actions of $\mathfrak{g}$ on smooth functions:

$$\begin{aligned} [R_X f](y) &= \frac{d}{dt}\Big|_{t=0} f(y\exp(tX)) \\ [L_X f](y) &= \frac{d}{dt}\Big|_{t=0} f(\exp(-tX)y) \, . \end{aligned}$$

If π is a representation of G then

$$R_g \Phi_{v,\hat{v}} = \Phi_{v,\hat{\pi}(g)\hat{v}}, \quad L_g \Phi_{v,\hat{v}} = \Phi_{\pi(g)v,\hat{v}} \, .$$

Thus for a fixed $\widehat{v}$ the map from V to $C^\infty(G)$ taking v to $\Phi_{v,\hat{v}}$ is L_G-equivariant. These equations translate directly to facts about the Lie algebra:

[right-action] **18.1. Lemma.** *For f in $C^\infty(G)$, X in $\mathfrak{g}$:*

$$R_X \Phi_{v,\hat{v}} = \Phi_{v,\hat{\pi}(X)\hat{v}}, \quad L_X \Phi_{v,\hat{v}} = \Phi_{\pi(X)v,\hat{v}} \, .$$

We shall soon find explicit formulas for certain matrix coefficients of the representations $\mathrm{DS}_n^{\pm}$, and somewhat later discuss the matrix coefficients of other representations. We begin with something relatively simple. I'll work with just DS_n^+ to avoid redundancy, and at first assume only that $n \ge 1$. Swapping $+$ with $-$ will cause little trouble.

We know that the dual of DS_n^+ is DS_n^-. Fix a vector $\widehat{v}$ in DS_n^- such that $\pi(k)\widehat{v} = \varepsilon^{-n}(k)\widehat{v}$ and therefore also $\pi(x_+)\widehat{v} = 0$. Define the map Φ from V into the space of functions on G:

$$\Phi\colon\ v \mapsto \Phi_v, \quad [\Phi_v](g) = \Phi_{v,\hat{v}}(g) = \langle \widehat{v}, \pi(g^{-1})v \rangle \, .$$

[holo-ds-mc] **18.2. Theorem.** *Suppose $n > 0$. The map taking v to Φ_v is an isomorphism of DS_n^+ with the space of K-finite functions F on G such that (a) $R_k F = \varepsilon^{-n}(k) F$ and (b) $R_{x_+} v = 0$.*

Beginning of the *Proof*. The necessity of (a) and (b) is straightforward. For k in K we have

$$\Phi_v(gk) = \langle \widehat{v}, \pi(k^{-1})\pi(g^{-1})v \rangle = \langle \pi(k)\widehat{v}, \pi(g^{-1})v \rangle = \varepsilon^{-n}(k)\Phi_v(g) \, ,$$

and

$$R_{x_+}\Phi_v = \Phi_{v,R_{x_+}\hat{v}} = 0 \, .$$

Since there exists a vector v in V such that $\langle \widehat{v}, v \rangle \ne 0$ the map Φ does not vanish identically. Since $\mathrm{DS}_n^{\pm}$ is irreducible, it is injective. So what remains is to show that every function Φ satisfying these conditions is equal to some Φ_v. Therefore, in order to prove that it is an isomorphism it suffices to show that the spectrum of κ on this space of F in the Proposition is $n, n+2, \ldots$ Proving this will introduce some new ideas, and will take up the next few sections. Along the way I shall find an explicit formula for Φ_v, and then show that for $n \ge 2$ it lies in $\mathrm{L}^2(G)$.

19. Complex projective space

We have seen the representations $D_n^{\pm}$ embedded into certain principal series representations, and also embedded into a space of functions on G via matrix coefficients. We shall soon see another realization that is closely related to classical automorphic forms.

In the rest of this part, let

$$\begin{aligned}
G_{\mathbb{R}} &= \mathrm{SL}_2(\mathbb{R}) \\
\mathfrak{X}_{\mathbb{R}} &= \mathbb{R}^2 - \{0\} \\
\mathbb{P}_{\mathbb{R}} &= \mathbb{P}^1(\mathbb{R}) \quad (\,\text{space of lines in } \mathbb{R}^2\,) \\
&= \mathfrak{X}_{\mathbb{R}}/\mathbb{R}^\times \\
G_{\mathbb{C}} &= \mathrm{SL}_2(\mathbb{C}) \\
\mathfrak{X}_{\mathbb{C}} &= \mathbb{C}^2 - \{0\} \\
\mathbb{P}_{\mathbb{C}} &= \mathbb{P}^1(\mathbb{C}) \quad (\,\text{space of complex lines in } \mathbb{C}^2\,) \\
&= \mathfrak{X}_{\mathbb{C}}/\mathbb{C}^\times \, .
\end{aligned}$$

The real projective space $\mathbb{P}_{\mathbb{R}}$ embeds into the complex projective space $\mathbb{P}_{\mathbb{C}}$. The action of $G_{\mathbb{R}}$ on the first leads to the construction of the principal series. Its action on the second turns out also to lead to interesting representations.

I can motivate the constructions to come by giving a new characterization of the principal series. The group $G_{\mathbb{R}}$ acts on $\mathfrak{X}_{\mathbb{R}}$ by linear transformations, and this induces in the natural way its action on $\mathbb{P}_{\mathbb{R}}$. If χ is a character of $\mathbb{R}^\times$, a function f on $\mathfrak{X}_{\mathbb{R}}$ is said to be χ-homogeneous if

$$f(cv) = \chi(c)f(v)$$

for all v in $\mathfrak{X}_{\mathbb{R}}$ and c in $\mathbb{R}^\times$.

Recall that P is the group of upper triangular matrices in $\mathrm{SL}_2(\mathbb{R})$ and N is its subgroup of unipotent matrices. The group N takes the point $(1,0)$ to itself, and therefore any function f on $\mathfrak{X}_{\mathbb{R}}$ gives rise to a function F on $N\backslash G$ according to the formula

$$F(g) = f\left(g^{-1}\begin{bmatrix}1\\0\end{bmatrix}\right).$$

I leave as an exercise:

[ps-homogeneous] **19.1. Proposition.** *The map taking f on $\mathfrak{X}_{\mathbb{R}}$ to F on $N\backslash G$ is an isomorphism of the smooth functions on $\mathfrak{X}_{\mathbb{R}}$ that are χ-homogeneous with the smooth principal series* $\mathrm{Ind}(\chi^{-1}\delta^{-1/2})$.

Among other things, this should make it clear that certain principal series representations contain the irreducible finite-dimensional representations of G, since if $\chi(x) = x^m$ for a non-negative integer m the space of homogeneous functions contains the homogeneous polynomials of degree m in $\mathbb{R}^2$, a space on ♡ [ps-homogeneous] which G acts as it does on FD_m. Proposition 19.1 will play a role in what's to come, but for the moment it will serve mostly as motivation for the next step. Another virtue of this result is that it defines the smooth principal series representations without making a choice of P—it is, in effect, a coordinate-free treatment.

We shall be interested mostly in functions on certain subsets of $\mathfrak{X}_{\mathbb{C}}$ that are homogeneous. But there will be a new restriction—I want to look only at functions that are holomorphic. This requires that the character χ be holomorphic as well, which in turn means that $\chi(c) = c^m$ with m now an arbitrary integer.

We shall need eventually some elementary results about the action of the complex group $G_{\mathbb{C}}$ on $\mathfrak{X}_{\mathbb{C}}$. It acts on $\mathfrak{X}_{\mathbb{C}}$ by linear transformations and since it commutes with scalars it acts also on the space $\mathcal{C}_m(\mathfrak{X}_{\mathbb{C}})$ of holomorphic functions on $\mathfrak{X}_{\mathbb{C}}$ of degree m. The projection from $\mathfrak{X}_{\mathbb{C}}$ to $\mathbb{P}_{\mathbb{C}}$ taking $(u,v) \mapsto u/v$ specifies $\mathfrak{X}_{\mathbb{C}}$ as a fibre space over $\mathbb{P}_{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ with fibre equal to $\mathbb{C}^\times$.

The map taking z to the point $(z,1)$ in $\mathfrak{X}_{\mathbb{C}}$ is a section of the projection over the embedded copy of $\mathbb{C}$. This does not extend continuously to ∞. Since the group $G_{\mathbb{C}}$ commutes with scalar multiplication, it preserves the fibring over $\mathbb{C}$. Elements of $G_{\mathbb{C}}$ do not take the image of the section into itself. Formally, we have

$$g\left(\begin{bmatrix}z\\1\end{bmatrix}\right) = \begin{bmatrix}az+b\\cz+d\end{bmatrix} = (cz+d)\begin{bmatrix}(az+b)/(cz+d)\\1\end{bmatrix} = j(g,z)\begin{bmatrix}g(z)\\1\end{bmatrix} \quad \text{if} \quad g = \begin{bmatrix}a & b\\c & d\end{bmatrix}.$$

The term $j(g,z) = cz+d$, when it is non-zero, is called the **automorphy factor**. It does not in fact vanish as long as $\mathrm{IM}(z) \neq 0$ and g is real. This is consistent with the fact that $\mathrm{GL}_2(\mathbb{R})$ takes the image of $\mathbb{P}_{\mathbb{R}}$ into itself, hence also its complement. At any rate, from this formula follows immediately the first of our elementary results:

[automorphy-product] **19.2. Lemma.** *Whenever both terms on the right hand side are non-zero we have*

$$j(gh,z) = j(g,h(z))j(h,z).$$

Since $\mathrm{SL}_2(\mathbb{R})$ is connected, it must take each component of the complement of $\mathbb{P}_\mathbb{R}$ in $\mathbb{P}_\mathbb{C}$ into itself. We can be more explicit.

[ytrans] **19.3. Lemma.** *If*

$$g = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{GL}_2(\mathbb{R}) \ \textit{then}\ \mathrm{IM}\big(g(z)\big) = \frac{\det(g)}{|cz+d|^2}\,.$$

Proof. Straightforward calculation:

$$\begin{aligned} \mathrm{IM}\left(\frac{az+b}{cz+d}\right) &= \frac{1}{2i}\left(\frac{az+b}{cz+d} - \frac{a\overline{z}+b}{c\overline{z}+d}\right) \\ &= \frac{1}{|cz+d|^2}\,\mathrm{IM}(z)\,. \end{aligned}$$

Let $\mathfrak{X}^*$ be the inverse image in $\mathfrak{X}_\mathbb{C}$ of the complement of $\mathbb{P}_\mathbb{R}$, where $\mathrm{IM}(z/w) \ne 0$. It has two components

$$\mathfrak{X}^\pm = \{(z,w) \,|\, \mathrm{sign}\ \mathrm{IM}(z/w) = \pm\}\,.$$

♡ [ytrans] As a consequence of Lemma 19.3, the group $G = \mathrm{SL}_2(\mathbb{R})$ takes each component $\mathfrak{X}^\pm$ into itself.

20. The holomorphic and anti-holomorphic discrete series

The group $G_\mathbb{R}$ acts on the space of smooth functions on $\mathfrak{X}^*$:

$$F \longmapsto L_gF, \quad [L_gF](x) = F(g^{-1}x)\,.$$

The space $\mathfrak{X}^*$ is an open subset of $\mathfrak{X}_\mathbb{C}$, hence possesses an inherited complex structure. Since $G_\mathbb{R}$ preserves this complex structure, it takes the subspace of holomorphic functions on $\mathfrak{X}^*$ into itself. Define C^∞_m to be the space of smooth functions on $\mathfrak{X}^*$ that are homogeneous of degree m with respect to $\mathbb{C}^\times$, and $\mathcal{C}_m$ the subspace of holomorphic ones. In effect, the functions in C^∞_m (respectively $\mathcal{C}_m$) are smooth (holomorphic) sections of a holomorphic line bundle on the complement $\mathcal{H}^*$ of $\mathbb{P}_\mathbb{R}$ in $\mathbb{P}_\mathbb{C}$. Since G commutes with scalar multiplication it takes $\mathcal{C}_m$ to itself. Let π_m be the representation of G on $\mathcal{C}_m$. If $m \ge 0$ this space contains a finite-dimensional subspace of m-homogeneous polynomials. Rather than investigate this case, *I'll assume now that* $m < 0$. Since $\mathfrak{X}^*$ is the union of its two connected components $\mathfrak{X}^\pm$, π_m is the direct sum of two components $\pi^\pm_m$.

A function in C^∞_m is determined by its restriction to the embedded copy of $\mathcal{H}^*$ in $\mathfrak{X}^*$. If F lies in C^∞_m define

$$f(z) = F\left(\begin{bmatrix} z \\ 1 \end{bmatrix}\right)\,.$$

Of course F is holomorphic if and only if f is. We can recover F from f:

$$F\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = w^m F\left(\begin{bmatrix} z/w \\ 1 \end{bmatrix}\right) = w^m f(z/w)\,.$$

How does the action of $G_\mathbb{R}$ on F translate to an action of $G_\mathbb{R}$ on f?

$$\begin{aligned} [\pi_m(g)F]\left(\begin{bmatrix} z \\ 1 \end{bmatrix}\right) &= F\left(\begin{bmatrix} az+b \\ cz+d \end{bmatrix}\right) \quad \text{where } g^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= (cz+d)^m F\left(\begin{bmatrix} (az+b)/(cz+d) \\ 1 \end{bmatrix}\right) \\ &= (cz+d)^m f\left(\frac{az+b}{cz+d}\right) \\ &= j(g^{-1},z)^m f\big(g^{-1}(z)\big)\,. \end{aligned}$$

Hence:

[gonf] **20.1. Lemma.** *For f in $\mathbb{C} - \mathbb{R}$,*

$$[\pi_m(g)f](z) = j(g^{-1}, z)^m f\big(g^{-1}(z)\big)\,.$$

What is the restriction of this representation to K? In particular, what are its K-finite vectors—in effect, its K-eigenvectors?

The space $\mathfrak{X}^*$ is the union of its two components, and as I have already observed the space $\mathcal{C}_m$ is therefore the sum of two components, one each of functions on $\mathfrak{X}^{\pm}$. I'll look at first at $\mathfrak{X}^+$, the inverse image in $\mathfrak{X}^*$ of the upper half plane $\mathcal{H}$.

In considering the action of $G_{\mathbb{R}}$ restricted to K, it is often a good idea to work with the action of $G_{\mathbb{R}}$ on the unit disk, which is obtained *via* the Cayley transform. We do that here. Recall that

$$C(z) = \frac{z-i}{z+i}, \quad C^{-1}(z) = i \cdot \frac{1+z}{1-z}$$

corresponding to matrices

$$\begin{bmatrix} 1 & -i \\ 1 & i \end{bmatrix}, \quad \frac{1}{2i}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}.$$

Thus for z in $\mathbb{D}$, g in $\mathrm{SL}_2(\mathbb{R})$, g takes

$$z \longmapsto \big(C \cdot g \cdot C^{-1}\big)(z)\,,$$

and in particular

$$\begin{bmatrix} c & -s \\ s & c \end{bmatrix} \longmapsto \begin{bmatrix} c - is & 0 \\ 0 & c + is \end{bmatrix}.$$

We can define a representation of G on the inverse image of $\mathbb{D}$ in $\mathfrak{X}_{\mathbb{C}}$ according to the formula

$$\rho_m(g) = (C^{-1})^* \pi_m(g)\, C^* \quad \big([C^* f](z) = f(C(z))\big) j(C, z)^m\,.$$

The product formula equation for the automorphy factor tells us that this also has the form

$$[\rho_m(g)f](z) = f(g^{-1}(z)) j(g^{-1}, z)^m\,,$$

In this realization, K acts by 'twisted' rotations around the origin. More explicitly, an eigenfunction f for character ε^ℓ must satisfy the equation

$$[\rho_m(k)f](z) = \varepsilon^\ell f(z) = f(k^{-1}(z)) j(k^{-1}, z)^m$$

for all k in K. But in this realization K is represented by matrices

$$r_\theta = \begin{bmatrix} 1/u & 0 \\ 0 & u \end{bmatrix} \quad (u = e^{i\theta})\,.$$

Since $k^{-1}z = u^2 z$ and $j(k, z) = u^{-1}$, f must satisfy the equation

$$f(u^2 z)u^{-m} = u^\ell f(z), \quad f(u^2 z) = u^{\ell+m} f(z)\,.$$

Here $|z| < 1$. If we set $u = -1$ in this equation we see that $m + \ell$ must be even, say $\ell + m = 2p$. If we set $z = c$ with $0 < c < 1$ and $u = e^{i\theta/2}$ this equation gives us $f(ce^{i\theta}) = e^{ip\theta} f(c)$. The following is a basic fact about holomorphic functions:

[eigen-holo] **20.2. Lemma.** *If $f(z)$ is holomorphic in the unit disk $|z| < 1$ and*

$$f(ce^{i\theta}) = e^{ip\theta}$$

for some $0 < c < 1$ then $f(z) = (z/c)^p$.

Proof. Apply Cauchy's integral formula:

$$\begin{aligned} f(z) &= \frac{1}{2\pi i} \int_{|\zeta|=c} \frac{f(\zeta)}{\zeta - z}\, d\zeta \\ &= \frac{c}{2\pi} \int_0^{2\pi} \frac{f(ce^{i\theta})}{ce^{i\theta} - z} e^{i\theta}\, d\theta \\ &= \frac{f(c)}{2\pi} \int_0^{2\pi} \frac{e^{ip\theta}}{1 - (z/c)e^{-i\theta}}\, d\theta \,. \end{aligned}$$

Now express the integral as a geometric series, and integrate term by term. ∎

In other words, the eigenfunctions of K in the representation on $\mathbb{D}$ are the monomials z^p, with eigencharacter ε^{2p-m} (which is positive, since I have assumed $m < 0$). Reverting to the action on $\mathcal{H}$:

[zholok] **20.3. Proposition.** *Assume $n > 0$. The eigenfunctions of K in π^+_{-n} are the functions*

$$\frac{(z-i)^p}{(z+i)^{p+n}}$$

for $p \ge 0$, with eigencharacter ε^{2p+n} of K.

Similarly, the eigenfunctions in π^-_{-n} are the functions

$$\frac{(z+i)^p}{(z-i)^{p+n}}$$

So we see:

[picm] **20.4. Proposition.** *The representation $\pi^\pm_{-n}$ for $n > 0$ is isomorphic to $\mathrm{DS}^\pm_n$.*

Note that I have used only the characterization of $\mathrm{DS}^\pm_n$ by K-spectrum, and not specified the way in which $\mathfrak{sl}_2$ acts in $\pi^\pm_m$ so as to obtain an explicit isomorphism.

21. Relation with functions on the group

♡ [holo-ds-mc] In order to prove Theorem 18.2, it now remains to identify $\mathcal{C}^+_m$ with the space of functions Φ in that Theorem.

THE ACTION OF K. Suppose F to be in $C^\infty_m(\mathfrak{X}^+)$. Let $f(z)$ be the corresponding function on $\mathcal{H}$, the restriction of F to the image of $\mathcal{H}$ in $\mathfrak{X}$. Define the function $\Phi = \Phi_F$ on $G_\mathbb{R}$ by the formula

[PhiFG] **(21.1)**
$$\Phi_F(g) = F\left(g \begin{bmatrix} i \\ 1 \end{bmatrix}\right) = j(g,i)^m f\big(g(i)\big)\,.$$

If

$$k = \begin{bmatrix} c & -s \\ s & c \end{bmatrix}$$

then

$$k \begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} ci - s \\ si + c \end{bmatrix} = (c + is) \begin{bmatrix} i \\ 1 \end{bmatrix} = \varepsilon(k) \begin{bmatrix} i \\ 1 \end{bmatrix} .$$

Therefore for all k in K and all g in G

[fgk] **(21.2)**
$$\Phi(gk) = \varepsilon^m(k)\Phi(g)\,.$$

Conversely, suppose given Φ in $C^\infty(G)$ such that (21.2) holds. Then equation (21.1) and the condition of homogeneity defines F in $C^\infty_m(\mathfrak{X}^+)$ uniquely.

We have proved:

21.3. Lemma. *The map taking F to the function Φ_F defined in (21.1) is an isomorphism of $C^\infty_m(\mathfrak{X}^+)$ with the space of all smooth functions Φ on G satisfying (21.2).* [fgk]

Next, we must see how to characterize the holomorphicity of F (or f) in terms of Φ.

THE CAUCHY EQUATIONS. Before I state the result, I'll recall some elementary facts of complex analysis. A smooth $\mathbb{C}$-valued function $f = u(x,y) + iv(x,y)$ on an open subset of $\mathbb{C}$ is holomorphic if and only if the real Jacobian matrix of f

$$\begin{bmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{bmatrix}$$

considered as a map from $\mathbb{R}^2$ to itself lies in the image of $\mathbb{C}$ in $M_2(\mathbb{R})$. (Since this image generically coincides with the group of orientation-preserving similitudes, this means precisely that it is conformal.) This condition is equivalent to the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}\,.$$

Holomorphicity may also be expressed by the single equation

$$\frac{\partial f}{\partial \overline{z}} = 0$$

where

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\,\frac{\partial}{\partial y}\right).$$

When f is holomorphic, its complex derivative is

$$\frac{\partial f}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\,\frac{\partial}{\partial y}\right).$$

The notation is designed so that for an arbitrary smooth function

$$df = \frac{\partial f}{\partial z}\,dz + \frac{\partial f}{\partial \overline{z}}\,d\overline{z}$$

where $dz = dx + i\,dy$.

HOLOMORPHICITY AND THE LIE ALGEBRA. The approach I take requires a slight digression. Let GL_2^+ [ytrans] be the subgroup of $\mathrm{GL}_2(\mathbb{R})$ consisting of g such that $\det(g) > 0$. According to Lemma 19.3, it takes each $\mathfrak{X}^\pm$ to itself. It also commutes with the scalar action on $\mathfrak{X}$, hence acts on $\mathcal{C}^\pm_m$, extending the representation of $\mathrm{SL}_2(\mathbb{R})$. The point of shifting to GL_2^+ is that if

[gl2p] **(21.4)**
$$p = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix}$$

then $j(p,z)=1$ for all z, and p takes the copy of $\mathcal{H}$ in $\mathfrak{X}$ to itself.

Given F in $C^\infty_m(\mathfrak{X}^+)$, now define Φ_F to be a function on GL_2^+ by the same formula as before:

$$\Phi_F(g) = F\left(g\begin{bmatrix} i \\ 1 \end{bmatrix}\right).$$

If

[gl2c] **(21.5)**
$$\lambda = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

then

$$\lambda\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} ai-b \\ bi+a \end{bmatrix} = (a+ib)\begin{bmatrix} i \\ 1 \end{bmatrix} = \varepsilon(\lambda)\begin{bmatrix} i \\ 1 \end{bmatrix} \quad (\varepsilon(\lambda)=a+ib).$$

Hence for all g

$$\Phi(g\lambda) = \varepsilon^m(\lambda)\Phi(g).$$

♡ **[gl2p]** Every g in $\mathrm{GL}_2^+(\mathbb{R})$ can be uniquely expressed as $p\lambda$ with p as in (21.4) and λ as in (21.5).

[lifting-hol] **21.6. Lemma.** *For F in $C^\infty_m(\mathfrak{X}^+)$ we have*

$$[R_{x_+}\Phi_F](g) = -4iy\,\varepsilon^2(\lambda/|\lambda|)\,\varepsilon^m(\lambda)\,\frac{\partial f(z)}{\partial \overline{z}}$$

if

$$g = p\lambda, \quad p = \begin{bmatrix} y & x \\ 0 & 1 \end{bmatrix} \qquad (p(i)=z=x+iy).$$

Proof. We have

$$\begin{aligned}[R_{x_+}\Phi_F](p\lambda) = [R_k R_{x_+}\Phi_F](p) &= [R_{\mathrm{Ad}(\lambda)x_+}R_\lambda\Phi_F](p) \\ &= \varepsilon^2(\lambda/|\lambda|)[R_{x_+}R_k\Phi_F](p) \\ &= \varepsilon^2(\lambda/|\lambda|)\varepsilon^m(\lambda)[R_{x_+}\Phi_F](p),\end{aligned}$$

so it suffices to prove the claim for $g=p$.

The Lie algebra of GL_2^+ has as basis:

$$\begin{aligned}\eta &= \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \\ \nu_+ &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ \zeta &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ \kappa &= \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},\end{aligned}$$

♡ **[gl2p]** The first two span the Lie algebra of matrices (21.4), the second those of (21.5), and

$$\begin{aligned}x_+ &= \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix} \\ &= 2(\eta+i\nu_+) - (\zeta+i\kappa).\end{aligned}$$

But now

$$\begin{aligned} R_{x_+}F(p) &= (R_\alpha - 2iR_{\nu_+} - iR_\kappa)\Phi_F(p) \\ &= (2R_\eta - R_\zeta - 2iR_{\nu_+} - iR_\kappa)\Phi_F(p) \\ &= (2R_\eta - 2iR_{\nu_+} + m - i(-mi))\Phi_F(p) \\ &= (2R_\eta - 2iR_{\nu_+})\Phi_F(p) \end{aligned}$$

Apply the basic formula $R_X f(g) = [\Lambda_{gXg^{-1}} f](g)$ (where $\Lambda = -L$) to get

$$(2R_\eta - 2iR_{\nu_+})\Phi_F(p) = (2\Lambda_{p\eta p^{-1}} - 2i\Lambda_{p\nu_+ p^{-1}})\Phi_F(p)\,.$$

But

$$\Lambda_\eta = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$$

and

$$p\eta p^{-1} = y\eta - x\nu_+$$

so

$$\begin{aligned} (2\Lambda_{p\eta p^{-1}} - 2i\Lambda_{p\nu_+ p^{-1}})\Phi_F(p) &= 2y\frac{\partial f}{\partial y} - 2iy\frac{\partial f}{\partial x} \\ &= -2iy\left(\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y}\right) \\ &= -4iy\,\frac{\partial f}{\partial \bar z}\,. \end{aligned}$$

[holo-cor] **21.7. Corollary.** *The function F in C^∞_m is holomorphic if and only if $R_{x_+}\Phi_F = 0$.*

♡[illegible] Finally, combining Lemma 21.3 and Corollary 21.7:

[lift-hola] **21.8. Theorem.** *The map $F \mapsto \Phi_F$ is a G-equivariant isomorphism of $\mathcal{C}^+_m$ with the space of all smooth functions Φ in $C^\infty(G)$ such that*

(a) $\Phi\big(gk)\big) = \varepsilon^m(k)\Phi(g)$ for all k in K;
(b) $R_{x_+}\Phi = 0$.

♡ [holo-ds-mc] This concludes the proof of Theorem 18.2. Something similar holds for $\mathcal{C}^-_m$.

22. Square-integrability

Assume $n \ge 1$.

In this section, I'll show that for $n \ge 2$ matrix coefficients of $\mathrm{DS}^\pm_n$ are square-integrable on G. There are two basic steps involved in this. The first is to find interpret the map from $\mathcal{C}^+_m$ to functions on G in terms of matrix coefficients, and the second is to use this construction to see that the functions you get are square-integrable.

SQUARE-INTEGRABILITY IN HOLOMORPHIC REALIZATIONS. For every multiplicative character χ of $\mathbb{C}^\times$ define $C^\infty_\chi(\mathfrak{X}^+)$ to be that of all smooth functions F on $\mathfrak{X}^+$ such that

$$F(\lambda v) = \chi(\lambda)F(v)\,.$$

This space is taken into itself by G. As before, such a function is determined by its restriction f to the embedded copy of $\mathcal{H}$ in $\mathfrak{X}$, and the effect of G on such functions is defined by the formula

$$[\pi_\chi f](z) = \chi(j(g^{-1}, z))f(g^{-1}(z))\,.$$

In particular, if $\chi(\lambda) = |\lambda|^n$ then

$$[\pi_\chi f](z) = |j(g^{-1}, z)|^n f(g^{-1}(z))\,.$$

♡ [ytrans] Lemma 19.3 tells us that if $n = 2$ then $\mathrm{IM}(z)$ is invariant under this action, which suggests defining the function

$$\mathrm{IM}\left(\begin{bmatrix} z \\ w \end{bmatrix}\right) = |w|^2 \mathrm{IM}(z/w)\,.$$

Now suppose F to be in $C^\infty_{-n}(\mathfrak{X}^+)$. $|F|^2 = F\overline{F}$ satisfies the functional equation

$$|F|^2(x\lambda) = |\lambda|^{-2n} F(x)\,.$$

for all x in $\mathfrak{X}$. The product

$$F(v)\mathrm{IM}^n(v)$$

is therefore invariant under scalar multiplication by λ, hence a function on $\mathcal{H}$. On $\mathcal{H}$ the Riemannian metric $(dx^2 + dy^2)/y^2$ is G-invariant, and defines on $\mathcal{H}$ a non-Euclidean geometry. The associated G-invariant measure is

$$dx\,dy/y^2\,.$$

We may therefore integrate $|F|^2$ against $dx\,dy/y^2$, at least formally, What we deduce is this:

[H-measure] **22.1. Proposition.** *The measure $y^{n-2}\,dx\,dy$ is G-invariant for π_{-n}.*

A concrete interpretation of this is that for f in $C_c^\infty(\mathcal{H})$ and $f_* = \pi_{-n}(g^{-1})f$ we have

$$\int_{\mathcal{H}} \big|f_*(z)\big|^2 y^n\,\frac{dx\,dy}{y^2} = \int_{\mathcal{H}} \big|f(z)\big|^2 y^n\,\frac{dx\,dy}{y^2}\,.$$

Proof. We can prove this directly. The first integrand is

$$\big|f\big(g(z)\big)\big|^2 \big|j(g,z)\big|^{-2n} y^n(z) = \big|f\big(g(z)\big)\big|^2 y^n\big(g(z)\big)\,,$$

so the result follows from the invariance of $dx\,dy/y^2$. ∎

Now comes the interesting fact. Take

$$F = \frac{(z-i)^p}{(z+i)^{p+n}}\,,$$

which is an eigenfunction of K in π^+_{-n}. Since $(x-i)^p/(x+ui)^p$ is bounded on $\mathcal{H}$ and $1/(z+i)^n$ is square-integrable with respect to the measure $y^{n-2}\,dx\,dy$ for $n > 1$, integration defines a G-invariant Hilbert norm on $\mathcal{C}^+_{-n}$. If we take

$$\varphi(z) = \frac{1}{(z+i)^n}$$

then each function

[mc-sqint] **(22.2)** $$\Phi_\varphi(g) = \pi(g^{-1})f \bullet \overline{\varphi}$$

♡ [lift-hola] is a matrix coefficient satisfying conditions (a) and (b) of Theorem 21.8.

♡ [mc-sqint] SQUARE-INTEGRABILITY ON G. It remains to be seen that the matrix coefficient (22.2) is square-integrable on G. This will be straight forward, once I recall an integral formula on G.

The group K acts by non-Euclidean rotation on $\mathcal{H}$, fixing the point i. Under these rotations, the vertical ray $[i, i\infty)$ sweep out all of $\mathcal{H}$. Since $\mathcal{H} = G/K$, this tells us:

[cartan-decomp] **22.3. Lemma.** *Every g in G may be factored as $g = k_1 a k_2$ with each k_i in K and*

$$a = \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \quad (t > 0)\,.$$

The factorization can be found explicitly. If such an expression is valid, then

$$ {}^t g\, g = k_2^{-1} a^2 k_2 \,. $$

Thus a^2 is the diagonal matrix of eigenvalues of the positive definite matrix ${}^t g\, g$, and k_2 is its eigenvector matrix. After finding these, set

$$ k_1 = g a^{-1} k_2^{-1} \,. $$

In non-Euclidean radial coordinates this measure is

$$ 2\pi \sinh(r)\, dr\, d\theta \,, $$

or, replacing r by coordinate on A, and taking into account that $\{\pm 1\}$ in K acts trivially:

[radial-volume] **(22.4)**
$$ \int_G f(g)\, dg = \frac{1}{2} \int_{K\times A^+ \times K} f(k_1 a k_2)(\alpha(a) - 1/\alpha(a))\, dk_1\, da\, dk_2 \,. $$

Here A^+ is the set of diagonal matrices

$$ \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} $$

with $t > 1$, and

$$ \alpha\colon ; \begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \longmapsto t^2 \,. $$

This is the same as δ, but for various reasons I prefer new notation here. Very roughly speaking, this formula is true because at iy we have

$$ \frac{\partial}{\partial\theta} = (y^2 - 1)\, \frac{\partial}{\partial x} \,, $$

so that (also at iy)

$$ \frac{dx\, dy}{y^2} = \frac{(y^2 - 1)\, d\theta\, dy}{y^2} \,. $$

♡ [mc-sqint] I now leave as an exercise the verification that for $n \geq 2$ the function defined by (22.2) lies in $\mathrm{L}^2(G)$.

23. Matrix coefficients and differential equations

Matrix coefficients

$$ \Phi_{v,\hat{v}}(g) = \langle \widehat{v}, \pi(g^{-1}) v \rangle $$

can be defined for any admissible representation of $(\mathfrak{g}, K)$. There exist explicit formulas for them in case π is irreducible, but these are not as important as a less precise qualitative description. The point is that matrix coefficients of an admissible representation of $(\mathfrak{g}, K)$ are solutions to certain ordinary differential ♡ [cartan-decomp] equations. According to Lemma 22.3, $G = KAK$. If $\widehat{v}$ and v are chosen to be eigenfunctions of K, then the corresponding matrix coefficient is determined by its restriction to A. But if π is irreducible, it is taken into a scalar multiple of itself by the Casimir operator $\pi(\Omega)$. Because of this, the restriction to A satisfies an ordinary differential equation, which I shall exhibit explicitly.

THE EUCLIDEAN LAPLACIAN. I first look at a relatively familiar example of the phenomenon we shall encounter. Consider the Laplacian in the Euclidean plane:

$$ \Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \,, $$

and let's suppose we want to find solutions of the eigenvalue equation

$$\Delta f = \lambda f$$

with circular symmetry—for example, want to solve the Dirichlet problem in the circle. The first step is separation of variables in polar coordinates. To do this one first expresses Δ in those coordinates:

$$\Delta f = \frac{\partial^2 f}{\partial r^2} + \frac{1}{r}\frac{\partial f}{\partial r} + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2},$$

and then looks for solutions of the form

$$f(r,\theta) = \varphi(r)e^{in\theta}.$$

This leads to the ordinary differential equation

[laplacian-lambda] **(23.1)**
$$\varphi''(r) + \frac{1}{r}\cdot\varphi'(r) - \frac{n^2}{r^2}\cdot\varphi(r) = \lambda\varphi$$

It has a **regular singularity** at the origin and an **irregular** singularity at infinity. Near 0 solutions behave like solutions to Euler's equation

$$\varphi''(r) + \frac{1}{r}\cdot\varphi'(r) - \frac{n^2}{r^2}\cdot\varphi(r) = 0,$$

♡ [laplacian-lambda] which is a first order approximation to (23.1). It can be rewritten as

$$r^2\varphi''(r) + r\varphi'(r) - n^2\varphi(r) = D^2\varphi(r) - n^2\varphi(r) = 0,$$

with $D = xd/dx$, the multiplicatively invariant derivative. A change of variables $x = e^r$ turns it into a more familiar second order equation with constant coefficients

$$\Phi''(y) - n^2\Phi(y) = 0.$$

This has a basis of solutions $e^{\pm nr} = x^{\pm n}$ unless $n = 0$, in which case 1 and $r = \log x$ form a basis. What this means for the original equation is that there exist solutions

$$x^n f_+(x),\ x^{-n} f_-(x) \ \text{ or } \ f_+(x),\ f_-(x)\log x$$

in which $f_\pm(x)$ are entire functions. In all cases, there is just a one-dimensional family of solutions that are not singular at 0.

Near infinity the equation looks more or less like the equation with constant coefficients

[bessel-infinity] **(23.2)**
$$\varphi''(r) = \lambda\varphi(r).$$

The case $\lambda = 0$ can be solved in a simple form, but in general the solutions are Bessel functions. These possess
♡ [bessel-infinity] solutions whose asymptotic behaviour is suggested by the approximating equation (23.2). Note that λ plays a role in this asymptotic behaviour but not in the behaviour of solutions near 0.

In general, a linear differential equation

$$\varphi''(x) + \frac{f_1(x)}{x}\cdot\varphi'(x) + \frac{f_0(x)}{x^2}\cdot\varphi(x) = 0$$

is said to have a regular singularity at $x = 0$ when $f_1(x)$, $f_2(x)$ are real analytic functions in the neighbourhood of 0. The associated Euler's equation is

$$\varphi''(x) + \frac{f_1(0)}{x} \cdot \varphi'(x) + \frac{f_0(0)}{x^2} \cdot \varphi(r) = 0\,,$$

which is equivalent to

$$D^2\varphi + (f_0 - 1)D\varphi + f_1(0)\varphi = 0\,.$$

For Euler's equation we look for solutions of the form x^λ, getting the indicial equation

$$\lambda(\lambda - 1) + f_1(0)\lambda + f_0(0) = 0$$

If this has distinct solutions λ_1, λ_2 then the functions x^{λ_1}, x^{λ_2} are a basis of solutions. Otherwise there is just one root λ and the functions x^λ, $x^\lambda \log x$ are a basis of solutions. Solutions of the original equation are linear comvinations of these whose coefficients are real analytic in the neighbourhood of the singularity.

Differential equations with regular singularities are the ones that occur in analyzing the behaviour of matrix coefficients. Irregular singularities occur when looking at Whittaker models.

The standard reference on ordinary differential equations with singularities is [Coddington-Levinson:1955]. It is clear if dense. A more readable account, but without all details, is [Brauer-Nohel:1967].

THE NON-EUCLIDEAN LAPLACIAN. In non-Euclidean geometry, the Laplacian

$$\Delta_{\mathcal{H}} = \frac{1}{y^2} \cdot \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right)$$

on the upper half plane $\mathcal{H}$ becomes in non-Euclidean polar coordinates

$$\frac{\partial^2}{\partial r^2} + \frac{1}{\tanh r} \cdot \frac{\partial}{\partial r} + \frac{1}{\sinh^2 r} \cdot \frac{\partial^2}{\partial \theta^2}\,.$$

For very small values of r this looks approximately like the Euclidean Laplacian. An eigenfunction of Δ and the rotation group satisfies

$$f''(r) + \frac{1}{\tanh r}\, f'(r) - \frac{n^2}{\sinh^2 r}\, f(r) = \lambda f(r)\,.$$

This has a regular singularity at 0 and an irregular one at ∞. But there is a major difference between this case and the Euclidean one—if we change variables $x = e^r$ we obtain the equation

$$D^2 f - \left(\frac{1+x^2}{1-x^2} \right) Df - \frac{n^2 x^2}{(1-x^2)^2}\, f = \lambda f\,.$$

This has a regular singularity at both 1 *and* ∞. While both Euclidean and non-Euclidean Laplacian equations have irregular singularities at infinity, the 'asymptotic' series of the second are actually convergent.

Let's look more closely at what happens. The point $x = 0$ is at effectively at infinity on G, and At 1 the associated Euler's equation is

$$D^2 f - Df = \lambda f$$

with solutions $x^{\pm r}$ with

$$r = 1/2 \pm \sqrt{\lambda + 1/4}$$

as long as $\lambda \neq -1/4$, and

$$x^{1/2}, x^{1/2} \log x$$

otherwise. Now $\lambda = -1/4 - t^2$ is the eigenvalue of a unitary principal series parametrized by it, and in this case we have series solutions

$$f_+(x)x^{1/2+it} + f_-(x)x^{1/2}x^{1/2-it}$$

as long as $t \neq 0$, and otherwise

$$f_0(x)x^{1/2} + f_1(x)x^{1/2}\log x\,.$$

♡ [radial-volume] According to the formula (22.4) these just fail to be in $\mathrm{L}^2(G)$.

At $x = 1$, we need to make a change of variables $t = y - 1$. I leave it as an exercise to see that the solutions here behave exactly as for the Euclidean Laplacian.

DERIVING RADIAL COMPONENTS. Here's a preview of what's to come. Suppose π to be an irreducible admissible representation of $(\mathfrak{g}, K)$. I now want to describe a matrix coefficient

$$\Phi(g) = \langle \widehat{\pi}(g)\widehat{v}, v\rangle$$

with $\widehat{v}$ and v eigenfunctions of K—say

$$\pi(k)\widehat{v} = \varepsilon^m(k)\widehat{v}, \quad \pi(k)v = \varepsilon^n(k)v\,.$$

Since $G = KAK$, the function Φ is determined by its restriction to A. Since π is irreducible, the Casimir operator acts on $\widehat{V}$ and V by some scalar γ, and this means that Φ is an eigenfunction of Ω. In short, the situation looks much like it did when searching for radial solutions of Laplace's equation, and we expect to find an ordinary differential equation satisfied by the restriction of Φ to A.

What is the ordinary differential equation satisfied by the restriction of Φ to A? What we know about Φ can be put in three partial differential equations satisfied by Φ:

(23.3)
[des-satisfied]

$$\begin{aligned} R_\Omega\Phi &= \ \gamma\,\Phi \\ L_\kappa\Phi &= mi\,\Phi \\ R_\kappa\Phi &= \ ni\,\Phi\,. \end{aligned}$$

♡ [cartan-decomp] The Cartan decomposition Lemma 22.3 says that the product map from $K \times A \times K$ to G is surjective. looking at what happens on the unit disk, it is apparently a non-singular map except near K. That is to say, for $a \neq 1$ in $|A|$, we are going to verify that L_κ, R_h, and R_κ for a basis of the tangent space at a (recall the notation of §2).

The basic fact comes from a trivial computation. For any g in G and X in $\mathfrak{g}$

$$\begin{aligned} [\mathrm{L}_{-X}f](g) &= \frac{d}{dt}f(\exp(tX)g) \\ &= \frac{d;}{dt}f(g\cdot g^{-1}\exp(tX)g) \\ &= [R_{X^g}f](g)\,. \end{aligned}$$

♡ [L-R] (We have seen this before in (8.5).) Thus our remark about the three vectors spanning the tangent space at a reduce to this:

[infinitesimal-cartan] **23.4. Lemma.** (Infinitesimal Cartan decomposition) *For $a \neq \pm 1$ in A*

$$\mathfrak{g} = \mathfrak{k}^a \oplus \mathfrak{a} \oplus \mathfrak{k}\,.$$

Proof. Since we know that

$$\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{a} \oplus \mathfrak{k} \,,$$

it suffices to see that ν_+ can be expressed can be expressed as a linear combination of κ^a and κ. Let α be the character of A taking

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} \longmapsto t^2 \,.$$

We shall need both of the following formulas:

$$\begin{aligned} \nu_+ &= \frac{\alpha(a)}{1-\alpha^2(a)} \cdot (\kappa^a - \alpha(a)\kappa) \\ \nu_- &= \frac{1}{1-\alpha^2(a)} \cdot (\kappa - \alpha(a)\kappa^a) \,. \end{aligned}$$

These can be easily deduced (I write α for $\alpha(a)$, ν for ν_+):

$$\begin{aligned} \kappa \; &= \nu_- - \nu \\ \kappa^a &= \alpha\nu_- - \alpha^{-1}\nu \\ \alpha\kappa &= \alpha\nu_- - \alpha\nu \\ \kappa^a - \alpha\kappa &= (\alpha - \alpha^{-1})\nu \\ \nu &= \frac{\kappa^a - \alpha\kappa}{\alpha - \alpha^{-1}} \\ \alpha^{-1}\kappa &= \alpha^{-1}\nu_- - \alpha^{-1}\nu \\ \nu_- &= \frac{\kappa^a - \alpha^{-1}\kappa}{\alpha^{-1} - \alpha} \,. \end{aligned} \qquad \blacksquare$$

If X is any element of $\mathfrak{g}$ we can write

$$\begin{aligned} \nu X &= \frac{\alpha(a)}{1-\alpha^2(a)} \cdot \kappa^a X - \frac{\alpha^2(a)}{1-\alpha^2(a)} \cdot \kappa X \\ &= \frac{\alpha(a)}{1-\alpha^2(a)} \cdot \kappa^a X - \frac{\alpha^2(a)}{1-\alpha^2(a)} \cdot X\kappa - \frac{\alpha^2(a)}{1-\alpha^2(a)} \cdot [\kappa, X] \,. \end{aligned}$$

Since

$$\Omega = \frac{h^2}{4} - \frac{h}{2} + \nu\nu_- \,,$$

this leads to:

[restriction-satisfied] **23.5. Proposition.** *If the equations (23.3) are satisfied, then the restriction φ of Φ to A satisfies*

$$\varphi''(x) - \left(\frac{1+x^2}{1-x^2}\right)\varphi'(x) - \left(\frac{x^2(m^2+n^2) - x(1+x^2)mn}{(1-x^2)^2}\right)\varphi(x) = \lambda\varphi(x) \,.$$

Here $x = t^2$ and $\partial/\partial x = (1/2)\,\partial/\partial t$.

24. The canonical pairing

Refer to [Casselman:2011].

m x

Part IV. Langlands' classification

25. The Weil group

In a later section I'll explain how the **Weil group** $W_{\mathbb{R}}$ plays a role in the representation theory of $G = \mathrm{SL}_2(\mathbb{R})$. In this one I'll recall some of its properties.

DEFINITION OF THE WEIL GROUPS. If F is any local field and E/F a finite Galois extension, the Weil group $W_{E/F}$ fits into a short exact sequence

$$1 \longrightarrow E^\times \longrightarrow W_{E/F} \longrightarrow \mathcal{G}\mathrm{al}(E/F) \longrightarrow 1\,.$$

It is defined by a certain non-trivial cohomology class in the Brauer group $H^2(\mathcal{G}\mathrm{al}(E/F), E^\times)$ determined by local class field theory. If F is $\mathfrak{p}$-adic and E_{ab} is a maximal abelian extension of E, then local class field theory asserts that $W_{E/F}$ may be identified with the subgroup of the Galois group of E_{ab}/F projecting modulo $\mathfrak{p}$ onto powers of the Frobenius. If $F = \mathbb{C}$ or $\mathbb{R}$ there is no such interpretation, and I think it is fair to say that, although the groups $W_{\mathbb{C}} = W_{\mathbb{C}/\mathbb{C}}$ and $W_{\mathbb{R}} = W_{\mathbb{C}/\mathbb{R}}$ are very simple, there is some mystery about their significance.

The group $W_{\mathbb{C}}$ is just $\mathbb{C}^\times$. The group $W_{\mathbb{R}}$ is an extension of $\mathbb{C}^\times$ by $\mathcal{G} = \mathcal{G}\mathrm{al}(\mathbb{C}/\mathbb{R}) = \{1, \sigma\}$, fitting into an exact sequence

$$1 \longrightarrow \mathbb{C}^\times \longrightarrow W_{\mathbb{R}} \longrightarrow \mathcal{G} \longrightarrow 1\,.$$

It is generated by the copy of $\mathbb{C}^\times$ and an element σ mapping onto the non-trivial element of $\mathcal{G}$, with relations

$$z \cdot \sigma = \sigma \cdot \overline{z}, \quad \sigma^2 = -1$$

Since 1 is not the norm of a complex number, the extension does not split. It is not a coincidence that it is isomorphic to the normalizer of a copy of $\mathbb{C}^\times$ in the unit group of the Hamilton quaternions $\mathbb{H}$. If E/F is any finite Galois extension of local fields, then $W_{E/F}$ may be embedded into the normalizer of a copy of $E^\times$ in the multiplicative group of the division algebra over F defined by the class in the Brauer group mentioned above.

The **norm map** NM from $W_{\mathbb{R}}$ to $\mathbb{R}^\times$ extends the norm on its subgroup $\mathbb{C}$, mapping

$$\begin{aligned} z &\longmapsto |z|^2 \\ \sigma &\longmapsto -1\,. \end{aligned}$$

It is a surjective homomorphism.

CHARACTERS OF THE WEIL GROUP. What are the characters of $W_{\mathbb{R}}$? They factor through the maximal abelian quotient of $W_{\mathbb{R}}$, and this is simple to describe.

[weil-abelian] **25.1. Proposition.** *The norm map* NM *identifies* $\mathbb{R}^\times$ *with the maximal abelian quotient of* $W_{\mathbb{R}}$.

Proof. Since $|z|^2 > 0$ for every z in $\mathbb{C}^\times$, the kernel of the norm map is the subgroup $\mathbb{S}$ of s in $\mathbb{C}^\times$ of norm 1. Every one of these can be expressed as $z/\overline{z}$, and since

$$z\sigma z^{-1}\sigma^{-1} = z/\overline{z}\,,$$

it is a commutator. ∎

As one consequence:

[weil-chars] **25.2. Corollary.** *The characters of* $W_{\mathbb{R}}$ *are those of the form* $\chi(\mathrm{NM}(w))$ *for* χ *a character of* $\mathbb{R}^\times$.

Recall that in this essay a character (elsewhere in the literature sometimes called a quasi-character) of any locally compact group is a homomorphism from it to $\mathbb{C}^\times$, and that characters of $\mathbb{R}^\times$ are all of the form $x \mapsto |x|^s (x/|x|)^n$ (depending only on the parity of n). The characters of the compact torus $\mathbb{S}$ are of the form

$$\varepsilon^n\colon\ z \longmapsto z^n$$

for some integer n, and this implies that all characters of $\mathbb{C}^\times$ are of the form

[complex-chars] **(25.3)**
$$z \longmapsto |z|^s (z/|z|)^n\,.$$

♡ [weil-chars] Together with Corollary 25.2, this tells us explicitly what the characters of $W_\mathbb{C}$ and $W_\mathbb{R}$ are.

It will be useful later on to know that another, more symmetrical, way to specify a character of $\mathbb{C}^\times$ is as

[alt-complex-chars] **(25.4)**
$$z \longmapsto z^\lambda \overline{z}^\mu\,,$$

♡ [alt-complex-chars] where λ and μ are complex numbers such that $\lambda - \mu$ lies in $\mathbb{Z}$. The two forms (25.3) and (25.4) are related by the formulas

$$\begin{aligned}\lambda &= (s+n)/2\\ \mu &= (s-n)/2\,.\end{aligned}$$

IRREDUCIBLE REPRESENTATIONS OF THE WEIL GROUP. Since $\mathbb{C}^\times$ is commutative, any continuous finite-dimensional representation of $\mathbb{C}^\times$ must contain an eigenspace with respect to some character. This tells us that all irreducible continuous representations of $W_\mathbb{C}$ are characters.

Now suppose (ρ, U) to be an irreducible continuous representation of $W_\mathbb{R}$ that is not a character. The space U must contain an eigenspace for $\mathbb{C}^\times$, say with character χ. If the restriction of χ to $\mathbb{S}$ is trivial, this eigenspace must be taken into itself by σ. It decomposes into eigenspaces of σ, and each of these becomes an eigenspace for a character of all of $W_\mathbb{R}$. The irreducible representation ρ must then be one of the two possible characters.

Otherwise, suppose the restriction of χ to $\mathbb{S}$ is not trivial, and that $V \subseteq U$ is an eigenspace for χ. Then $\sigma(V)$ is an eigenspace for $\overline{\chi}$, and U must be the direct sum of V and $\sigma(V)$.

We can describe one of these representations of dimension 2 in a simple manner. Consider the two-dimensional representation of $W_\mathbb{R}$ induced by χ from $\mathbb{C}^\times$. One conclusion of my recent remarks is:

[two-dim-weil] **25.5. Proposition.** *Every irreducible representation of $W_\mathbb{R}$ that is not a character is of the form $\mathrm{Ind}(\chi\,|\,\mathbb{C}^\times, W_\mathbb{R})$ for some character χ of $\mathbb{C}^\times$ that is not trivial on $\mathbb{S}$.*

Only χ and $\overline{\chi}$ give rise to isomorphic representations of $W_\mathbb{R}$.

If χ is trivial on $\mathbb{S}$ then the induced representation decomposes into the direct sum of the two characters of $W_\mathbb{R}$ extending those of $\mathbb{C}^\times$.

Proof. Frobenius reciprocity gives a G-equivariant map from ρ to $\mathrm{Ind}(\chi\,|\,\mathbb{C}^\times, W_\mathbb{R})$. ▯

There is a natural basis $\{f_1, f_\sigma\}$ of the space $\mathrm{Ind}(\chi)$. Here f_x has support on $\mathbb{C}^\times x$ and $f_x(zx) = \chi(z)$. For this basis

(25.6)
[irred-2]
$$\begin{aligned}R_z &= \begin{bmatrix}\chi(z) & 0\\ 0 & \overline{\chi}(z)\end{bmatrix}\\ R_\sigma &= \begin{bmatrix}0 & \chi(-1)\\ 1 & 0\end{bmatrix}.\end{aligned}$$

PROJECTIVE REPRESENTATIONS. Later on we shall be interested in continuous maps from $W_\mathbb{R}$ to $\mathrm{PGL}_2(\mathbb{C})$. If ρ is a two-dimensional representation of $W_\mathbb{R}$, it determines also a continuous homomorphism into $\mathrm{PGL}_2(\mathbb{R})$. There are two types:

(1) Direct sums $\chi_1 \oplus \chi_2$ of two characters. One of these gives rise, modulo scalars, to the homomorphism

$$w \longmapsto \begin{bmatrix} [\chi_1/\chi_2](w) & 0 \\ 0 & 1 \end{bmatrix} .$$

That is to say, these correspond to the homomorphisms (also modulo scalars)

$$w \longmapsto \begin{bmatrix} \chi(w) & 0 \\ 0 & 1 \end{bmatrix}$$

for characters χ of $W_{\mathbb{R}}$.

♡ [irred-2] (2) Irreducible representations $\mathrm{Ind}(\chi \,|\, \mathbb{C}^\times, W_{\mathbb{R}})$ with $\overline{\chi} \neq \chi$. In this case, we are looking at the image modulo scalar matrices of the representation defined by (25.6). But modulo scalars all complex matrices

$$\begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix}$$

are conjugate by diagonal matrices. Hence if $\chi(z) = |z|^s (z/|z|)^n$, this becomes the representation (modulo scalars)

$$\begin{aligned} z &\longmapsto \begin{bmatrix} (z/\overline{z})^n & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma &\longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} . \end{aligned}$$

In fact:

[projective-weil] **25.7. Proposition.** *Any continuous homomorphism from $W_{\mathbb{R}}$ to $\mathrm{PGL}_2(\mathbb{C})$ is the image of one to $\mathrm{GL}_2(\mathbb{C})$.*

Given what we know about the classification of representations of $W_{\mathbb{R}}$, this will classify all homomorphisms from $W_{\mathbb{R}}$ to $\mathrm{PGL}_2(\mathbb{R})$. In fact, the proof will do this explicitly.

Proof. Suppose given a homomorphism φ from $W_{\mathbb{R}}$ to $\mathrm{PGL}_2(\mathbb{C})$. We embed $\mathrm{PGL}_2(\mathbb{C})$ into $\mathrm{GL}_3(\mathbb{C})$ by means of the unique 3-dimensional representation induced by that of $\mathrm{GL}_2(\mathbb{C})$ taking

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \longmapsto \begin{bmatrix} a/b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b/a \end{bmatrix} .$$

The image of $\mathrm{PGL}_2(\mathbb{C})$ in $\mathrm{GL}_3(\mathbb{C})$ can be identified with the special orthogonal group of the matrix

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

This can be seen most easily by considering conjugation of the 2×2 matrices of trace 0, which leaves invariant the quadratic form $\det$.

It is elementary to see by eigenvalue arguments that every commuting set of semi-simple elements in any GL_n may be simultaneously diagonalized. This is true of the image of $\varphi(\mathbb{C}^\times)$ in $\mathrm{SL}_3(\mathbb{C})$. The diagonal matrices on $\mathrm{SO}(Q)$ are those of the form

$$\begin{bmatrix} t & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/t \end{bmatrix} ,$$

which is the image of the group of diagonal matrices in $\mathrm{PGL}_2(\mathbb{C})$. Hence $\varphi(\mathbb{C}^\times)$ itself can be conjugated into the diagonal subgroup of $\mathrm{PGL}_2(\mathbb{C})$. It is then (modulo scalars) of the form

$$z \longmapsto \begin{bmatrix} \rho(z) & 0 \\ 0 & 1 \end{bmatrix}$$

for some character ρ. There are now two cases to look at: (a) the character ρ is trivial, or (b) it is not. In the first case, $\varphi(\sigma)$ can be any element of $\mathrm{PGL}_2(\mathbb{C})$; they are all conjugate. In this case, lifting φ to $\mathrm{GL}_2(\mathbb{C})$ is immediate. In the second case, $\varphi(\sigma)$ must lie in the normalizer of the diagonal matrices. Either it lies in the torus itself, in which case φ is essentially a character of $W_\mathbb{R}$, and again lifting is immediate; or it is of the form

$$\begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \text{ which is conjugate by a diagonal matrix to } \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} .$$

In this case $\varphi(\sigma)$ has order two. But since $\sigma^2 = -1$, this means that the restriction of φ to $\mathbb{C}^\times$ has -1 in its kernel. So

$$\rho(z) = (z/\overline{z})^n .$$

Again we can lift. **0**

Remark. This is a very special case of a more general result first proved in [Langlands:1974] (Lemma 2.10) for real Weil groups, and extended to other local fields by [Labesse:1984]. That $\mathbb{C}^\times$ has image in a torus is a special case of Proposition 8.4 of [Borel:1991]. That the image of $W_\mathbb{R}$ is in the normalizer of a torus follows from Theorem 5.16 of [Springer-Steinberg:1970].

26. Langlands' classification for tori

Quasi-split reductive groups defined over $\mathbb{R}$ are those containing a Borel subgroup rational over $\mathbb{R}$. They are classified (up to isomorphism) by pairs $\mathcal{L}, \Phi)$ where $\mathcal{L}$ is a based root datum $(L, \Delta, L^\vee, \Delta^\vee)$ and Φ an involution of Φ of $\mathcal{L}$. The associated **dual group** is the complex group $\widehat{G}$ associated to the dual root datum $\mathcal{L}^\vee$, and its L-group (at least in one incarnation) is the semi-direct product $\widehat{G} \rtimes \mathcal{G}$. More precisely, $\widehat{G}$ is assigned an épinglage, and σ acts on $\widehat{G}$ by an automorphism of the épinglage in which σ acts according to how Φ acts on $\mathcal{L}$.

If G is split, then ${}^LG$ is just the direct product $\widehat{G} \times \mathcal{G}$. For a non-split example, let G be the special unitary group of the Hermitian matrix

$$H = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} ,$$

the group of 3×3 matrices g such that $g = {}^t\overline{g}^{-1}$. Here $\widehat{G}$ is PGL_3, and σ acts on it by the involution $g \mapsto w_\ell \, g^\theta w_\ell^{-1}$, where

$$g^\theta = {}^tg^{-1}, \quad w_\ell = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} .$$

A homomorphism from $W_\mathbb{R}$ into ${}^LG$ is called **admissible** if (a) its image lies in the set of semi-simple elements of ${}^LG$ and (b) the map is compatible with the canonical projection from ${}^LG$ onto $\mathcal{G}$. [Langlands:1989] classifies admissible representations of $(\mathfrak{g}, K)$ by such admissible homomorphisms. That is to say, each admissible homomorphism corresponds to a finite set of irreducible $(\mathfrak{g}, K)$-modules, called an L-packet, and all representations of G arise in such packets. We shall see later what packets look like for $\mathrm{SL}_2(\mathbb{R})$. One point of this correspondance is **functoriality**—whenever we are given an algebraic homomorphism from ${}^LH$ to ${}^LG$ compatible with projections onto $\mathcal{G}$, each L-packet of representations of H will give rise to one of G.

In particular, characters of a real torus ${}^LT$ are classified by conjugacy classes of admissible homomorphisms from $W_{\mathbb{R}}$ to ${}^LT$. I'll not explain the most general result, but just exhibit how it works for the three tori of relevance to representations $\mathrm{SL}_2(\mathbb{R})$.

If T is a torus defined over $\mathbb{R}$, let $X^*(T)$ be its lattice of algebraic characters over $\mathbb{C}$. The involution Φ that defines T as a real torus may be identified with an involution of $X^*(T)$.

I first state what happens for complex tori. Here $W_{\mathbb{C}} = \mathbb{C}^\times$ and every torus is T just a product of copies of $\mathbb{C}$. Let $X_*(T)$ be its lattice of algebraic homomorphisms from $\mathbb{C}^\times$ to T (its algebraic cocharacters) and $X^*(T)$ the lattice of algebraic characters. These are dual lattices. If λ and μ are in $\mathbb{C} \otimes X^*(T)$ and $\lambda - \mu \in X^T$, then one can specify a character of T by the formula

$$\chi_{\lambda,\mu}\colon\ t \longmapsto t^\lambda \overline{t}^\mu$$

It is characterized and even defined by the property that if α is any algebraic cocharacter then

$$\chi_{\lambda,\mu}(\alpha(z)) = z^{\langle\lambda,\alpha\rangle} \overline{z}^{\langle\mu,\alpha\rangle} \,.$$

The is gives an isomorphism of of the group of characters of T with such pairs (λ, μ). The effect of this on $\mathbb{S}$ is to take s to $s^{\lambda-\mu}$. I have already pointed this out for T of dimension one, and the general result follows easily from that. But $X^*(T)$ is $X_*(\widehat{T})$, and continuous homomorphisms from $\mathbb{C}^\times$ are parametrized by exactly the same data, so Langlands' classification is verified in this case.

Now I look at some simple real tori.

Case 1. Let $T = \mathbb{G}_m$ over $\mathbb{R}$. Its group of rational points is $\mathbb{R}^\times$. The lattice L is just $\mathbb{Z}$, and $\Phi = I$. Its dual group is just $\mathbb{C}^\times$. Admissible homomorphisms from $W_{\mathbb{R}}$ to ${}^LT$ may be identified with characters of $W_{\mathbb{R}}$,
♡ **[weil-chars]** which may be identified with characters of $\mathbb{R}^\times = T(\mathbb{R})$ according to Corollary 25.2.

Case 2. Let $T = \mathbb{S}$. Again $X^*(T) = \mathbb{Z}$, but now $\Phi = -I$. The L-group of T is the semi-direct product of $\widehat{T} = \mathbb{C}^\times$, with σ acting on it as $z \mapsto z^{-1}$. Suppose φ to be an admissible homomorphism from $W_{\mathbb{R}}$ to ${}^LT$. It must take

$$\begin{aligned} z &\longmapsto z^\lambda \overline{z}^\mu \\ \sigma &\longmapsto c \times \sigma \end{aligned}$$

for some λ, μ in $\mathbb{C}$ with $\lambda - \mu$ in $\mathbb{Z}$, some c in $\mathbb{C}^\times$.

Conjugating $c \times \sigma$ changes c by a square, so that one may as well assume $c = 1$.

The condition $\varphi(\sigma)^2) = \varphi(-1)$ implies that $\varphi(-1) = 1$, or that $\lambda - \mu$ must be an even integer. That $\varphi(z\sigma) = \varphi(\sigma\overline{z})$ means that $\mu = -\lambda$. Therefore λ itself must be an integer n. This map corresponds to the character $z \mapsto z^n$ of $\mathbb{S}$.

Case 3. Now let T be the real torus obtained from the multiplicative group over $\mathbb{C}$ by restriction of scalars. Its character group, as we have seen, may be identified with all maps $z^\lambda \overline{z}^\mu$ with $\lambda - \mu \in \mathbb{Z}$. In effect I am going to check in this case that Langlands classification is compatible with restriction of scalars.

The dual group $\widehat{T}$ is now $\mathbb{C}^\times \times \mathbb{C}^\times$, and Φ swaps factors. Suppose φ to be an admissible homomorphism from $W_{\mathbb{R}}$ to ${}^LT$. Since $z\sigma = \sigma\overline{z}$ in $W_{\mathbb{R}}$, the restriction of φ to $\mathbb{C}$ takes

[zim] (26.1)
$$z \mapsto (z^\lambda \overline{z}^\mu, z^\lambda \overline{z}^\mu) \quad (\lambda - \mu = n \in \mathbb{Z})$$

for some λ, μ with $\lambda - \mu \in \mathbb{Z}$. Then $\varphi(\sigma) = (x, y) \times \sigma$ for some $(x, y) \in \widehat{T}$. The equation $\sigma^2 = -1$ translates to the condition $xy = (-1)^n$. Up to conjugation we may take $(x, y) = ((-1)^n, 1)$, and then

[sigim] (26.2)
$$\sigma \longmapsto ((-1)^n, 1) \times \sigma \,.$$

27. Langlands' classification for SL(2)

Following a phenomenon occurring for unramified representations of $\mathfrak{p}$-adic groups, Langlands has offered a classification of irreducible representations of real reductive groups that plays an important role in the global theory of automorphic forms. It is particularly simple, but nonetheless instructive, in the case of $G = \mathrm{SL}_2(\mathbb{R})$. The point is to associate to every admissible homomorphism from $W_\mathbb{R}$ to $\mathrm{PGL}_2(\mathbb{C})$ a set of irreducible admissible $(\mathfrak{sl}_2, K)$ modules. Furthermore, the structure of the finite set concerned will be described.

The best way to see the way things work is to relate things to the embeddings of the L-groups of tori into ${}^LG$. There are two conjugacy classes of tori in G, split and compact.

Case 1. Suppose T is split. Then ${}^LT$ is just the direct product of $\mathbb{C}^\times$ and $\mathcal{G}$, and we may map it into the L-group of G:

$$z \longmapsto \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix}, \quad \sigma \longmapsto I \times \sigma\,.$$

Case 2. Suppose T is compact. Then ${}^LT$ is the semi-direct product of $\mathbb{C}^\times$ by $\mathcal{G}\mathrm{al}$, with σ acting as $z \mapsto 1/z$. We may embed this also into the L-group of G:

$$\begin{aligned} z &\longmapsto \begin{bmatrix} z & 0 \\ 0 & 1 \end{bmatrix} \\ \sigma &\longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \end{aligned}$$

But now if we look back at the classification of admissible homomorphisms from $W_\mathbb{R}$ to $\mathrm{PGL}_2(\mathbb{C})$2, we see that *all admissible homomorphisms from $W_\mathbb{R}$ into $\widehat{G}$ factor through some ${}^LT$.* The principal series for the character χ of A corresponds to the admissible homomorphism into ${}^LA$ that parametrizes χ, and the representations $\mathrm{DS}_{n,\pm}$ make up the L-packet parametrized by the admissible homomorphisms that parametrize the character $z^{\pm n}$ of the compact torus.

One feature of Langlands' classification that I have not yet mentioned is evident here—tempered representation of $\mathrm{SL}_2(\mathbb{R})$ are those for which the image of $W_\mathbb{R}$ in $\widehat{G}$ is bounded.

This parametrization might seem somewhat arbitrary, but it can be completely justified by seeing'what happens globally, where the choices I have made are forced by a matching of L-functions associated to automorphic forms.

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