Clifford algebras and spinors

Bill Casselman
University of British Columbia
cass@math.ubc.ca

This essay will present a brief outline of the theory of Clifford algebras, together with a small amount of material about quadratic forms. I follow loosely the well known book **Geometric algebra** by Emil Artin, but with elegant modifications that I saw originally in some lecture notes by Raoul Bott, (dating from 1962), subsequently included in [Atiyah-Bott-Shapiro:1964].

In the first section, I'll recall a few facts about quadratic forms and orthogonal groups that will be needed eventually. In the second, I'll discuss quaternion algebras, which will play a major role later on. In the third, I'll look at Clifford algebras, and spin groups. In the fourth, I'll exhibit a number of examples, mostly in low dimensions.

Throughout, unless specified otherwise, F will be an arbitrary field of characteristic other than two, and Q will be a nondegenerate quadratic form on the F-vector space V.

Contents

1. Quadratic spaces

In this essay, a quadratic form on a vector space V over F is a function Q such that the function

$$u \circ v = (1/2)(Q(u+v) - Q(u) - Q(v))$$

is bilinear. (This requires that the characteristic not be equal to 2.) If V is given a basis $\{v_i\}$, the form Q corresponds to a symmetric matrix M_Q such that

$$Q(v) = {}^t v \, M_Q v \, .$$

The (i,j) entry in M_Q is $v_i \circ v_j$. The determinant of M_Q modulo $(F^\times)^2$ is a well defined invariant of the form, called its **discriminant**.

Thus

$$Q(u+v) = Q(u) + 2(u \circ v) + Q(v), \quad u \circ u = Q(u) \, .$$

If X is any subset of V, let

$$X^\perp = \{v \in V \mid v \circ x = 0 \text{ for all } x \in X\} \, .$$

The intersection $X \cap X^\perp$ may be non-trivial, in which case it will be made up of **isotropic** elements, on which Q vanishes. It is always a vector subspace. The assumption that Q is nondegenerate means that $V^\perp = \{0\}$. Equivalently, the map

$$\mathfrak{B}\colon \ V \longrightarrow \mathrm{Hom}(V, F), \quad v \longmapsto [u \longmapsto v \circ u]$$

is an isomorphism.

1.1. Lemma. *If V has dimension n and U is a linear subspace of V of dimension d, then $U^\perp$ has dimension $n-d$.*

Proof. It is the kernel of the composition of $\mathfrak{B}$ with restriction to U.

1.2. Lemma. *If U is a subspace of V on which the restriction of Q is nondegenerate, then*

$$V = U \oplus U^{\perp}\,.$$

Proof. Because $U \cap U^{\perp} = \{0\}$.

As a special case:

1.3. Lemma. *If $Q(v) \neq 0$ then every vector w in V can be expressed as $c \cdot v + u$ with u in $v^{\perp}$.*

Proof. Familiar. Explicitly

$$c = \frac{w \circ v}{Q(v)}\,.$$

With our assumption that the characteristic of F is not two, one can always choose a basis that makes M_Q diagonal:

1.4. Proposition. *There exists in V a basis (v_i) such that $v_i \circ v_j = 0$ if $i \neq j$.*

Proof. By induction on the dimension of V. If $n = 1$, the result is trivial. Otherwise, choose v_n such that $Q(v_n) \neq 0$, and applying the previous Lemma write $V = F \cdot v_n \oplus v_n^{\perp}$. Apply induction.

QUADRATIC FIELD EXTENSIONS. Suppose E/F to be a separable quadratic field extension, considered as a vector space over F. The norm map $x \mapsto x\overline{x}$ is a quadratic form on E.

HYPERBOLIC SPACES. In general, V will contain isotropic vectors. One quadratic space with lots of them is the hyperbolic space of dimension $2m$. If $m = 1$, this is the quadratic space H_2 whose matrix is

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

The space H_{2m} is the orthogonal sum of m copies of H_2. In a suitable coordinate system, its form matrix is

$$\begin{bmatrix} 0 & I_m \\ I_m & 0 \end{bmatrix}.$$

A subspace of V is called isotropic if Q vanishes identically on it.

1.5. Lemma. *If $\{u_i\}$ is the basis of an isotropic subspace U of V, there exist vectors w_i in V spanning an isotropic subspace W such that*

$$u_i \circ w_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases}$$

In these circumstances,

$$V = U \oplus W \oplus (U \oplus W)^{\perp}\,.$$

Proof. By induction. It suffices to show that there exists w such that

$$w \circ u_i = \begin{cases} 1 & \text{if } i = 1 \\ 0 & \text{otherwise.} \end{cases}$$

I leave this as an exercise.

THE ORTHOGONAL GROUP. The orthogonal group $\mathrm{O}(Q)$ is that of all X in $\mathrm{GL}(V)$ preserving Q, and $\mathrm{SO}(Q)$ is the intersection $\mathrm{O}(Q) \cap \mathrm{SL}(V)$. If V is given a coordinate system, the group $\mathrm{O}(Q)$ is that of all matrices X such that

$${}^{t}X\, M_Q X = M_Q\,.$$

It follows from this formula that $\det(X) = \pm 1$, so that $\mathrm{O}(Q)/\mathrm{SO}(Q)$ has at most two elements. If $Q(v) \neq 0$ the map

$$r_v\colon\ u \longmapsto u - 2\left(\frac{u \circ v}{Q(v)}\right) \cdot v$$

is an orthogonal reflection in the hyperplane $v^\perp$. The determinant of a reflection is -1, so $\mathrm{O}(Q)/\mathrm{SO}(Q) = \{\pm 1\}$.

The following is Theorem 3.20 of [Artin:1966].

1.6. Theorem. *In a non-degenerate quadratic space of dimension n, every transformation σ in $\mathrm{O}(Q)$ can be expressed as a product of at most n reflections.*

Proof. The proof will be by induction on n, and will take a while.

If σ fixes an **anisotropic** vector v (that is to say, such that $Q(v) \neq 0$) then it is essentally an isometry in the space $v^\perp$ of dimension $n-1$, so we may apply induction to see that it is a product of at most $n-1$ reflections.

If there exists a vector v such that $u = \sigma(v) - v$ is anisotropic, let ρ be the reflection r_u, which will take v to $\sigma(v)$, and $\rho\sigma$ will be an isometry that fixes v, so we may apply the earlier case.

So now we are reduced to the case in which (i) all vectors fixed by σ are isotropic and (ii) if v is anisotropic then $\sigma(v) - v$ is isotropic. I claim:

1.7. Lemma. *In these circumstances, σ lies in $\mathrm{SO}(Q)$.*

Proof of the Lemma. In several steps.

Step 1. *No matter whether v is anisotropic or not, $\sigma(v) - v$ is isotropic.*

This should be intuitive. For example, suppose $F = \mathbb{R}$. The set of all $\sigma(v) - v$ is a linear subspace W of V. The anisotropic vectors in V are dense, and therefore their image in W is also dense. But Q is continous, so it vanishes in all of W.

If a subset Ω of $\mathbb{R}^n$ is open and v is an arbitrary vector in $\mathbb{R}^n$, then perturbations of v will generically lie in Ω. This suggests the following general proof.

Suppose ν to be isotropic. Then for any anisotropic v and ε in F

$$Q(\nu + \varepsilon v) = \varepsilon\, \nu \circ v + \varepsilon^2 Q(v)\,.$$

But since V is nondegenerate the vector space $\nu^\perp$ has dimension $n - 1 > 1$, so we may find an anisotropic v in it, and then

$$Q(\nu + \varepsilon v) = \varepsilon^2 Q(v)\,.$$

Thus $Q(\nu + \varepsilon v)$ is anisotropic for all $\varepsilon \neq 0$. For $\varepsilon \neq 0$

$$\begin{aligned} 0 &= Q(\sigma(\nu + \varepsilon v) - (\nu + \varepsilon v)) \\ &= Q((\sigma(\nu) - \nu) + \varepsilon\,(\sigma(v) - v)) \\ &= Q(\sigma(\nu) - \nu) + \varepsilon\,(\sigma(\nu) - \nu) \circ (\sigma(v) - v)\,. \end{aligned}$$

But $|F| > 2$, so each term in this linear function of ε must vanish, which proves the claim.

Step 2. Let W be the image of the linear map $v \mapsto \sigma(v) - v$. The form Q vanishes identically on W, so is has dimension $d \leq n/2$. The claim now is that *σ fixes every v in $W^\perp$*, which is of dimension $n - d$.

Suppose v to lie in $W^\perp$. Then for every u in V

$$\begin{aligned} (\sigma(v) - v) \circ u &= \sigma(v) \circ u - v \circ u \\ &= v \circ \sigma^{-1}(u) - v \circ u \\ &= v \circ (\sigma^{-1}(u) - u) \\ &= 0 \end{aligned}$$

since if $u = \sigma(x)$ then $\sigma^{-1}(u) - u = x - \sigma(x)$ lies in W. But since $\mathrm{RAD}(V) = 0$, $v = 0$. ▯

Step 3. Since σ fixes no anisotropic vector, all vectors in $W^\perp$ are isotropic, and must have dimension at most $\lfloor n/2 \rfloor$. Therefore $n - d = d$, $n = 2d$, and $W = W^\perp$ is a maximal isotropic subspace of V. *The space V itself must be a hyperbolic space of dimension* $2d$, a direct sum of hyperbolic planes.

Step 4. Now choose a basis $\{w_i\}$ of W and a dual basis $\widehat{w}_i$ of an isotropic complement U. The matrix of σ must be of the form

$$\begin{bmatrix} I & S \\ 0 & I \end{bmatrix} .$$

But this lies in $\mathrm{SO}(Q)$, and the Lemma is proved. ▯

Now to conclude the proof of the Theorem. First of all, we know rom the Lemma that it is true for any σ in $\mathrm{O}(Q)$ with $\det(\sigma) = -1$.

Let ρ be any reflection in $O(Q)$. Then $\rho\sigma$ has $\det = -1$, and hence we may write

$$\rho\sigma = r_1 \ldots r_m$$

with $m \le 2d$. But since $\rho\sigma$ has $\det = -1$, m must be odd, hence $m \le 2d - 1$. Write $\sigma = \rho r_1 \ldots r_m$. ▯

2. Quaternion algebras

In this section F is allowed to have characteristic two.

Suppose E/F to be a separable quadratic extension. It might even be the algebra $F \oplus F$. For α in $F^\times$, let $B = B_{E,\alpha}$ be the algebra generated over F by E and an element σ, with relations

$$x\sigma = \sigma\overline{x}, \quad \sigma^2 = \alpha .$$

The field F embeds into it, and its image is the centre of B.

If c lies in $E^\times$, then changing σ to $c\sigma$ determines an isomorphic algebra, and α changes to $c\overline{c}\alpha$, so the isomorphism class of $B_{E,\alpha}$ depends only on the image of α in $F^\times/NE^\times$.

The field E acts on the right on B, so the identification with E^2 is the map

$$(x, y) \longmapsto x + \sigma y .$$

Acting by multiplication on the left, B commutes with this right action of E. This gives us an embedding of B into $M_2(E)$. Explicitly, $x + \sigma y$ takes

$$\begin{aligned} 1 &\longmapsto x + \sigma y \\ \sigma &\longmapsto x + \sigma y\sigma \\ &= x\sigma + \sigma^2\overline{y} \\ &= \sigma\overline{x} + \alpha\overline{y} . \end{aligned}$$

In other words, it corresponds to the matrix

$$\mu(x + \sigma y) = \begin{bmatrix} x & \alpha\overline{y} \\ y & \overline{x} \end{bmatrix} .$$

In conformity with the theory of Galois descent, these are the matrices such that $\gamma^{-1}\overline{X}\gamma = X$, with

$$\gamma = \begin{bmatrix} 0 & \alpha \\ 1 & 0 \end{bmatrix} .$$

The embedding of B into $M_2(E)$ determines by restriction the trace operator $X \mapsto \operatorname{trace}(X)$. The determinant of $\mu(x+\sigma y)$ is

$$N_{B/F}(x) = x\overline{x} - \alpha y\overline{y}\,.$$

Both the trace and the norm lie in F, and the norm is a homomorphism from $B^\times$ to $F^\times$. Considering B as a vector space over F, this gives us a non-degenerate quadratic form of dimension four.

Define the **conjugate** of $x+\sigma y$ to be $\overline{x}+\overline{y}\sigma$. Conjugation is an involutory anti-automorphism. The norm map can then be expressed as

(2.1) $$N(x+\sigma y) = (x+\sigma y)(\overline{x}-\overline{y}\sigma) = (x+\sigma y)(\overline{x+\sigma y})\,.$$

2.2. Proposition. *If α lies in $NE^\times$ then $B_{E,\alpha}$ is isomorphic to $M_2(F)$, and otherwise it is a division algebra.*

In particular, if $E = F\oplus F$ then B is isomorphic to $M_2(F)$. If E is a field, then $E\otimes E$ is isomorphic to $E\oplus E$, so that $B\otimes K$ is isomorphic to $M_2(K)$ for any extension field K/F into which E embeds.

Proof. Suppose λ to be a generator of E/F, satisfying the quadratic equation

$$\lambda^2 - a\lambda + b = 0\,.$$

Take 1, λ as a basis of E/F. Since

$$\lambda\cdot 1 = \lambda$$
$$\lambda\cdot\lambda = a\lambda - b$$

we get an embedding of E into $M_2(F)$ taking

$$\lambda \longmapsto \begin{bmatrix} 0 & -b \\ 1 & a \end{bmatrix}.$$

But then

$$\overline{\lambda} \longmapsto \begin{bmatrix} a & b \\ -1 & 0 \end{bmatrix}$$

If

$$\sigma = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$$

then $\sigma^2 = I$ and

$$\sigma \begin{bmatrix} a & b \\ -1 & 0 \end{bmatrix} \sigma^{-1} = \begin{bmatrix} 0 & -b \\ 1 & a \end{bmatrix}$$

so that $B_{E,1}$ is isomorphic to $M_2(F)$.

Now suppose α does not lie in $NE^\times$. Then from (2.1) one can see that $N(z)\neq 0$ unless $z=0$, and $z\neq 0$ has as inverse $\overline{z}/N(z)$. ∎

Example. Suppose that F does not have characteristic two. Then $E = F(\sqrt{\beta})$ for some β. Say $\lambda = \sqrt{\beta}$. Then B has as basis

$$1,\ \lambda,\ \sigma,\ \sigma\lambda$$

whose squares are

$$1,\ \beta,\ \alpha,\ -\alpha\beta\,.$$

The norm form is

$$N(x,y,z,w) = x^2 - \beta y^2 - \alpha z^2 + \alpha\beta w^2\,.$$

This form has discriminant 1, and in fact this property characterizes the four-dimensional forms defined by quaternion algebras.

If $F = \mathbb{R}$, $E = \mathbb{C}$, and $\alpha = -1$ then B is Hamilton's quaternions $\mathbb{H}$.

The data (E, α) determining B are not at all unique. If K/F is any quadratic extenion embedded into B there exists some β in B such that (k, β) also define B. It commonly happens that B contains lots of quadratic extensions. For $\mathfrak{p}$-adic fields there is exactly one quaternion division algebra, into which all quadratic extensions embed.

What is the orthogonal group of $N_{B/F}$? First of all, if ν lies in $N^1_{E/F}$ then multiplication

(***a***) $$X \longmapsto \nu X$$

preserves the norm. So does conjugation by any element of $B^\times$:

(***b***) $$X \longmapsto \mu X \mu^{-1}\,.$$

Since

$$\mu \cdot \nu X \cdot \mu^{-1} = \mu\nu\mu^{-1} \cdot \mu X \mu^{-1}$$

the two types of orthogonal transformations in (a) and (b) give rise to a homomorphism from the semi-direct product $N^1_{B/F} \rtimes (B^\times/F^\times)$ to the special orthogonal group of $N_{B/F}$.

2.3. Proposition. *This homomorphism from $N^1_{B/F} \rtimes (B^\times/F^\times)$ to $\mathrm{SO}(N_{B/F})$ is an isomorphism.*

The orthogonal group contains in addition the conjugation operator.

Associated to B is also a quadratic space of dimension three. The kernel has dimension three, on which one may consider the restriction of $N_{B/F}$, which I'll call $N_{B,0}$. Conjugations by elements of $B^\times$ take this space to itself and preserve $N_{B,0}$.

2.4. Proposition. *The canonical homomorphism from $B^\times/F^\times$ to $\mathrm{SO}(N_{B,0})$ is an isomorphism.*

ALGEBRAIC GROUPS. The group $F^\times$ may be considered as the F-rational points of a one-dimensional algebraic subvariety

$$\{(x, y) \,|\, xy = 1\}$$

of F^2. Similarly $E^\times$ may be regarded as the F-rational points of a two-dimensional variety, and $B^\times$ as those on an algebraic variety of dimension four. Related algebraic varieties defined over F include $B^\times/F^\times$ and $N^1_{B/F}$. The embedding of $N^1_{B/F}$ into $B^\times$ induces a homomorphism from $N^1_{B/F}$ to $B^\times/F^\times$, whose kernel is $\{\pm 1\}$. As a homomorphism of algebraic groups it is a surjection, which means that the associated map of points rational over an algebraic closure is surjective, but it is not generally a surjection of F-rational points.

For example, if B is the real quaternion algebra $\mathbb{H}$ the map of $\mathbb{R}$-rational points is surjective, whereas if $B = M_2(\mathbb{R})$ it is not. Of course the homomorphism of $\mathbb{C}$-rational points, from $\mathrm{SL}_2(\mathbb{C})$ to $\mathrm{GL}_2(\mathbb{C})/\mathbb{C}^\times$, is surjective.

3. Spinors

The **Clifford algebra** $C = C(V, Q)$ is the quotient of $\bigotimes^\bullet V$ by the two-sided ideal generated by tensors of the form $v \otimes v - Q(v)$. It inherits a multiplication from the tensor algebra. Now

$$Q(u + v) - Q(u) - Q(v) = 2(u \circ v) = (u + v)^2 - u^2 - v^2 = u \cdot v + v \cdot u\,.$$

For example, if $u \circ v = 0$ then $u \cdot v = -v \cdot u$. Thus u and v do not generally commute.

3.1. Lemma. *Suppose V to have dimension n, and let (e_i) be a basis. Then the algebra C has dimension 2^n, and a basis is made up of the images e_S of the tensors*

$$e_{i_1} \otimes \ldots \otimes e_{i_k}$$

as S runs through ordered subsets $\{i_1 < \ldots < i_k\}$ of $[1,n]$.

Proof. This is because C may be filtered by order, and the graded module is the exterior algebra. ▯

There is a simple formula for the product of two basis elements. Define an operation on ordered subsets of $[1,n]$:

$$S \dot{+} T = (S \cup T) - (S \cap T)\,.$$

In effect, this is bit-wise addition modulo 2. Define also a function on pairs from $[1,n]$:

$$(s,t) = \begin{cases} 1 & \text{if } s \le t \\ -1 & \text{if } s > t. \end{cases}$$

3.2. Lemma. *For S, T ordered subsets of $[1,n]$*

$$e_S \cdot e_T = \prod_{S \times T} (s,t) \cdot \prod_{S \cap T} Q(e_u) \cdot e_{S \dot{+} T}\,.$$

Proof. By induction on $|S|$, since $e_s \cdot e_t = -e_t \cdot e_s$ if $s \ne t$. ▯

The ring C is itself graded by parity of degree, defining C^0 and C^1. There exists a canonical embedding of V into C^1, and I shall identify V with its image. It generates C.

The Clifford algebra possesses a natural universal property, characterizing homomorphisms from the tensor algebra that factor through $C(V)$. Because of this, and since $Q(-v) = Q(v)$, there exists a unique **involutory automorphism**

$$\alpha\colon\ C \longrightarrow C, \quad v \longmapsto -v\,.$$

The parity grading of $C(V)$ is the eigenspace decomposition of α.

There is also the **transpose**

$$x \mapsto {}^t x, \quad e_1 \otimes \ldots \otimes e_k \mapsto e_k \otimes \ldots \otimes e_1\,,$$

which is an **anti-automorphism**. And finally is also the **conjugation** anti-automorphism

$$x \mapsto \overline{x} = \alpha({}^t x) = {}^t\alpha(x)\,.$$

For every x in $C(Q)$, define its **norm**

$$N(x) = x \cdot \overline{x}\,.$$

Example. Suppose V to be $\mathbb{R}^2$ with basis i, j and $Q(xi + yj) = -(x^2 + y^2)$. If $k = ij$ in $C(Q)$, then $k^2 = -1$ and the algebra $C(Q)$ may then be identified with Hamiltonian's quaternions $\mathbb{H}$. The conjugate of $w + xi + yj + zk$ is $w - xi - yj - zk$,

$$N(w + xi + yj + zk) = w^2 + x^2 + y^2 + z^2\,,$$

and the norm map is a homomorphism. In general, the norm map does not behave in such a simple fashion, as we shall see.

Let $C = C(V)$. The **Clifford group** of C is the subgroup Γ of x in $C^\times$ such that the map

$$\rho(x)\colon\ v \mapsto \alpha(x) v x^{-1}$$

takes V to itself. It is a group, stable under transpose and conjugation, and ρ is a group homomorphism into $\mathrm{GL}(V)$.

Remark. [Artin:1966] defines a version of ρ by ordimnary conjugation, not this twisted conjugation. This causes him a fair amount of difficulty with signs. It is here that I am following [Bott:1962-3]. There is an illuminating discussion of the point in §3 of Brian Conrad's Math 210 lecture notes. The basic problem with Artin's defintion is that, when n is odd, the center of the Clifford algebra—and hence the kernel of his ρ—is a linear combination of elements in C^0 and C^1. Lack of homogeneity makes many formulations and proofs rather awkward.

What is the image of ρ? What is its kernel? We shall work slowly towards proving:

3.3. Theorem. *The homomorphism ρ is a surjection from Γ onto $\mathrm{O}(Q)$. The kernel of the map consists of the scalars $F^\times$.*

There are three things to be shown: (a) that the image contains $\mathrm{O}(Q)$, (b) that its kernel is the scalars, and (c) that the image is contained in $\mathrm{O}(Q)$. I show these in several steps.

Step 1. The vector v is a unit in C if and only if $Q(v) \neq 0$, in which case

$$v^{-1} = v/Q(v)\,.$$

Then $\alpha(v) = -v$ and

$$\begin{aligned} \alpha(v)uv^{-1} &= -vuv/Q(v) \\ &= -v(2u\circ v - vu)/Q(v) \\ &= -2\cdot\frac{u\circ v}{v\circ v}\cdot v + u \\ &= r_v(u)\,. \end{aligned}$$

Hence:

3.4. Lemma. *If $Q(v) \neq 0$ then v lies in Γ and the corresponding linear transformation is the orthogonal reflection r_v.*

With Artin's definition the image is $-r_v$. This causes trouble because whether or not this lies in $\mathrm{SO}(Q)$ depends on the dimension of V.

By Theorem 1.6, this implies claim (a).

Step 2. Next is (b):*The kernel of ρ is the group of scalars $F^\times$.*

This is Proposition 3.2 in [Bott:1962]. The proof is straightforward if a bit long. Suppose

$$\alpha(x)v = vx$$

for all v in V. If $x = x_0 + x_1$ with $x_i \in C^i$, then upon matching parities this translates to

$$x_0v = vx_0, \quad -x_1v = vx_1\,.$$

Suppose $e = e_i$ to be a basis element of V. We can write x_0 uniquely as $a + eb$ in terms of the basis e_S, with a in C^0 and b in C^1, neither of them involving any terms with a factor e. Setting $v = e$ we get

$$ae + ebe = ea + Q(e)b\,.$$

Since neither a nor b contains any factor e, Lemma 3.2 tells us that $ae = ea$ and $be = -eb$. We can hence cancel $ea = ae$ from both sides. We then deduce that $Q(e)b = -Q(e)b$, which implies that $b = 0$. Hence the expansion of x_0 cannot involve any e_S with i in S. Since e_i was arbitrary, x_0 must be a scalar.

Suppose similarly that $x_1 = a + eb$ with a in C^1, b in C^0. Now

$$av + ebv = -va - veb$$

for all v. Setting $v = e$ we conclude immediately this time that $b = 0$. Now set $a = ec$ with c in C^0. We have

$$vec = -ecv$$

for all v in V. Setting $v = e$ we deduce that $c = 0$, hence $x_1 = 0$.

Step 3. Recall the **norm** map

$$N(x) = x \cdot \overline{x} = x \cdot \alpha({}^t x)\,.$$

I claim that $N(x) \in F^\times$ *for* x *in* Γ.

Proof. It suffices to show that it is in the kernel of ρ. Let u in V be given, and let $v = \rho(x)u$. Then

$$\alpha(x)ux^{-1} = v\,.$$

Apply the transpose map to get

$${}^t x^{-1}\, u\, {}^t\alpha(x) = v = \alpha(x)ux^{-1}$$

so that

$$\alpha(\alpha({}^t x)x)u(\alpha({}^t x)x)^{-1} = u\,.$$

Therefore $\overline{x}x = c$ lies in $F^\times$ and $x^{-1} = \overline{x}/c$ so $x\overline{x} = c$ as well.

Step 4. *The map* N *is a homomorphism from* Γ *to* $F^\times$.

Proof. By the previous step

$$N(xy) = xy \cdot \overline{yx} = x \cdot N(y) \cdot \overline{x} = N(x)N(y)\,.$$

Step 5. *The image of* ρ *is contained in* $\mathrm{O}(Q)$.

Proof. By the previous step

$$N(\rho(x)v) = N(\alpha(x)yx^{-1}) = N(\alpha(x))N(v)N(x)^{-1} = N(v)\,.$$

This concludes the proof of the Theorem.

3.5. Proposition. *The group* Γ *is the union of* $\Gamma \cap C^0$ *and* $\Gamma \cap C^1$.

Proof. Follows from Theorem 1.6 and Theorem 3.3.

Define

$$\mathrm{GSpin}(Q) = \Gamma \cap C^0\,.$$

The image of $\mathrm{GSpin}(Q)$ with respect to ρ is equal to $\mathrm{SO}(Q)$. More precisely:

3.6. Theorem. *This sequence is exact:*

$$1 \longrightarrow F^\times \longrightarrow \mathrm{GSpin}(Q) \longrightarrow \mathrm{SO}(Q) \longrightarrow 1\,.$$

Let $\mathrm{Spin}(Q)$ be the subgroup of $\mathrm{GSpin}(Q)$ on which $N = 1$. In general, the map

$$\mathrm{Spin}(Q) \longrightarrow \mathrm{SO}(Q)$$

is not surjective.

4. Examples

In this section I'll exhibit a number of examples. They will all be of dimension of at most four. In these low dimensions, there is a very simple criterion for deciding whether or not a given x in $C^0(Q)$ actually lies in Γ.

4.1. Lemma. *If $n \le 4$, an element x of $C^0(Q)$ lies in Γ if and only if $N(x)$ lies in $F^\times$.*

Proof. We have already proved one half of this.

Supposing x to lie in C^0 and $N(x)$ to lie in $F^\times$, we must show that $u = xvx^{-1}$ lies in V whenever v lies in V. But since $N(x)$ lies in $F^\times$, $x^{-1} = {}^t c/N(x)$, from which we deduce that ${}^t u = u$. However, under the assumption that $n \le 4$ the subspace of x in $C1(Q)$ with ${}^t x = x$ coincides with V.

This is enhanced by the observation that *if $n \le 3$ then $N(x)$ lies in $F^\times$ for all x in $C^0(Q)$.*

Example. Take $V = F \cdot v$, $Q(x \cdot v) = ax^2$. Then C is spanned by 1 and v, with $v \cdot v = a$, and $C(Q)$ is isomorphic to $F(\sqrt{a})$. If a is a square, this is isomorphic as a ring to the direct sum $F \oplus F$.

Example. Take $F = \mathbb{R}$, $V = \mathbb{R} \cdot j + \mathbb{R} \cdot k$, with $Q(xj + yk) = -(x^2 + y^2)$. If $i = jk$, then also $i^2 = -1$. Embed $\mathbb{C}$ into C^0, taking $\sqrt{-1}$ to i. Then C is the space $\mathbb{C} + \mathbb{C}j$ and

$$jz = \overline{z}j \,.$$

We may identify C^1 with $\mathbb{C}j$. We have relations

$$\begin{aligned} i^2 &= j^2 = k^2 = -1 \\ ij &= k,\ jk = i,\ ki = j \,, \end{aligned}$$

so C is isomorphic to the quaternion algebra $\mathbb{H}$. The group GSpin is isomorphic to $\mathbb{C}^\times$ and if V is identified with $\mathbb{C}j$, z acts as rotation by $z/\overline{z}$.

Example. Take $F = \mathbb{R}$, $V = \mathbb{R}^3$ with basis e_i, $Q(x, y, z) = x^2 + y^2 + z^2$. Set

$$\begin{aligned} i &= e_2 e_3 \\ j &= e_3 e_1 \\ k &= e_1 e_2 \end{aligned}$$

Then C^0 is isomorphic to $\mathbb{H}$, GSpin is $\mathbb{H}^\times$ and Spin is the kernel of $N_{\mathbb{H}/\mathbb{R}}$.

Example. Let F be arbitrary, (V, Q) the hyperbolic plane. If u, v are a hyperbolic basis, then

$$e = (u + v)/2, \quad f = (u - v)/2$$

satisfy

$$Q(e) = 1, \quad Q(f) = -1 \quad e \circ f = 0 \,.$$

Mapping

$$e \longmapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad f \longmapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

determines an isomorphism of $C(Q)$ with $M_2(F)$. It takes

$$ef \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad fe \longmapsto \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} .$$

The group Spin is that of diagonal matrices

$$\begin{bmatrix} t & 0 \\ 0 & 1/t \end{bmatrix} = t \cdot \frac{(1 + ef)}{2} + (1/t) \cdot \frac{(1 - ef)}{2} \,,$$

which act as hyperbolic rotations $(x, y) \mapsto (t^2 x, y/t^2)$.

Example. Suppose $(V, Q) = H_{2d}$, with orthogonal basis e_i, f_i. Then $\mathrm{Spin}(Q)$ contains the elements

$$t \cdot \frac{(1 + e_i f_i)}{2} + (1/t) \cdot \frac{(1 - e_i f_i)}{2}$$

which generate a torus isomorphic to $(F^\times)^d$. Relating this algebraic torus to one in $\mathrm{SO}(Q)$ will eventually show that Spin is a Zariski-connected algebraic variety.

Example. Let F be arbitrary, v the space of 2×2 matrices of trace 0, $Q = -\det$. If we take as coordinates

$$\begin{bmatrix} z & x \\ y & -z \end{bmatrix}$$

the form Q becomes $xy + z^2$. The group $\mathrm{GL}_2(F)$ acts on this by conjugation, preserving Q. The group $\mathrm{PGL}_2(F)$ therefore embeds into $\mathrm{O}(Q)$. This is the unique irreducible representation of $\mathrm{PGL}_2(F)$. The image of $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ (suitably rearranged) is

$$\begin{bmatrix} a/b & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & b/a \end{bmatrix} .$$

and the image of PGL_2 therefore lies in $\mathrm{SO}(Q)$, and in fact is isomorphic to it. The group $\mathrm{O}(Q)$ is generated by its image and $-I$.

As a mild generalization of this, take (V, Q) to be any three-dimensional quadratic space with isotropic vectors. It will be equivalent to $H \oplus ax^2$ for some $a \neq 0$.

Say

$$v_1^2 = 1, \quad v_2^2 = -1, \quad v_3^2 = a\,.$$

Set

$$\gamma_1 = v_2 v_3, \quad \gamma_2 = v_3 v_1, \quad \gamma_3 = \gamma_1 \gamma_2 = -a\, v_1 v_2\,.$$

These, together with 1, form a basis of C^0. We have

$$\begin{aligned} \gamma_1^2 &= v_2 v_3 v_2 v_3 \\ &= -v_2^2 v_3^2 \\ &= a \\ \gamma_2^2 &= v_3 v_1 v_3 v_1 \\ &= -v_1^2 v_3^2 \\ &= -a \\ \gamma_3^2 &= a^2 v_1 v_2 v_1 v_2 \\ &= -a^2 v_1^2 v_2^2 \\ &= a^2\,. \end{aligned}$$

The algebra $C^0(Q)$ is therefore a quaternion algebra. Its norm form is

$$x^2 - ay^2 + az^2 - a^2 w^2\,.$$

It is in fact isomorphic to $M_2(F)$, since the norm form is isotropic. Consequently, $\Gamma = \mathrm{GL}_2(F)$ and the spin group is $\mathrm{SL}_2(F)$.

Example. Let F be arbitary, E/F quadratic. Let V be the vector space of all 2×2 Hermitian matrices

$$h = \begin{bmatrix} y + x & z \\ \overline{z} & y - x \end{bmatrix}$$

Let Q be the quadratic form

$$(-1/2)\det(h) = z\overline{z} + x^2 - y^2 \,,$$

which takes values in F. Then $\mathrm{Spin}(Q)$ is $\mathrm{SL}_2(E)$. The group $\mathrm{SL}_2(E)$ acts on this:

$$h \mapsto gh{}^t\overline{g}\,,$$

and we get therefore an embedding of $\mathrm{SL}_2(E)$ into $\mathrm{SO}(Q)$. In fact, this identifies $\mathrm{SL}_2(E)$ with $\mathrm{Spin}(Q)$.

How to see this? Let $E = F(\sqrt{\alpha})$,

$$\begin{aligned} v_1^2 &= 1 \\ v_2^2 &= -1 \\ v_3^2 &= 1 \\ v_4^2 &= -\alpha \end{aligned}$$

The ring $C^0(Q)$ therefore contains the Clifford algebra associated to the quadratic space $H \oplus F$, which we know to be $M_2(F)$. If

$$\gamma = v_1 v_2 v_3 v_4 \,.$$

then

$$\gamma^2 = \alpha \,.$$

This means that $F \oplus F \cdot \gamma$ generates a ring isomorphic to E. Furthermore, γ generates the center of C^0. This gives an embedding of $M_2(E)$ into C^0, which turns out to be an isomorphism. The group Γ is that of all x in $\mathrm{GL}_2(K)$ such that $\det(x)$ lies in $F^\times$, and $\mathrm{Spin}(Q)$ must be $\mathrm{SL}_2(K)$.

Example. In a later version, I'll explain the cases $(V, Q) = H_4 \oplus x^2$, in which the spin group is $\mathrm{Sp}_4(F)$. Also let $(V, Q) = H_6$, in which the spin group is $\mathrm{SL}_4(F)$.

LANGLANDS DUALITY.

4.2. Proposition. *The groups* GSp_{2n} *and* GSpin_{2n+1} *are Langlands duals.*

This is Proposition 2.4 of [Asgari:2002].

As an example of a low-dimensional accident:

4.3. Proposition. *The group* GSpin_5 *is isomorphic to* GSp_4*.*

Proof. This is not quite immediate. We need a suitable isomorphism τ of $X^*(T)$ with $X_*(T)$. It must take Δ to $\Delta^\vee$, as must also its transpose ${}^tT$. The transpose ${}^tT$ is defined by the equation

$$\langle {}^t\tau(u), v\rangle = \langle u, \tau(v)\rangle \,.$$

But if $\gamma = \varepsilon_1$ the pairing matrix is

$$\begin{bmatrix} \langle\alpha, \alpha^\vee\rangle & \langle\alpha, \beta^\vee\rangle & \langle\alpha, \gamma^\vee\rangle \\ \langle\beta, \alpha^\vee\rangle & \langle\beta, \beta^\vee\rangle & \langle\beta, \gamma^\vee\rangle \\ \langle\gamma, \alpha^\vee\rangle & \langle\gamma, \beta^\vee\rangle & \langle\gamma, \gamma^\vee\rangle \end{bmatrix} = \begin{bmatrix} 2 & -1 & 0 \\ -2 & 2 & -1 \\ -1 & 0 & 0 \end{bmatrix} ,$$

so the map

$$\begin{aligned} \tau \colon \alpha &\longmapsto \beta^\vee \\ \beta &\longmapsto \alpha^\vee \\ \gamma &\longmapsto \gamma^\vee \,. \end{aligned}$$

is the one we are looking for.

The isomorphism is not unique, since there exists an involution of GSp_{2n} acting as I in Sp_{2n} but taking μ to $-\mu$:

$$X \longmapsto {}^{*}X^{-1} .$$

Therefore there are two possible identifications of the dual of GSp_4 with GSp_4, both taking B to B, T to T. (This caused me some confusion in verifying the Proposition.)

5. References

1. Emil Artin, **Geometric algebra**, Wiley, 1966.

2. Mahdi Asgari, 'Local L-functions for split spinor groups', *Canadian Journal of Mathematics* **54** (2002), 673–693.

3. Michael Atiyah, Raoul Bott, and Arnold Shapiro, 'Clifford modules', *Topology* **3** (supplementary issue) (1964), 3–38.

4. ——, ——, and Isadore Singer, Harvard lecture notes from a seminar of 1962–3 on the Index Theorem.

5. Raoul Bott, 'The spinor groups', in [Atiyah-Bott-Singer:1962–3].

6. Brian Conrad, 'Clifford algebras and spin groups', lecture notes from Math 210C, available at

`http://math.stanford.edu/~conrad/210CPage/handouts.html`