Arithmetic Theory of Symmetrizable Split Maximal Kac-Moody Groups

by

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Abstract

In this thesis we present a reduction theory for the symmetrizable split maximal Kac-Moody groups. However there are many technical difficulties before one can even formulate a reduction theorem. Combining the two main approaches commonly seen in the literature we define a group, first over any field of characteristic zero and then on any commutative ring of characteristic zero. Then we prove a number of structural properties of the group such as representation in the highest weight modules, existence of a Tits system and an Iwasawa decomposition over $\mathbb{R}$ and $\mathbb{C}$. Finally we arrive at reduction theory which can only hold for part of the group.

Table of Contents

Appendices

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I would like to thank H. Garland for providing me with original copies of [10] and [13], J. Humphreys for mailing me hard copies of [26] and [27] (which are very hard to find) and S. Kumar for scanning and emailing [23] (another text which is not widely available). While these are not cited in the body of the thesis, reading them was very helpful to me as it enabled me to find the right frame to deal with the problems that are the subject of the thesis.

Chapter 1

Introduction

1.1 The Aim

The goal of this Thesis is to state and prove a reduction theorem for Kac-Moody groups which arise from generalized Cartan matrices (GCMs) which are symmetrizable and invertible. In fact the thesis generalizes the earlier work by H. Garland in [11] and [12] which only deals with untwisted affine GCMs. The only exceptions are §18 and §21 in [12]. The former constructs a fundamental domain for the unipotent subgroup while the latter deals with intersection of the $\mathbf{\Gamma}$-orbits with the Siegel set. The question of constructing a fundamental domain does not generalize since his construction relies on the realization of untwisted affine Kac-Moody algebras as central extensions of loop algebras. As for the intersection of the $\mathbf{\Gamma}$-orbits with the Siegel set there is a analogous theory in the general symmetrizable case however this is not covered in the thesis.

1.2 Structure of the Thesis

The thesis has seven chapters beside the current one and each chapter in the thesis begin with short overview of what will follow. The seven chapters can be divided in four parts:

- Chapters 2 and 3 chapters provide an introduction to theory of Kac-Moody algebras and their representation theory.
- The material in chapter 4 corresponds to [11], it provides an arithmetic theory for Kac-Moody algebras.
- In chapters 5, 6 and 7 we define the maximal Kac-Moody group. Most of

the material is from chapter I of [17]. After defining a number of subgroups (including the minimal parabolics and the Borel) O. Mathieu employs techniques from algebraic geometry to define a group by constructing every single Bruhat cell and constructing global product and inverse maps. Instead of doing this we simply use Tits's work and take the maximal Kac-Moody group to be the product of parabolic subgroups amalgamated along their intersections.

- Chapter 8 deals with reduction theory of the group constructed earlier and contains all the new results. First we encounter the problem that unlike the finite dimensional case we can not have a reduction theory for the entire group. So one has to find a $\mathbf{\Gamma}$-invariant proper subset of the group which is well behaved. There are various candidates, we examine two in detail, prove reduction theory for one of them and show that it contains a large subset. Finally we conclude by listing some open problems.

1.3 Notation

Here is some of the notation used in the thesis:

- $\mathbb{C}, \mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}$: the usual suspects.
- $\otimes$ always indicates $\otimes_{\mathbb{Z}}$.
- $\mathscr{F}$ is a field of characteristic zero.
- $\mathfrak{R}$ is a commutative ring with a unit.
- For any $\mathscr{F}$-vector space $V_{\mathscr{F}}$ we let $V_{\mathscr{F}}^*$ denote its linear dual: $\text{HOM}_{\mathscr{F}}(V_{\mathscr{F}}, \mathscr{F})$.
- $\langle\cdot,\cdot\rangle : V_{\mathscr{F}}^* \times V_{\mathscr{F}} \to \mathscr{F}$ is the natural pairing between $V_{\mathscr{F}}$, and its dual, $V_{\mathscr{F}}^*$.
- $\text{HOM}_{\mathfrak{R}}(\cdot,\cdot)$: the set of $\mathfrak{R}$-module homomorphisms.
- $\text{HOM}_{\mathfrak{R}-\text{alg}}(\cdot,\cdot)$: the set of $\mathfrak{R}$-algebra homomorphisms.
- $\text{HOM}_{\mathfrak{R}-\text{lie}}(\cdot,\cdot)$: the set of $\mathfrak{R}$-Lie algebra homomorphisms.

- $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{l})$ the universal enveloping algebra of the $\mathfrak{R}$-Lie algebra $\mathfrak{l}_{\mathfrak{R}}$.
- $\widetilde{\mathfrak{U}}_{\mathfrak{R}}(\mathfrak{l})$ is the augmentation ideal in $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{l})$.

There is also an index of notation which gives the page on which the symbol was first introduced or defined.

Chapter 2

Kac-Moody Algebras: A Primer

Kac-Moody algebras are a class of Lie algebras generalizing the notion of finite dimensional semi-simple Lie algebras (see Theorem 2.11 below). In this chapter we define the Kac-Moody algebras, and introduce some elementary concepts.

Our references for basic theory of Kac-Moody algebras are [7] (chapters 14 - 16 and 19) and [15].

2.1 Definitions

Notation 2.1. Let $n \in \mathbb{N}$ and set $I = \{1, \cdots, n\}$.

Definition 2.2. A *generalized Cartan matrix* (or *GCM* for short) is a square matrix $\mathsf{A} = \left(\mathsf{A}_{ij}\right)_{i,j \in I}$ satisfying the following:

(1) $\mathsf{A}_{ij} \in \mathbb{Z}$ for all $i, j \in I$.

(2) $\mathsf{A}_{ii} = 2$ for all $i \in I$.

(3) $\mathsf{A}_{ij} \leq 0$ if $i \neq j$.

(4) $\mathsf{A}_{ij} \neq 0$ if and only if $\mathsf{A}_{ji} \neq 0$ for all $i, j \in I$.

Definition 2.3. Let A be a GCM of size n and corank r. A *realization* of A is a triplet: $(\mathfrak{a}_{\mathbb{C}}, \Pi, \Pi^{\vee})$, where:

(1) $\mathfrak{a}_{\mathbb{C}}$ is a $\mathbb{C}$-vector space of dimension $n + r$.

(2) $\Pi = \{\alpha_1, \cdots, \alpha_n\}$ is a linearly independent subset of $\mathfrak{a}_{\mathbb{C}}^*$.

(3) $\Pi^{\vee} = \left\{\alpha_1^{\vee}, \cdots, \alpha_n^{\vee}\right\}$ is a linearly independent subset of $\mathfrak{a}_{\mathbb{C}}$.

(4) $\left\langle \alpha_i, \alpha_j^{\vee} \right\rangle = \mathsf{A}_{ji}$ for all $i, j \in I$.

Remark 2.4. If $\dim(\mathfrak{a}_\mathbb{C}) < n+r$ then one can not find linearly independent subsets $\Pi \subset \mathfrak{a}_\mathbb{C}^*$ and $\Pi^\vee \subset \mathfrak{a}_\mathbb{C}$ which satisfy (4) in Definition 2.3.

Remark 2.5. For a given GCM, a realization always exists and is unique up to isomorphism of vector spaces, see Proposition 14.2 and 14.3 in [7].

Definition 2.6. Let A be a GCM with realization $(\mathfrak{a}_\mathbb{C}, \Pi, \Pi^\vee)$. The associated Kac-Moody algebra $\mathfrak{g}_\mathbb{C}$ is the $\mathbb{C}$-Lie algebra generated by $\mathfrak{a}_\mathbb{C}$ and $2n$ generators $\{e_{\pm i} \,:\, i \in I\}$, subject to the following relations:

$$\begin{aligned} [\mathfrak{a}_\mathbb{C}, \mathfrak{a}_\mathbb{C}] &= 0 \\ \left[e_i, e_{-j}\right] &= \delta_{ij}\alpha_i^\vee && \forall i, j \in I \\ [z, e_{\pm i}] &= \pm \langle \alpha_i,\, z\rangle\, e_{\pm i} && \forall i \in I, \forall z \in \mathfrak{a}_\mathbb{C} \end{aligned}$$

And the *Serre relations*:

$$\mathrm{ad}(e_{\pm i})^{1-\mathsf{A}_{ij}}(e_{\pm j}) = 0, \qquad \forall i, j \in I : i \neq j.$$

Remark 2.7. Note that this is not the definition given in [7] and [15], however for the particular class of GCMs we are interested in (symmetrizable Kac-Moody algebras) the two definitions coincide (Theorem 19.30 in [7] or Theorem 9.11 in [15]).

Proposition 2.8 ([7] Proposition 14.17)**.** *There is an automorphism ω of $\mathfrak{g}_\mathbb{C}$ satisfying $\omega^2 = 1$ determined by:*

$$\omega(e_{\pm i}) = -e_{\mp i}, \quad \omega|_{\mathfrak{a}_\mathbb{C}} = -1_{\mathfrak{a}_\mathbb{C}}.$$

2.2 A Classification of GCMs

Definition 2.9. Two GCMs, A and B are called *equivalent* if they have the same size n and there is a permutation τ of I such that:

$$\mathsf{B}_{ij} = \mathsf{A}_{\tau(i)\tau(j)}, \qquad \forall i, j \in I.$$

Definition 2.10. A GCM, A, is called *decomposable* if it is equivalent to a diagonal sum

$$\begin{pmatrix} \mathsf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathsf{A}_2 \end{pmatrix}$$

of smaller GCMs $\mathsf{A}_1, \mathsf{A}_2$. A GCM that is not decomposable is called *indecomposable*.

If A is a GCM so is its transpose. Moreover A is indecomposable if and only if its transpose is indecomposable.

Let $v = (v_1, \cdots, v_n)$ be a vector in $\mathbb{R}^n$. We write $v \geq 0$ ($v > 0$) if $v_i \geq 0$ ($v_i > 0$) for each i. Then one has the following classification of indecomposable GCMs (see [7] Corollary 15.11) into three classes each of which is closed under taking transpose:

(1) A has finite type if and only if there exists $u > 0$ with $\mathsf{A}u > 0$.

(2) A has affine type if and only if there exists $u > 0$ with $\mathsf{A}u = 0$.

(3) A has indefinite type if and only if there exists $u > 0$ with $\mathsf{A}u < 0$.

The following justifies our claim that Kac-Moody algebras generalize the notion of finite dimensional simple Lie algebras:

Theorem 2.11 ([7] Theorem 15.19)**.** *Let A be an indecomposable GCM. Then A has finite type if and only if it is the Cartan matrix of a finite dimensional simple Lie algebra.*

At any rate from now on we will assume that the GCM is invertible, in particular $r = 0$. This is not an essential assumption for what we want to do but it will simplify our task.

2.3 Root System

Definition 2.12. We define two discrete additive subgroup of $\mathfrak{a}_{\mathbb{C}}^*$ and $\mathfrak{a}_{\mathbb{C}}$ generated by Π and $\Pi^\vee$:

$$\mathcal{Q} = \mathbb{Z}\alpha_1 \oplus \cdots \oplus \mathbb{Z}\alpha_n,$$
$$\mathcal{Q}^\vee = \mathbb{Z}\alpha_1^\vee \oplus \cdots \oplus \mathbb{Z}\alpha_n^\vee.$$

$\mathcal{Q}$ is called the *root lattice*, and $\mathcal{Q}^\vee$ the *coroot lattice*. Finally set:

$$\mathcal{Q}_\pm = \mathbb{Z}_\pm\alpha_1 \oplus \cdots \oplus \mathbb{Z}_\pm\alpha_n.$$

Definition 2.13. For $\alpha = \sum k_i\alpha_i \in \mathcal{Q}$ the *height* of α is the number $\mathrm{ht}(\alpha) = \sum k_i$.

Definition 2.14. Introduce a partial ordering $\preceq$ on $\mathfrak{a}_{\mathbb{C}}^*$ by setting $\mu \preceq \lambda$ if $\lambda - \mu \in \mathcal{Q}_+$.

Definition 2.15. For every $\mathfrak{a}_{\mathbb{C}}$-module $V_{\mathbb{C}}$ and every $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ we define the *weight space* associated to λ as :

$$V_{\mathbb{C},\lambda} = \{v \in V_{\mathbb{C}} : z \cdot v = \langle \lambda, z \rangle v, \text{ for all } z \in \mathfrak{a}_{\mathbb{C}}\}.$$

The elements of $V_{\mathbb{C},\lambda}$ are called the *weight vectors* corresponding to λ and the dimension of $V_{\mathbb{C},\lambda}$ is the *multiplicity* of the weight λ. The set:

$$\left\{\lambda \in \mathfrak{a}_{\mathbb{C}}^* \setminus \{0\} : V_{\mathbb{C},\lambda} \neq \{0\}\right\},$$

is called the *weights of* $V_{\mathbb{C}}$.

Definition 2.16. With the *adjoint action*, $z \cdot x = [z, x]$, $\mathfrak{g}_{\mathbb{C}}$ becomes an $\mathfrak{a}_{\mathbb{C}}$-module. In this particular case the weight vectors and weight spaces are referred to as *roots* and *root spaces* respectively, while the weights of $\mathfrak{g}_{\mathbb{C}}$ will be denoted by Δ and called the *root system* of $\mathfrak{g}_{\mathbb{C}}$.

Proposition 2.17 ([7] Proposition 14.18)**.**

(1) $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{\alpha \in \mathcal{Q}} \mathfrak{g}_{\mathbb{C},\alpha}$.

(2) $\dim\left(\mathfrak{g}_{\mathbb{C},\alpha}\right) < \infty$ *for all* $\alpha \in \mathcal{Q}$.

(3) $\mathfrak{g}_{\mathbb{C},\mathbf{0}} = \mathfrak{a}_{\mathbb{C}}$.

(4) *If* $\alpha \neq 0$ *then* $\mathfrak{g}_{\mathbb{C},\alpha} = 0$ *unless* $\alpha \in \mathcal{Q}$.

(5) $\left[\mathfrak{g}_{\mathbb{C},\alpha}, \mathfrak{g}_{\mathbb{C},\beta}\right] \subset \mathfrak{g}_{\mathbb{C},\alpha+\beta}$ *for all* $\alpha, \beta \in \mathcal{Q}$.

Definition 2.18. Proposition 2.17 shows that $\Delta \subset \mathcal{Q}$. The roots in $\mathcal{Q}_+$ are called *positive roots* and denoted by Δ_+ and the roots in $\mathcal{Q}_-$ are called *negative roots* and denoted by Δ_-. If we set:

$$\mathfrak{n}_{\mathbb{C}}^{\pm} = \bigoplus_{\alpha \in \Delta_\pm} \mathfrak{g}_{\mathbb{C},\alpha}$$

then we have a direct sum of $\mathbb{C}$-vector spaces:

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{n}_{\mathbb{C}}^{-} \oplus \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}^{+},$$

This direct sum decomposition is referred to as the *triangular decomposition of* $\mathfrak{g}_{\mathbb{C}}$. The subspaces $\mathfrak{n}_{\mathbb{C}}^{-}$, $\mathfrak{a}_{\mathbb{C}}$ and $\mathfrak{n}_{\mathbb{C}}^{+}$ are in fact Lie subalgebras of $\mathfrak{g}_{\mathbb{C}}$. In accordance with the finite dimensional case $\mathfrak{a}_{\mathbb{C}}$ is called the *Cartan subalgebra* of $\mathfrak{g}_{\mathbb{C}}$.

Proposition 2.19 ([7] Proposition 14.19)**.**

(1) $\dim\left(\mathfrak{g}_{\mathbb{C},\alpha_i}\right) = \dim\left(\mathfrak{g}_{\mathbb{C},-\alpha_i}\right) = 1$.

(2) *If* $k > 1$ *then* $\dim\left(\mathfrak{g}_{\mathbb{C},k\alpha_i}\right) = \dim\left(\mathfrak{g}_{\mathbb{C},-k\alpha_i}\right) = 0$.

Definition 2.20. For each $i \in I$ we define the following subalgebras:

$$\mathfrak{n}_{\pm\alpha_i,\mathbb{C}} = \mathbb{C}e_{\pm i} = \mathfrak{g}_{\mathbb{C},\pm\alpha_i}.$$

The subalgebra $\mathfrak{b}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}^{+}$ is called the *Borel algebra* of $\mathfrak{g}_{\mathbb{C}}$. For each $i \in I$ the corresponding *minimal parabolic* algebra is defined as: $\mathfrak{p}_{i,\mathbb{C}} = \mathfrak{n}_{-\alpha_i,\mathbb{C}} \oplus \mathfrak{b}_{\mathbb{C}}$.

2.4 An Analogue of the Killing Form

For the finite dimensional semi-simple Lie algebras the *Killing form* on a finite dimensional semi-simple Lie algebra is defined as:

$$(x|y) = \mathrm{trace}(\mathrm{ad}(x) \circ \mathrm{ad}(y)).$$

The Killing form has very desirable properties: it is a non-degenerate symmetric bilinear form that is *invariant*, i.e. one has: $([x, y]|z) = (x|[y, z])$. We would like to define an analogue of the Killing form for Kac-Moody algebras that are not of finite type. However since these are infinite dimensional, the expression $\mathrm{trace}(\mathrm{ad}(x) \circ \mathrm{ad}(y))$ is not always defined. Therefore we will impose a further restriction on our GCM so that $\mathfrak{g}_{\mathbb{C}}$ has an analogue of the Killing form.

Definition 2.21. A GCM, A, is called *symmetrizable* if there exists a nonsingular diagonal matrix $\mathsf{D} = \mathrm{diag}(\mathsf{d}_1, \cdots, \mathsf{d}_n)$ and a symmetric matrix B, such that $\mathsf{A} = \mathsf{DB}$.

Remark 2.22. Indecomposable GCMs of finite or affine type are symmetrizable, see Theorem 15.17 in [7].

Definition 2.23. On $\mathfrak{a}_{\mathbb{C}}$ define:

$$\left(\alpha_i^\vee \middle| \alpha_j^\vee\right) = \left(\alpha_j^\vee \middle| \alpha_i^\vee\right) = \mathsf{d}_i \mathsf{d}_j \mathsf{B}_{ij}.$$

Since $\Pi^\vee$ is a basis for $\mathfrak{a}_{\mathbb{C}}$ by extending using linearity we obtain a symmetric bilinear form on the Cartan subalgebra.

Proposition 2.24 ([7] Proposition 16.1)**.** *The symmetric bilinear form on $\mathfrak{a}_{\mathbb{C}}$ defined above is non-degenerate.*

Based on Proposition 2.24 we define a bijection $\mathfrak{a}_{\mathbb{C}}^* \to \mathfrak{a}_{\mathbb{C}}$ given by: $\alpha \mapsto z_\alpha$ where z_α is defined by:

$$(z_\alpha|z) = \langle \alpha, z \rangle, \qquad \forall z \in \mathfrak{a}_{\mathbb{C}},$$

in particular we have $\alpha_i^\vee = \mathsf{d}_i z_{\alpha_i}$. Using this bijection we can define the induced bilinear form on $\mathfrak{a}_{\mathbb{C}}^*$:

$$(\lambda|\mu) := \left(z_\lambda \middle| z_\mu\right).$$

In particular we have: $\left(\alpha_i \middle| \alpha_j\right) = \mathsf{B}_{ij}$.

Theorem 2.25 ([15] Theorem 2.2, Exercise 2.2)**.** *If* A *is symmetrizable then* $\mathfrak{g}_{\mathbb{C}}$ *has a non-degenerate symmetric bilinear* $\mathbb{C}$*-valued form such that:*

(1) $(\cdot|\cdot)$ *is invariant.*

(2) When restricted on $\mathfrak{a}_{\mathbb{C}}$*,* $(\cdot|\cdot)$ *is given by Definition 2.23.*

(3) $\left(\mathfrak{g}_{\mathbb{C},\alpha} \middle| \mathfrak{g}_{\mathbb{C},\beta}\right) = 0$ *unless* $\alpha + \beta = 0$.

(4) Suppose $x \in \mathfrak{g}_{\mathbb{C},\alpha}$, $y \in \mathfrak{g}_{\mathbb{C},-\alpha}$ *then* $[x, y] = (x|y)\, z_\alpha$.

(5) The pairing $\mathfrak{g}_{\mathbb{C},\alpha} \times \mathfrak{g}_{\mathbb{C},-\alpha}$ *given by* $(x, y) \mapsto (x|y)$ *is non-degenerate.*

(6) For each $0 \neq x \in \mathfrak{g}_{\mathbb{C},\alpha}$ *there exists* $y \in \mathfrak{g}_{\mathbb{C},-\alpha}$ *with* $[x, y] \neq 0$.

(7) $(\cdot|\cdot)$ *is uniquely determined by (1) and (2).*

Definition 2.26. The form of Theorem 2.25 is called the *standard invariant form* on $\mathfrak{g}_{\mathbb{C}}$.

Remark 2.27. When A is of finite type the standard invariant form is a multiple of the Killing form.

2.5 Integrable Modules

Definition 2.28. A linear endomorphism F of vector space $V_{\mathbb{C}}$ is called *locally nilpotent* if for every vector $v \in V_{\mathbb{C}}$ there exists $N \in \mathbb{N}$ such that $F^N(v) = 0$.

Definition 2.29. A representation, $\pi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V_{\mathbb{C}})$, is called *integrable* if:

$$V_{\mathbb{C}} = \bigoplus_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} V_{\mathbb{C},\lambda},$$

and if $\pi(e_{\pm i})$ are locally nilpotent endomorphisms of $V_{\mathbb{C}}$ for all $i \in I$.

Lemma 2.30 ([7] Proposition 7.17)**.** *For all $i \in I$, $\mathrm{ad}(e_{\pm i})$ are locally nilpotent endomorphisms of $\mathfrak{g}_{\mathbb{C}}$, in other words the adjoint module is integrable.*

Proof. Since we already know that $\mathfrak{g}_{\mathbb{C}}$ decomposes into $\mathfrak{a}_{\mathbb{C}}$ weight spaces, we only need to show that $\mathrm{ad}(e_{\pm i})$ are locally nilpotent linear endomorphisms of $\mathfrak{g}_{\mathbb{C}}$ for all $i \in I$. We will show $\mathrm{ad}(e_i)$ is locally nilpotent, the proof for $\mathrm{ad}(e_{-i})$ is similar. First we claim that if $\mathrm{ad}(e_i)$ acts locally nilpotently on x and y then it also locally nilpotently on $[x, y]$ to see this consider:

$$\mathrm{ad}(e_i)^N([x, y]) = \sum_{k=0}^{N} \binom{N}{k} \left[\mathrm{ad}(e_i)^k(x), \mathrm{ad}(e_i)^{N-k}(y)\right],$$

$\mathrm{ad}(e_i)^k(x)$ will be 0 if k is sufficiently large and $\mathrm{ad}(e_i)^{N-k}(y)$ will be 0 if $N - k$ is sufficiently large. Thus $\mathrm{ad}(e_i)^N([x, y])$ will be 0 if N is sufficiently large. Therefore the set of elements of $\mathfrak{g}_{\mathbb{C}}$ on which $\mathrm{ad}(e_i)$ acts locally nilpotently is a subalgebra. However

$$\begin{aligned}
\mathrm{ad}(e_i)(e_i) &= 0 \\
\mathrm{ad}(e_i)^{1-\mathsf{A}_{ij}}(e_j) &= 0 && i \neq j \\
\mathrm{ad}(e_i)^2(\alpha_j^\vee) &= 0 \\
\mathrm{ad}(e_i)^3(e_{-i}) &= 0 \\
\mathrm{ad}(e_i)(e_{-j}) &= 0 && i \neq j
\end{aligned}$$

and therefore the subalgebra contains all the generators, so it is the whole of $\mathfrak{g}_{\mathbb{C}}$. □

2.6 Weyl Group

2.6.1 Definition

Definition 2.31. For a locally nilpotent endomorphism F of a vector space $V_{\mathbb{C}}$ we define its exponential, $\textsc{Exp}(F)$, as the formal sum:

$$\textsc{Exp}(F) = \sum_{k=0}^{\infty} \frac{F^k}{k!}.$$

Definition 2.32. If $\pi : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{gl}(V_{\mathbb{C}})$ is an integrable module for $\mathfrak{g}_{\mathbb{C}}$ then for each $i \in I$ we can define:

$$r_i^{\pi} = \textsc{Exp}(\pi(e_i))\,\textsc{Exp}(\pi(-e_{-i}))\,\textsc{Exp}(\pi(e_i)) \in \mathbf{GL}(V_{\mathbb{C}}).$$

In particular since the adjoint module is integrable, for each $i \in I$ we have an automorphism $r_i^{\mathrm{ad}} \in \mathbf{GL}(\mathfrak{g}_{\mathbb{C}})$.

Lemma 2.33 ([7] Propositions 16.11)**.** $r_i^{\mathrm{ad}}(\mathfrak{a}_{\mathbb{C}}) = \mathfrak{a}_{\mathbb{C}}$. *For $z \in \mathfrak{a}_{\mathbb{C}}$ we have:*

$$r_i^{\mathrm{ad}}(z) = z - \langle \alpha_i, z\rangle\, \alpha_i^{\vee}.$$

Proof. Let $z \in \mathfrak{a}_{\mathbb{C}}$, we will compute the action of r_i^{ad} term by term. First we have:

$$\textsc{Exp}(\mathrm{ad}(e_i))(z) = (1 + \mathrm{ad}(e_i))(z) = z + [e_i, z] = z - \langle \alpha_i, z\rangle\, e_i.$$

Based on this we add the second term:

$$\begin{aligned}
\textsc{Exp}(\mathrm{ad}(-e_{-i}))\,\textsc{Exp}(\mathrm{ad}(e_i))(z) &= \textsc{Exp}(\mathrm{ad}(-e_{-i}))\big(z - \langle \alpha_i, z\rangle\, e_i\big)\\
&= \left(1 - \mathrm{ad}(e_{-i}) + \frac{\mathrm{ad}(e_{-i})^2}{2}\right)\big(z - \langle \alpha_i, z\rangle\, e_i\big)\\
&= z - \langle \alpha_i, z\rangle\, e_i - [e_{-i}, z] + \langle \alpha_i, z\rangle\, [e_{-i}, e_i]\\
&\quad + \tfrac{1}{2}\,\mathrm{ad}(e_{-i})\big([e_{-i}, z] - \langle \alpha_i, z\rangle\, [e_{-i}, e_i]\big)\\
&= z - \langle \alpha_i, z\rangle\, e_i - \langle \alpha_i, z\rangle\, e_{-i} - \langle \alpha_i, z\rangle\, \alpha_i^{\vee}\\
&\quad + \tfrac{1}{2}\,\mathrm{ad}(e_{-i})\big(\langle \alpha_i, z\rangle\, e_{-i} + \langle \alpha_i, z\rangle\, \alpha_i^{\vee}\big)
\end{aligned}$$

$$
\begin{aligned}
&= z - \langle \alpha_i, z \rangle e_i - \langle \alpha_i, z \rangle e_{-i} - \langle \alpha_i, z \rangle \alpha_i^\vee \\
&\quad + \tfrac{1}{2}\big(0 + \langle \alpha_i, z \rangle (2e_{-i})\big) \\
&= z - \langle \alpha_i, z \rangle e_i - \langle \alpha_i, z \rangle \alpha_i^\vee
\end{aligned}
$$

Finally:

$$
\begin{aligned}
r_i^{\mathrm{ad}}(z) &= \mathrm{Exp}(\mathrm{ad}(e_i))\,\mathrm{Exp}(\mathrm{ad}(-e_{-i}))\,\mathrm{Exp}(\mathrm{ad}(e_i))(z) \\
&= \mathrm{Exp}(\mathrm{ad}(e_i))\big(z - \langle \alpha_i, z \rangle e_i - \langle \alpha_i, z \rangle \alpha_i^\vee\big) \\
&= (1 + \mathrm{ad}(e_i))\big(z - \langle \alpha_i, z \rangle e_i - \langle \alpha_i, z \rangle \alpha_i^\vee\big) \\
&= \big(z - \langle \alpha_i, z \rangle e_i - \langle \alpha_i, z \rangle \alpha_i^\vee\big) \\
&\quad + \big([e_i, z] - 0 - \langle \alpha_i, z \rangle \big[e_i, \alpha_i^\vee\big]\big) \\
&= z - \langle \alpha_i, z \rangle \alpha_i^\vee - 2\langle \alpha_i, z \rangle e_i + \langle \alpha_i, z \rangle (2e_i) \\
&= z - \langle \alpha_i, z \rangle \alpha_i^\vee \qquad \square
\end{aligned}
$$

Definition 2.34. Let r_i denote the restriction of r_i^{ad} to $\mathfrak{a}_{\mathbb{C}}$, then one gets: $r_i^2 = 1_{\mathfrak{a}_{\mathbb{C}}}$ and $r_i(\alpha_i^\vee) = -\alpha_i^\vee$. In fact we have:

$$
r_i(z) = z - \langle \alpha_i, z \rangle \alpha_i^\vee .
$$

r_i are called the *fundamental reflections*, the group they generate (as a subgroup of $\mathbf{GL}(\mathfrak{a}_{\mathbb{C}})$) is called the *Weyl group* and is denoted by $\mathbf{W}$.

Proposition 2.35 ([7] Proposition 16.13)**.** *The bilinear form* $(\cdot|\cdot)$ *on* $\mathfrak{a}_{\mathbb{C}}$ *is* $\mathbf{W}$-*invariant.*

Proof. Let $z, z' \in \mathfrak{a}_{\mathbb{C}}$. Then:

$$
\begin{aligned}
\big(r_i(z)\big|r_i(z')\big) &= \big(z - \langle \alpha_i, z \rangle \alpha_i^\vee \big| z' - \langle \alpha_i, z' \rangle \alpha_i^\vee\big) \\
&= \big(z\big|z'\big) - \langle \alpha_i, z \rangle \big(\alpha_i^\vee\big|z'\big) - \langle \alpha_i, z' \rangle \big(z\big|\alpha_i^\vee\big) + \langle \alpha_i, z' \rangle \langle \alpha_i, z \rangle \big(\alpha_i^\vee\big|\alpha_i^\vee\big) \\
&= \big(z\big|z'\big) - \langle \alpha_i, z \rangle \big(\mathsf{d}_i z_{\alpha_i}\big|z'\big) - \langle \alpha_i, z' \rangle \big(z\big|\mathsf{d}_i z_{\alpha_i}\big) + \langle \alpha_i, z' \rangle \langle \alpha_i, z \rangle (2\mathsf{d}_i) \\
&= \big(z\big|z'\big) - \langle \alpha_i, z \rangle \mathsf{d}_i \langle \alpha_i, z' \rangle - \langle \alpha_i, z' \rangle \mathsf{d}_i \langle \alpha_i, z \rangle + \langle \alpha_i, z' \rangle \langle \alpha_i, z \rangle (2\mathsf{d}_i) \\
&= \big(z\big|z'\big) \qquad \square
\end{aligned}
$$

In fact a converse is true that is if there exists a non-degenerate symmetric $\mathbf{W}$-invariant bilinear form on $\mathfrak{a}_{\mathbb{C}}$ then A is symmetrizable ([15] Exercise 3.3).

2.6.2 Action on the Weight Space

Definition 2.36. We define a $\mathbf{W}$-action on $\mathfrak{a}_{\mathbb{C}}^*$, let $w \in W$ and $\lambda \in \mathfrak{a}_{\mathbb{C}}^*$ then the weight $w(\lambda)$ is defined as follows:

$$\forall z \in \mathfrak{a}_{\mathbb{C}} : \quad \langle w(\lambda),\, z\rangle = \left\langle \lambda,\, w^{-1}(z)\right\rangle .$$

Lemma 2.37. *The $\mathbf{W}$-action on $\mathfrak{a}_{\mathbb{C}}^*$ is compatible with the bijection $\alpha \mapsto z_\alpha$ and hence the induced bilinear form $(\cdot|\cdot)$ on $\mathfrak{a}_{\mathbb{C}}^*$ is $\mathbf{W}$-invariant as well.*

Proof. Let $w(\lambda) = \mu$ for $\lambda, \mu \in \mathfrak{a}_{\mathbb{C}}^*$ and take $z \in \mathfrak{a}_{\mathbb{C}}$ to be arbitrary:

$$\begin{aligned}(w(z_\lambda)|z) &= \left(z_\lambda \middle| w^{-1}(z)\right) = \left\langle \lambda,\, w^{-1}(z)\right\rangle \\ &= \langle w(\lambda),\, z\rangle = \langle \mu,\, z\rangle = \left(z_\mu \middle| z\right) = \left(z_{w(\lambda)} \middle| z\right)\end{aligned}$$

Since $(\cdot|\cdot)$ is non-degenerate and $z \in \mathfrak{a}_{\mathbb{C}}$ is arbitrary we have: $w(z_\lambda) = z_{w(\lambda)}$.

For the $\mathbf{W}$-invariance, let $\lambda, \mu \in \mathfrak{a}_{\mathbb{C}}^*$ and observe:

$$\left(r_j(\lambda) \middle| r_j(\mu)\right) = \left(z_{r_j(\lambda)} \middle| z_{r_j(\mu)}\right) = \left(r_j(z_\lambda) \middle| r_j(z_\mu)\right) = \left(z_\lambda \middle| z_\mu\right) = (\lambda|\mu)\,. \quad \square$$

Lemma 2.38 ([7] Proposition 16.14)**.** *The action of r_i on $\mathfrak{a}_{\mathbb{C}}^*$ is given by: $r_i(\lambda) = \lambda - \left\langle \lambda,\, \alpha_i^\vee\right\rangle \alpha_i$.*

Proof. Let $\lambda \in \mathfrak{a}_{\mathbb{C}}^*, z \in \mathfrak{a}_{\mathbb{C}}$:

$$\begin{aligned}\langle r_i(\lambda),\, z\rangle &= \langle \lambda,\, r_i(z)\rangle \\ &= \left\langle \lambda,\, z - \langle \alpha_i,\, z\rangle\, \alpha_i^\vee\right\rangle \\ &= \langle \lambda,\, z\rangle - \left\langle \lambda,\, \alpha_i^\vee\right\rangle \langle \alpha_i,\, z\rangle \\ &= \left\langle \lambda - \left\langle \lambda,\, \alpha_i^\vee\right\rangle \alpha_i,\, z\right\rangle \qquad \square\end{aligned}$$

Proposition 2.39 ([7] Proposition 16.15)**.** *If $\alpha \in \Delta, w \in \mathbf{W}$ then $w(\alpha) \in \Delta$. Moreover* $\dim\left(\mathfrak{g}_{\mathbb{C},\alpha}\right) = \dim\left(\mathfrak{g}_{\mathbb{C},w(\alpha)}\right)$.

2.6.3 W as a Coxeter Group

Theorem 2.40 ([7] Theorem 16.17)**.** *The Weyl group* **W** *is a Coxeter group generated by* $r_1, \cdots, r_n$ *with relations:*

$$\begin{aligned} r_i^2 &= 1 & & \\ (r_i r_j)^2 &= 1 & & \text{if } \mathsf{A}_{ij}\mathsf{A}_{ji} = 0 \\ (r_i r_j)^3 &= 1 & & \text{if } \mathsf{A}_{ij}\mathsf{A}_{ji} = 1 \\ (r_i r_j)^4 &= 1 & & \text{if } \mathsf{A}_{ij}\mathsf{A}_{ji} = 2 \\ (r_i r_j)^6 &= 1 & & \text{if } \mathsf{A}_{ij}\mathsf{A}_{ji} = 3 \end{aligned}$$

2.7 Geometry of the Weyl Group

2.7.1 Real and Imaginary Roots

Definition 2.41. $\alpha \in \Delta$ is called a *real root* if there exist $\alpha_i \in \Pi$ and $w \in \mathbf{W}$ such that $\alpha = w(\alpha_i)$, the set of all real roots is denoted by Δ^{re}. A root that is not real is called *imaginary* and the collection of all imaginary roots is indicated by Δ^{im}. Finally $\Delta^{\text{re}}_{\pm}, \Delta^{\text{im}}_{\pm}$ are defined as the 4 possible intersections of real and imaginary roots with positive and negative ones.

Remark 2.42. Real roots behave very much like the roots of finite dimensional semi-simple Lie algebras: they have multiplicity 1 and the only multiples of a real root α that themselves are roots are $\pm\alpha$ ([7] Proposition 16.18). Imaginary roots on the other hand have no counterpart in the finite dimensional Lie algebras (see Proposition 16.27 in [7]). Moreover if $\alpha \in \Delta^{\text{im}}_{+}$ then $k\alpha \in \Delta^{\text{im}}_{+}$ for all $k \in \mathbb{Z}_{+}$ ([7] Corollary 16.25) which in turn implies that $\mathfrak{g}_{\mathbb{C}}$ is infinite dimensional exactly when A is not of finite type.

2.7.2 The Tits Cone

Definition 2.43. Given a GCM A of size n let $\mathfrak{a}_{\mathbb{R}}$ be an n-dimensional $\mathbb{R}$-vector space such that $\left(\mathbb{C} \otimes_{\mathbb{R}} \mathfrak{a}_{\mathbb{R}}, \Pi, \Pi^{\vee}\right)$ is a realization for A. In our case, since we assume A to be invertible, we may take $\mathfrak{a}_{\mathbb{R}}$ to be the $\mathbb{R}$-subspace in $\mathfrak{a}_{\mathbb{C}}$ generated by simple coroots.

Definition 2.44. We define the *Tits cone* and the *open Tits cone* as the following subsets in $\mathfrak{a}_{\mathbb{R}}$:

$$\mathbb{T} = \{z \in \mathfrak{a}_{\mathbb{R}} : \langle \xi, z \rangle < 0, \text{ for finitely many } \xi \in \Delta_+^{\text{re}}\}$$
$$\textsc{Int}(\mathbb{T}) = \{z \in \mathfrak{a}_{\mathbb{R}} : \langle \xi, z \rangle \leq 0, \text{ for finitely many } \xi \in \Delta_+^{\text{re}}\}$$

Here $\textsc{Int}(\mathbb{T})$ is the interior of $\mathbb{T}$ in the metric topology of $\mathfrak{a}_{\mathbb{R}}$.

Remark 2.45. A GCM A has finite type if and only if $\mathbb{T} = \textsc{Int}(\mathbb{T}) = \mathfrak{a}_{\mathbb{R}}$.

The subsets $\mathbb{T}$, $\textsc{Int}(\mathbb{T})$ are closely related to the action of the Weyl group:

Definition 2.46. For each subset $J \subset I$ the corresponding *face* in $\mathfrak{a}_{\mathbb{R}}$ is defined as follows:

$$\mathbb{F}_J = \{z \in \mathfrak{a}_{\mathbb{R}} : \langle \alpha_i, z \rangle = 0, \forall i \in J \text{ and } \langle \alpha_i, z \rangle > 0, \forall i \notin J\}.$$

Given any subset $J \subset I$ we say J has finite type if the principal submatrix corresponding to J has finite type. Now define:

$$\mathbb{D} = \bigcup_{J \subset I} \mathbb{F}_J$$
$$\mathbb{D}_{\text{fin}} = \bigcup_{\substack{J \subset I \\ J \text{ has finite type}}} \mathbb{F}_J$$

Proposition 2.47 ([2] Proposition 4.4.9)**.**

(1) $\mathbb{T} = \mathbf{W} \cdot \mathbb{D}$.

(2) $\textsc{Int}(\mathbb{T}) = \mathbf{W} \cdot \mathbb{D}_{\text{fin}}$.

The following characterization of the closure of the Tits cone in the metric topology of $\mathfrak{a}_{\mathbb{R}}$ will be useful later:

Proposition 2.48 ([29] Proposition 5.6)**.** *If* A *is of indefinite type then:*

$$\mathrm{CL}(\mathbb{T}) = \left\{ z \in \mathfrak{a}_{\mathbb{R}} : \langle \xi, z \rangle \geq 0, \ \forall \xi \in \Delta_{+}^{\mathrm{im}} \right\}.$$

Chapter 3

Kac-Moody Algebras: Highest Weight Modules

In this chapter we introduce the concept of a highest weight module. Almost all of the theory of this class of representations is similar to that of the finite dimensional case; one major difference is that when our GCM is not of finite type then the irreducible quotient of the Verma module is not finite dimensional.

One of the most important aspects of the theory is the *Shapovalov bilinear form* (see Definition 3.11), originally introduced in [22]. This provides us with a non-degenerate symmetric contravariant bilinear form on any irreducible highest weight module (see Proposition 3.13). Furthermore, if the irreducible highest weight module is integrable as well, we get a positive definite inner product.

3.1 Verma Modules

Definition 3.1. Let $\Lambda \in \mathfrak{a}_{\mathbb{C}}^*$ and define $\mathfrak{K}_{\mathbb{C}}^{\Lambda}$ to be the left ideal of $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$ generated by $\mathfrak{n}_{\mathbb{C}}^{+}$ and all elements of the form $z - \langle \Lambda, z \rangle$ where $z \in \mathfrak{a}_{\mathbb{C}}$. The *Verma module* with highest weight Λ is defined as:

$$M(\Lambda)_{\mathbb{C}} = \mathfrak{U}_{\mathbb{C}}(\mathfrak{g})/\mathfrak{K}_{\mathbb{C}}^{\Lambda}.$$

Proposition 3.2 ([7] §19.1)**.** *Let $\mathbb{1}_{\Lambda} \in M(\Lambda)_{\mathbb{C}}$ be the image of $1 \in \mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$. Then:*

(1) Every element of $M(\Lambda)_{\mathbb{C}}$ is uniquely expressible in the form of $u \cdot \mathbb{1}_{\Lambda}$ for some $u \in \mathfrak{U}_{\mathbb{C}}(\mathfrak{n}^{-})$.

(2) $M(\Lambda)_{\mathbb{C}} = \bigoplus_{\lambda \in \mathfrak{a}_{\mathbb{C}}^*} M(\Lambda)_{\mathbb{C},\lambda}$.

(3) $M(\Lambda)_{\mathbb{C},\lambda} \neq 0$ *if and only if* $\lambda \prec \Lambda$.

Another way of defining the Verma module is as follows: let $\mathbb{C}_\Lambda$ be a 1-dimensional vector space on which $\mathfrak{a}_\mathbb{C}$ acts via $\langle \Lambda, \cdot \rangle$. We extend this to a representation of $\mathfrak{b}_\mathbb{C} = \mathfrak{a}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C}^+$ by requiring that $\mathfrak{n}_\mathbb{C}^+$ act trivially. Now we have:

$$M(\Lambda)_\mathbb{C} := \mathrm{I{\scriptstyle ND}}_{\mathfrak{b}_\mathbb{C}}^{\mathfrak{g}_\mathbb{C}}\left(\mathbb{C}_\Lambda\right) = \mathfrak{U}_\mathbb{C}(\mathfrak{g}) \otimes_{\mathfrak{U}_\mathbb{C}(\mathfrak{b})} \mathbb{C}_\Lambda.$$

Lemma 3.3. *The product map gives us an isomorphism of* $\mathbb{C}$*-vector spaces:*

$$\mathfrak{U}_\mathbb{C}(\mathfrak{n}^-) \otimes_\mathbb{C} \mathfrak{U}_\mathbb{C}(\mathfrak{a}) \otimes_\mathbb{C} \mathfrak{U}_\mathbb{C}(\mathfrak{n}^+) \cong \mathfrak{U}_\mathbb{C}(\mathfrak{g}).$$

Proof. This follows from the triangular decomposition of $\mathfrak{g}_\mathbb{C}$ and the PBW Theorem. □

In particular we have: $\mathfrak{U}_\mathbb{C}(\mathfrak{g}) \cong \mathfrak{U}_\mathbb{C}(\mathfrak{n}^-) \otimes_\mathbb{C} \mathfrak{U}_\mathbb{C}(\mathfrak{b})$. Therefore as $\mathfrak{U}_\mathbb{C}(\mathfrak{n}^-)$-modules, $M(\Lambda)_\mathbb{C} \cong \mathfrak{U}_\mathbb{C}(\mathfrak{n}^-)$, where $\mathfrak{U}_\mathbb{C}(\mathfrak{n}^-)$ acts on itself via left multiplication. In other words as $\mathfrak{U}_\mathbb{C}(\mathfrak{n}^-)$-modules, all Verma module look the same.

3.2 The Irreducible Quotient

Definition 3.4. The Verma module has a unique maximal proper submodule ([7] Theorem 10.9) which we shall denote by $M'(\Lambda)_\mathbb{C}$. We define:

$$L(\Lambda)_\mathbb{C} = M(\Lambda)_\mathbb{C}/M'(\Lambda)_\mathbb{C}.$$

Then $L(\Lambda)_\mathbb{C}$ is an irreducible module and therefore it is called the *irreducible highest module with highest weight* Λ.

Remark 3.5. From Proposition 3.2 we see that

$$L(\Lambda)_\mathbb{C} = \bigoplus_{\lambda \in \mathfrak{a}_\mathbb{C}^*} L(\Lambda)_{\mathbb{C},\lambda}$$

and that every weight λ that appears in $L(\Lambda)_\mathbb{C}$ has to be of the form $\Lambda - \alpha$, where $\alpha \in \mathcal{Q}_+$.

Definition 3.6. The set of weights of $L(\Lambda)_{\mathbb{C}}$ will be denoted by $\mathscr{P}_{\Lambda}$. The *depth* of the weight $\lambda = \Lambda - \alpha \in \mathscr{P}_{\Lambda}$, denoted by $\mathrm{dp}(\lambda)$, is taken to be the $\mathrm{ht}(\alpha)$.

Definition 3.7. $\mu \in \mathfrak{a}^*_{\mathbb{C}}$ is called *integral* if $\langle \mu, \alpha_i^\vee \rangle \in \mathbb{Z}$ for all $i \in I$. It is called *dominant* if $\langle \mu, \alpha_i^\vee \rangle \geqslant 0$ for all $i \in I$.

Proposition 3.8 ([7] Proposition 19.14; [16] Corollary 2.1.8)**.** *$L(\Lambda)_{\mathbb{C}}$ is integrable if and only if Λ is dominant and integral.*

3.3 The Shapovalov Bilinear Form

Definition 3.9. Consider the involution $\omega : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{g}_{\mathbb{C}}$ we have from Proposition 2.8 and let $\mathfrak{U}(\omega) : \mathfrak{U}_{\mathbb{C}}(\mathfrak{g}) \to \mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$ be its lift to the universal enveloping algebra. Now define: $\boldsymbol{\sigma} = \mathfrak{U}(\omega) \circ \boldsymbol{\gamma}$, where $\boldsymbol{\gamma}$ is the principal anti-automorphism of $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$ (see Appendix C).

Definition 3.10. Based on Lemma 3.3 we can write $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$ as a direct sum of two vector spaces:

$$\mathfrak{U}_{\mathbb{C}}(\mathfrak{g}) = \mathfrak{U}_{\mathbb{C}}(\mathfrak{a}) \oplus \left(\mathfrak{n}^-_{\mathbb{C}} \cdot \mathfrak{U}_{\mathbb{C}}(\mathfrak{g}) + \mathfrak{U}_{\mathbb{C}}(\mathfrak{g}) \cdot \mathfrak{n}^+_{\mathbb{C}} \right).$$

Let $\boldsymbol{\eta}$ denote the projection on the first factor, this map is commonly referred to as the *Harish-Chandra map*.

Definition 3.11. The *Shapovalov bilinear form* is defined as follows:

$$\begin{cases} \mathbb{S} : \mathfrak{U}_{\mathbb{C}}(\mathfrak{g}) \times \mathfrak{U}_{\mathbb{C}}(\mathfrak{g}) \to \mathfrak{U}_{\mathbb{C}}(\mathfrak{a}) \\ \mathbb{S}(x, y) := \boldsymbol{\eta}(\boldsymbol{\sigma}(x) y) \end{cases}$$

Proposition 3.12 ([18] §2.8 Proposition 1)**.**

(1) $\mathbb{S}$ is symmetric.

(2) For all $x, y, u \in \mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$, $\mathbb{S}(ux, y) = \mathbb{S}(x, \boldsymbol{\sigma}(u) y)$.

(3) For $\alpha \neq \beta \in \mathfrak{Q}$, $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})_\alpha \perp \mathfrak{U}_{\mathbb{C}}(\mathfrak{g})_\beta$.

(4) $\mathbb{S}(1, 1) = 1$.

Since $\Pi^\vee$ is a basis for the abelian Lie algebra $\mathfrak{a}_\mathbb{C}$ (recall our assumption that the GCM A is of full rank), any weight $\Lambda \in \mathfrak{a}_\mathbb{C}^*$ can be extended to the polynomial ring: $\mathfrak{U}_\mathbb{C}(\mathfrak{a}) \cong \mathbb{C}\left[\alpha_1^\vee, \cdots, \alpha_n^\vee\right]$ in a natural way:

$$\Lambda\left(\alpha_i^\vee \alpha_j^\vee\right) = \langle \Lambda, \alpha_i^\vee \rangle \left\langle \Lambda, \alpha_j^\vee \right\rangle.$$

Now we define a bilinear form:

$$\mathbb{S}_\Lambda : M(\Lambda)_\mathbb{C} \times M(\Lambda)_\mathbb{C} \to \mathbb{C},$$

as follows. Let $v, w \in M(\Lambda)_\mathbb{C}$ then by Proposition 3.2 there exist $x, y \in \mathfrak{U}_\mathbb{C}(\mathfrak{n}^-)$ such that $v = x \cdot \mathbb{1}_\Lambda$, $w = y \cdot \mathbb{1}_\Lambda$ and set:

$$\mathbb{S}_\Lambda(v, w) = \Lambda(\mathbb{S}(x, y)).$$

Proposition 3.13 ([16] Proposition 2.3.2)**.**

(1) $\mathbb{S}_\Lambda$ *is symmetric.*

(2) $\mathbb{S}_\Lambda$ *is contravariant, that is:*

$$\mathbb{S}_\Lambda(x \cdot v, w) = \mathbb{S}_\Lambda(v, \boldsymbol{\sigma}(x) \cdot w),$$

for all $v, w \in M(\Lambda)_\mathbb{C}$ *and all* $x \in \mathfrak{U}_\mathbb{C}(\mathfrak{g})$.

(3) $\mathbb{S}_\Lambda(M(\Lambda)_{\mathbb{C},\mu}, M(\Lambda)_{\mathbb{C},\nu}) = 0$ *if* $\mu \neq \nu$.

(4) $\mathbb{S}_\Lambda(M'(\Lambda)_\mathbb{C}, M(\Lambda)_\mathbb{C}) = 0$.

(5) $\mathbb{S}_\Lambda$ *induces a non-degenerate symmetric contravariant bilinear form on* $L(\Lambda)_\mathbb{C}$ *also denoted by* $\mathbb{S}_\Lambda$.

(6) Any contravariant bilinear form on $L(\Lambda)_\mathbb{C}$ *is a scalar multiple of* $\mathbb{S}_\Lambda$ *and hence it is automatically symmetric.*

Remark 3.14 ([15] §9.4). For any highest weight vector $v_\Lambda \in L(\Lambda)_{\mathbb{C},\Lambda}$, we define a corresponding functional: $\mathsf{E}_{v_\Lambda}[\cdot] : L(\Lambda)_\mathbb{C} \to \mathbb{C}$ as follows:

$$v = \mathsf{E}_{v_\Lambda}[v]v_\Lambda + v', \quad v' \in \bigoplus_{\lambda \neq \Lambda} L(\Lambda)_{\mathbb{C},\lambda}.$$

Then one may write:

$$\mathbb{S}_\Lambda(x \cdot v_\Lambda,\ y \cdot v_\Lambda) = \mathsf{E}_{v_\Lambda}[\boldsymbol{\sigma}(x)y \cdot v_\Lambda].$$

A normalization such as $\mathbb{S}_\Lambda(\mathbb{1}_\Lambda,\ \mathbb{1}_\Lambda) = 1$ will determine the bilinear form uniquely, we will use this normalization from now and we will abbreviate $\mathsf{E}_{\mathbb{1}_\Lambda}[\cdot]$ to $\mathsf{E}[\cdot]$.

3.4 A Positive Definite Inner Product

Definition 3.15. Let $\mathfrak{g}_\mathbb{R}$ be the real subalgebra of $\mathfrak{g}_\mathbb{C}$ generated by $\{e_{\pm i} : i \in I\}$ and $\mathfrak{a}_\mathbb{R}$. This gives us a conjugate linear involution of $\mathfrak{g}_\mathbb{C}$ denoted by $u \mapsto \overline{u}$ which we lift to $\mathfrak{U}_\mathbb{C}(\mathfrak{g})$. Since the involution ω satisfies: $\omega(\mathfrak{g}_\mathbb{R}) \subseteq \mathfrak{g}_\mathbb{R}$ and we have: $\overline{\omega(u)} = \omega(\overline{u})$ we can define a conjugate linear anti-automorphism of $\mathfrak{U}_\mathbb{C}(\mathfrak{g})$ of order two by setting $\boldsymbol{\sigma_0} = \overline{\boldsymbol{\sigma}}$ ($\boldsymbol{\sigma}$ was introduced in Definition 3.9).

For any $\Lambda \in \mathfrak{a}^*_\mathbb{R}$ we get a real form: $L(\Lambda)_\mathbb{R} := \mathfrak{g}_\mathbb{R} \cdot \mathbb{1}_\Lambda \subset L(\Lambda)_\mathbb{C}$ and hence a conjugate linear involution of $L(\Lambda)_\mathbb{C}$, denoted by $v \mapsto \overline{v}$. We define a Hermitian form $\{\cdot,\ \cdot\}$ on $L(\Lambda)_\mathbb{C}$ by:

$$\{v,\ w\} = \mathbb{S}_\Lambda(v,\ \overline{w}).$$

Since $\mathbb{S}_\Lambda$ was a contravariant bilinear form, $\{\cdot,\ \cdot\}$ becomes a *contravariant Hermitian form*, this means that we have:

$$\{x \cdot v,\ w\} = \{v,\ \boldsymbol{\sigma_0}(x) \cdot w\}.$$

In order to get an inner product on $L(\Lambda)_\mathbb{C}$ we need $\{\cdot,\ \cdot\}$ to be positive definite. Using the contravariance of $\{\cdot,\ \cdot\}$ we can calculate some inner products. For example:

$$\begin{aligned}\{e_{-i}\cdot\mathbb{1}_\Lambda,\, e_{-i}\cdot\mathbb{1}_\Lambda\} &= \{\mathbb{1}_\Lambda,\, \boldsymbol{\sigma}_0(e_{-i})e_{-i}\cdot\mathbb{1}_\Lambda\}\\ &= \{\mathbb{1}_\Lambda,\, e_i e_{-i}\cdot\mathbb{1}_\Lambda\}\\ &= \{\mathbb{1}_\Lambda,\, ([e_i, e_{-i}] + e_{-i}e_i)\cdot\mathbb{1}_\Lambda\}\\ &= \left\{\mathbb{1}_\Lambda,\, \alpha_i^\vee\cdot\mathbb{1}_\Lambda\right\}\\ &= \left\{\mathbb{1}_\Lambda,\, \left\langle\Lambda,\, \alpha_i^\vee\right\rangle\mathbb{1}_\Lambda\right\}\\ &= \left\langle\Lambda,\, \alpha_i^\vee\right\rangle\{\mathbb{1}_\Lambda,\, \mathbb{1}_\Lambda\}\end{aligned}$$

If we use a normalization such as $\{\mathbb{1}_\Lambda,\, \mathbb{1}_\Lambda\} = 1$ we see that $\left\langle\Lambda,\, \alpha_i^\vee\right\rangle \geq 0$ is a necessary condition for $\{\cdot,\, \cdot\}$ being positive definite. In fact with similar calculations one can show that Λ being dominant and integral is necessary. However it turns out that this condition is sufficient as well:

Theorem 3.16 ([16] Theorem 2.3.13)**.** $\{\cdot,\, \cdot\}$ *is positive definite on* $L(\Lambda)_\mathbb{C}$ *if and only if* Λ *is dominant and integral.*

Notation 3.17. For $v \in L(\Lambda)_\mathbb{C}$ we set: $\|v\| = \sqrt{\{v,\, v\}}$.

Chapter 4

Kac-Moody Algebras: Arithmetic Theory

In any integrable highest weight module, $L(\Lambda)_{\mathbb{C}}$, we would like to define a lattice which is compatible with the inner product defined on $L(\Lambda)_{\mathbb{C}}$. That is, the inner product of any two elements in the lattice is an integer, in particular the length of any element is a positive integer. How should we define such a lattice? Recall that the highest weight vector generates the module: $L(\Lambda)_{\mathbb{C}} = \mathfrak{U}_{\mathbb{C}}(\mathfrak{g}) \cdot \mathbb{1}_{\Lambda}$. Based on this we may define $L(\Lambda)_{\mathbb{Z}} = \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) \cdot \mathbb{1}_{\Lambda}$ where $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$ itself is a lattice in $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$ and we show that this is the lattice in $L(\Lambda)_{\mathbb{C}}$ with the desired properties. This chapter is divided in two sections: in §1 we first define what we mean by a lattice in $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$ (see Definition 4.1) and then construct one by giving generators. In §2 we show that the subset defined in the highest weight module is a lattice with all the desired properties.

The material of §1 is based on [28] §4.4 while §2 follows [11], we note that while the [11] only deals with the specific case of affine GCMs the same proof can be used for the general case as we show here.

4.1 An Integral Form for $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$

4.1.1 Construction

Definition 4.1. An *integral form* of a $\mathbb{C}$-algebra $\mathfrak{A}_{\mathbb{C}}$ is a subring $\mathfrak{A} \subset \mathfrak{A}_{\mathbb{C}}$ such that the canonical map: $\mathbb{C} \otimes \mathfrak{A} \to \mathfrak{A}_{\mathbb{C}}$ is bijective.

Notation 4.2. In order to simplify our notation in this section we will use $\mathfrak{U}_{\mathbb{C}}$, $\mathfrak{U}^0_{\mathbb{C}}$ and $\mathfrak{U}^{\pm}_{\mathbb{C}}$ as a shorthand for $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$, $\mathfrak{U}_{\mathbb{C}}(\mathfrak{a})$ and $\mathfrak{U}_{\mathbb{C}}(\mathfrak{n}^{\pm})$ respectively.

Definition 4.3. We define the following subrings of $\mathfrak{U}_{\mathbb{C}}$ (for notation see Appendix A):

$$\begin{aligned} \mathfrak{U}^{\pm i} &= \bigoplus_{p \in \mathbb{N}} \mathbb{Z} e_{\pm i}^{[p]} \\ \mathfrak{U}^{0} &= \left\langle \binom{z}{p} : z \in \mathcal{Q}^{\vee}, p \in \mathbb{N} \right\rangle \\ \mathfrak{U}^{\pm} &= \langle \mathfrak{U}^{\pm 1}, \cdots, \mathfrak{U}^{\pm n} \rangle \\ \mathfrak{U} &= \langle \mathfrak{U}^{-}, \mathfrak{U}^{0}, \mathfrak{U}^{+} \rangle \end{aligned}$$

Theorem 4.4 ([28] §4.4)**.** $\mathfrak{U}$ *is an integral form for* $\mathfrak{U}_{\mathbb{C}}$.

The proof of Theorem 4.4 can be divided in two steps:

(1) $\mathfrak{U}^{-}, \mathfrak{U}^{0}$ and $\mathfrak{U}^{+}$ are integral forms for $\mathfrak{U}_{\mathbb{C}}^{-}, \mathfrak{U}_{\mathbb{C}}^{0}$ and $\mathfrak{U}_{\mathbb{C}}^{+}$ respectively.

(2) The product map: $\mathfrak{U}^{-} \otimes \mathfrak{U}^{0} \otimes \mathfrak{U}^{+} \to \mathfrak{U}$ is bijective.

(1) and (2) combined with Lemma 3.3 would imply that $\mathfrak{U}$ is an integral form for $\mathfrak{U}_{\mathbb{C}}$.

4.1.2 Proof of Theorem 4.4

Proof of (1)

Proposition 4.5. $\mathfrak{U}^{0}$ *is an integral form for* $\mathfrak{U}_{\mathbb{C}}^{0}$.

Proof. $\mathfrak{U}_{\mathbb{C}}^{0} \cong \mathbb{C}[\alpha_1^{\vee}, \cdots, \alpha_n^{\vee}]$ and $\alpha_1^{\vee}, \cdots, \alpha_n^{\vee} \in \mathfrak{U}^{0}$. □

Proposition 4.6. $\mathfrak{U}^{\pm}$ *is an integral form for* $\mathfrak{U}_{\mathbb{C}}^{\pm}$.

Proof. Since $\mathfrak{U}^{+}$ is a subring of $\mathfrak{U}_{\mathbb{C}}^{+}$, we only need to show that the canonical map $\mathbb{C} \otimes \mathfrak{U}^{+} \to \mathfrak{U}_{\mathbb{C}}^{+}$ is bijective. It is surjective because $\mathfrak{U}^{+}$ contains all the generators of $\mathfrak{U}_{\mathbb{C}}^{+}$. Assume it is not injective, that is:

$$0 \neq \sum_{i=1}^{k} c_i \otimes x_i \mapsto \sum_{i=1}^{k} c_i x_i = 0,$$

where x_i are monomials in $\mathfrak{U}^+$ and $c_i \in \mathbb{C}$ are all nonzero. Using the grading of $\mathfrak{U}_{\mathbb{C}}^+$ we see that all x_i have the same degree m. Therefore $\{x_1, \cdots, x_k\} \subset \mathfrak{U}_{\mathbb{C}}^+$ is linearly independent over $\mathbb{Z}$ but not over $\mathbb{C}$. This contradiction proves that the canonical map is injective as well and hence $\mathfrak{U}^+$ is an integral form for $\mathfrak{U}_{\mathbb{C}}^+$. The proof for $\mathfrak{U}^-$ is identical. □

Proof of (2)

Lemma 4.7. *For $z \in \mathfrak{a}_{\mathbb{C}}$ and $p, q \in \mathbb{N}$ we have:*

$$\binom{z}{p} e_{\pm i}^{[q]} = e_{\pm i}^{[q]} \binom{z \pm q \langle \alpha_i, z \rangle}{p}.$$

Proof. Note that $[z, e_{\pm i}] = \pm \langle \alpha_i, z \rangle e_{\pm i}$ and then use Lemma A.3 with $P(X) = \binom{X}{p}$. □

Lemma 4.8. $\mathfrak{U}^0 \mathfrak{U}^{\pm i} = \mathfrak{U}^{\pm i} \mathfrak{U}^0$.

Proof. This follows from Lemma 4.7. □

Lemma 4.9. $\mathfrak{U}^i \mathfrak{U}^0 \mathfrak{U}^{-i} = \mathfrak{U}^{-i} \mathfrak{U}^0 \mathfrak{U}^i$.

Proof. Use Lemma 4.8 to arrive at:

$$\mathfrak{U}^i \mathfrak{U}^0 \mathfrak{U}^{-i} = \mathfrak{U}^i \mathfrak{U}^{-i} \mathfrak{U}^0$$

Using Lemma A.6 with $x = e_i$, $y = e_{-i}$ and $z = \alpha_i^\vee$ we see that: $e_i^{[p]} e_{-i}^{[q]}$ can be written as a sum, where each summand belongs to $\mathfrak{U}^{-i} \mathfrak{U}^0 \mathfrak{U}^i$. Hence:

$$\mathfrak{U}^i \mathfrak{U}^{-i} \mathfrak{U}^0 = \mathfrak{U}^{-i} \mathfrak{U}^0 \mathfrak{U}^i \mathfrak{U}^0$$

Using Lemma 4.8 one more time gives us the result. □

Lemma 4.10. *If $j, i_1, \cdots, i_m$ is a sequence of $m+1$ elements in I then:*

$$\mathfrak{U}^{i_1} \mathfrak{U}^{i_2} \cdots \mathfrak{U}^{i_m} \mathfrak{U}^{-j} \subset \mathfrak{U}^{-j} \mathfrak{U}^0 \mathfrak{U}^+.$$

Proof. We prove this by induction on m:

- $\underline{m=1}$: If $j = i$ then $\mathfrak{U}^i\mathfrak{U}^{-i} \subset \mathfrak{U}^{-i}\mathfrak{U}^0\mathfrak{U}^+$ follows from Lemma 4.9. If $m = 1$ and $j \neq i$ then $\mathfrak{U}^i\mathfrak{U}^{-j} \subset \mathfrak{U}^{-j}\mathfrak{U}^0\mathfrak{U}^+$ is a consequence of the fact that e_i and e_{-j} commute.

- $\underline{m>1}$:

$$\begin{aligned}
\mathfrak{U}^{i_1}\mathfrak{U}^{i_2}\cdots\mathfrak{U}^{i_m}\mathfrak{U}^{-j} &= \mathfrak{U}^{i_1}\left(\mathfrak{U}^{i_2}\cdots\mathfrak{U}^{i_m}\mathfrak{U}^{-j}\right) \\
&\subset \mathfrak{U}^{i_1}\left(\mathfrak{U}^{-j}\mathfrak{U}^0\mathfrak{U}^+\right) && \text{induction hypothesis} \\
&= \left(\mathfrak{U}^{i_1}\mathfrak{U}^{-j}\right)\mathfrak{U}^0\mathfrak{U}^+ \\
&\subset \left(\mathfrak{U}^{-j}\mathfrak{U}^0\mathfrak{U}^+\right)\mathfrak{U}^0\mathfrak{U}^+ && \text{base of induction} \\
&= \mathfrak{U}^{-j}\mathfrak{U}^0\left(\mathfrak{U}^+\mathfrak{U}^0\right)\mathfrak{U}^+ \\
&= \mathfrak{U}^{-j}\mathfrak{U}^0\mathfrak{U}^0\mathfrak{U}^+\mathfrak{U}^+ && \text{Lemma 4.8} \\
&\subset \mathfrak{U}^{-j}\mathfrak{U}^0\mathfrak{U}^+
\end{aligned}$$

□

Proposition 4.11. *The product map* $\mathfrak{U}^- \otimes \mathfrak{U}^0 \otimes \mathfrak{U}^+ \to \mathfrak{U}$ *is bijective.*

Proof. Lemma 3.3 implies the injectivity of the product map. In order to prove surjectivity let $\mathfrak{U}'$ denote the the image of the product map. Then:

$$\begin{aligned}
\mathfrak{U}'\mathfrak{U}^{-i} &\subset \mathfrak{U}' && \text{Lemma 4.10} \\
\mathfrak{U}'\mathfrak{U}^{i} &\subset \mathfrak{U}' && \mathfrak{U}^i \subset \mathfrak{U}^+ \\
\mathfrak{U}'\mathfrak{U}^{0} &\subset \mathfrak{U}' && \text{Lemma 4.8}
\end{aligned}$$

So $\mathfrak{U}'\mathfrak{U} \subset \mathfrak{U}'$ which implies that $\mathfrak{U}'$ contains $\mathfrak{U}$. □

4.2 The Chevalley Lattice

4.2.1 Construction

Definition 4.12. Set:

$$\mathfrak{g}_{\mathbb{Z}} := \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) \cap \mathfrak{g}_{\mathbb{C}}$$

$$\mathfrak{n}_{\mathbb{Z}}^{\pm} := \mathfrak{U}_{\mathbb{Z}}(\mathfrak{n}^{\pm}) \cap \mathfrak{n}_{\mathbb{C}}^{\pm}$$
$$\mathfrak{a}_{\mathbb{Z}} := \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a}) \cap \mathfrak{a}_{\mathbb{C}} = \mathfrak{Q}^{\vee}$$

In particular this allows us to define all these Lie algebras over any commutative ring of characteristic zero with a unit.

Definition 4.13. Let Λ be an integral and dominant weight and define the *Chevalley lattice*, as $L(\Lambda)_{\mathbb{Z}} := \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) \cdot \mathbb{1}_{\Lambda} \subset L(\Lambda)_{\mathbb{C}}$. Then Chevalley lattice is a $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$-invariant $\mathbb{Z}$-module in $L(\Lambda)_{\mathbb{C}}$, below we will show that it is indeed a lattice in $L(\Lambda)_{\mathbb{C}}$ (see Theorem 4.18).

Lemma 4.14 ([11] Lemma 11.4)**.** $\mathsf{E}[\cdot]$ *takes integer values on* $L(\Lambda)_{\mathbb{Z}}$.

Proof. Let $v \in L(\Lambda)_{\mathbb{Z}}$, by definition there exists $a \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$ such that $v = a \cdot \mathbb{1}_{\Lambda}$. Define:

$$\widetilde{\mathfrak{U}}_{\mathbb{Z}}(\mathfrak{n}^{\pm}) := \widetilde{\mathfrak{U}}_{\mathbb{Q}}(\mathfrak{n}^{\pm}) \cap \mathfrak{U}_{\mathbb{Z}}(\mathfrak{n}^{\pm}),$$

which is the integral span of all monomials in $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{n}^{\pm})$ of strictly positive degree. Then we have:

$$\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) = \widetilde{\mathfrak{U}}_{\mathbb{Z}}(\mathfrak{n}^{-})\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) + \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})\widetilde{\mathfrak{U}}_{\mathbb{Z}}(\mathfrak{n}^{+}) + \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$$
$$\widetilde{\mathfrak{U}}_{\mathbb{Z}}(\mathfrak{n}^{+}) \cdot \mathbb{1}_{\Lambda} = 0$$
$$\widetilde{\mathfrak{U}}_{\mathbb{Z}}(\mathfrak{n}^{-})\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) \cdot \mathbb{1}_{\Lambda} \subset \bigoplus_{\lambda \neq \Lambda} L(\Lambda)_{\mathbb{C},\lambda}$$

Hence there exists $a_0 \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$ such that:

$$\mathsf{E}[v] = \mathsf{E}[a \cdot \mathbb{1}_{\Lambda}] = \mathsf{E}[a_0 \cdot \mathbb{1}_{\Lambda}].$$

But by definition $a_0 \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$ is an integral linear combination of products of elements of the form: $\binom{\alpha_i^{\vee}}{m}$ which act on v as follows:

$$\binom{\alpha_i^{\vee}}{m} \cdot \mathbb{1}_{\Lambda} = \binom{\langle \Lambda, \alpha_i^{\vee} \rangle}{m} \mathbb{1}_{\Lambda}.$$

Now since Λ is dominant and integral $\langle \Lambda, \alpha_i^{\vee} \rangle \in \mathbb{Z}_{+}$ for all $i \in I$. □

Theorem 4.15. *If $v, w \in L(\varLambda)_{\mathbb{Z}}$ then $\{v,\, w\} \in \mathbb{Z}$.*

Proof. By definition there exist $a, b \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$ such that $v = a \cdot \mathbb{1}_{\varLambda}, w = b \cdot \mathbb{1}_{\varLambda}$. Hence:

$$\begin{aligned}\{v,\, w\} &= \mathbb{S}_{\varLambda}(v,\, \overline{w}) \\ &= \mathbb{S}_{\varLambda}(v,\, w) \\ &= \varLambda(\mathbb{S}(a,\, b)) = \varLambda\big(\boldsymbol{\eta}(\boldsymbol{\sigma}(a)b)\big)\end{aligned}$$

where $\boldsymbol{\eta}$ is the Harish Chandra map. Since $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$ is $\boldsymbol{\sigma}$-invariant, $\boldsymbol{\sigma}(a)b \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$. Moreover we may define a map $\boldsymbol{\eta}_{\mathbb{Z}} : \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) \to \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$ such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) & \xrightarrow{\subset} & \mathfrak{U}_{\mathbb{C}}(\mathfrak{g}) \\ {\scriptstyle \boldsymbol{\eta}_{\mathbb{Z}}}\downarrow & & \downarrow{\scriptstyle \boldsymbol{\eta}} \\ \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a}) & \xrightarrow[\subset]{} & \mathfrak{U}_{\mathbb{C}}(\mathfrak{a}) \end{array}$$

Therefore we have:

$$\{v,\, w\} = \varLambda\big(\boldsymbol{\eta}(\boldsymbol{\sigma}(a)b)\big) = \varLambda\big(\boldsymbol{\eta}_{\mathbb{Z}}(\boldsymbol{\sigma}(a)b)\big).$$

But $\varLambda$ is dominant and integral so when extended to $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$ it will only produce integer values. $\square$

Corollary 4.16. *If $v \in L(\varLambda)_{\mathbb{Z}}$ then $\|v\| \geqslant 1$.*

Definition 4.17. An *admissible basis* for $L(\varLambda)_{\mathbb{C}}$ is an ordered basis consisting of weight vectors ordered such that the depth of basis elements is non-decreasing in this basis. That is if $\{v_1, v_2, \cdots\}$ is a an admissible basis with $v_k \in L(\varLambda)_{\mathbb{C},\lambda_k}$ then $i < j$ implies $\mathrm{dp}(\lambda_i) \leq \mathrm{dp}(\lambda_j)$. Note that the first basis element of any admissible basis has to be a *highest weight vector*, that is it belongs to $L(\varLambda)_{\mathbb{C},\varLambda}$.

Theorem 4.18 ([11] Theorem 11.3)**.** *$L(\varLambda)_{\mathbb{C}}$ has an admissible basis such that its $\mathbb{Z}$-span is $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$-invariant.*

4.2.2 Proof of Theorem 4.18

Set: $L(\Lambda)_{\mathbb{Z},\lambda} = L(\Lambda)_{\mathbb{Z}} \cap L(\Lambda)_{\mathbb{C},\lambda}$. Then $L(\Lambda)_{\mathbb{Z}}$ has a direct sum decomposition:

$$L(\Lambda)_{\mathbb{Z}} = \bigoplus_{\lambda \in \mathscr{P}_\Lambda} L(\Lambda)_{\mathbb{Z},\lambda}.$$

Let $\mathscr{B}_\lambda$ be a $\mathbb{Z}$-basis for $L(\Lambda)_{\mathbb{Z},\lambda}$ and set:

$$\mathscr{B} = \bigcup_{\lambda \in \mathscr{P}_\Lambda} \mathscr{B}_\lambda,$$

we claim $\mathscr{B}$ is the basis we are looking for. There are 3 points to prove:

(1) The $\mathbb{Z}$-span of $\mathscr{B}$ is $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$-invariant.

(2) $\mathscr{B}$ spans $L(\Lambda)_{\mathbb{C}}$.

(3) $\mathscr{B}$ is linearly independent over $\mathbb{C}$.

Since $L(\Lambda)_{\mathbb{Z}}$ is the $\mathbb{Z}$-span of $\mathscr{B}$, (1) is true. (2) follows from $L(\Lambda)_{\mathbb{C}} = \mathfrak{U}_{\mathbb{C}}(\mathfrak{g})\cdot \mathbb{1}_\Lambda$ and the fact that $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$ is a lattice in $\mathfrak{U}_{\mathbb{C}}(\mathfrak{g})$. That leaves (3), however this is equivalent to to proving that for any finite subset of $L(\Lambda)_{\mathbb{Z}}$ linear independence over $\mathbb{Z}$ implies linear independence over $\mathbb{C}$. We prove the latter by contradiction, so suppose there exist $v_1, \cdots, v_r \in L(\Lambda)_{\mathbb{Z}}$ which are linearly independent over $\mathbb{Z}$, but not over $\mathbb{C}$, that there exist $c_1, \cdots, c_r \in \mathbb{C}$ not all zero such that:

$$\sum_{j=1}^{r} c_j v_j = 0. \tag{4.19}$$

Moreover we assume that r is minimal, in other words any other subset of $L(\Lambda)_{\mathbb{Z}}$ of size smaller than r is not a counterexample to our claim. In order to derive a contradiction we first need the following Lemma:

Lemma 4.20. *Let $v_1, \cdots, v_r \in L(\Lambda)_{\mathbb{Z}}$ be such that $v_1 \neq 0$ and $\sum_{j=1}^{r} c_j v_j = 0$ with $c_j \in \mathbb{C}$. Then there exist integers $n_1, \cdots, n_r$ with $n_1 \neq 0$ satisfying:*

$$\sum_{j=1}^{r} c_j n_j = 0.$$

Proof. Choose $a \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$ such that $\mathsf{E}[a \cdot v_1] \neq 0$. Such an element exists, otherwise v_1 would generate a non-trivial submodule of $L(\Lambda)_{\mathbb{C}}$ which did not intersect $L(\Lambda)_{\mathbb{C},\Lambda}$, and by definition of $L(\Lambda)_{\mathbb{C}}$, this is impossible. Applying first a and then $\mathsf{E}[\cdot]$ to $\sum_{j=1}^{r} c_j v_j = 0$, we get: $\sum_{j=1}^{r} c_j \mathsf{E}\big[a \cdot v_j\big] = 0$. Since $v_j \in L(\Lambda)_{\mathbb{Z}}, a \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g})$ we have $\mathsf{E}\big[a \cdot v_j\big] \in \mathbb{Z}$ from Corollary 4.14. Now take $n_j := \mathsf{E}\big[a \cdot v_j\big]$, note that $n_1 \neq 0$ due to our choice of a. □

Applying Lemma 4.20 to (4.19) we see that there exist integers $n_1, \cdots, n_r \in \mathbb{Z}$ with $n_1 \neq 0$ satisfying:

$$\sum_{j=1}^{r} c_j n_j = 0. \tag{4.21}$$

Now from (4.19) and (4.21) we have:

$$0 = \left(\sum_{j=1}^{r} c_j v_j\right) n_1 = c_1 n_1 v_1 + \sum_{j=2}^{r} c_j n_1 v_j$$

$$0 = \left(\sum_{j=1}^{r} c_j n_j\right) v_1 = c_1 n_1 v_1 + \sum_{j=2}^{r} c_j n_j v_1$$

Eliminating $c_1 n_1 v_1$ using the two equations we get:

$$\sum_{j=2}^{r} c_j \left(n_1 v_j - n_j v_1\right) = 0$$

Set $w_j := n_1 v_j - n_j v_1$. Now $w_2, \cdots, w_r \in L(\Lambda)_{\mathbb{Z}}$ are linearly dependent over $\mathbb{Z}$ since $v_1, \cdots, v_r$ were linearly independent over $\mathbb{Z}$ however we have:

$$\sum_{j=2}^{r} c_j w_j = 0.$$

which implies $w_2, \cdots, w_r$ is linearly dependent over $\mathbb{C}$ which contradicts the minimality of r.

Chapter 5

Groups over $\mathbb{Q}$

In this chapter we give a definition of the split maximal Kac-Moody group over $\mathbb{Q}$ (and any field of characteristic zero) associated to a given GCM. One should note that given a GCM that is not of finite type one can associate at least two different groups with it (maximal vs minimal). Moreover in either case there are several definitions in the literature (see [20] for various definitions of both the maximal and minimal groups and their comparison). The definition given below is new but it is a synthesis of two main ways of approaching the subject. Our starting point is the result that any complex semi-simple Lie group can be expressed as the amalgamated product of the minimal parabolic subgroups and the normalizer of a maximal torus (see Theorem D.2). We use this theorem as our definition: first define the minimal parabolic subgroups and the normalizer of the maximal torus and then we define the split maximal Kac-Moody group as their amalgamated product.

The first step (defining the subgroups) is the subject of §1, for any subalgebra $\mathfrak{m}_{\mathbb{Q}} \subset \mathfrak{g}_{\mathbb{Q}}$ which is invariant under the adjoint action of $\mathfrak{a}_{\mathbb{Q}}$ we construct a $\mathbb{Q}$-algebra: $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$. Then we show that $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ is in fact a $\mathbb{Q}$-Hopf algebra and so we have an affine group scheme over $\mathbb{Q}$. In §2 we compute the Hopf algebra $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ in terms of the representations of $\mathfrak{m}_{\mathbb{Q}}$ where $\mathfrak{m}_{\mathbb{Q}} = \mathfrak{a}_{\mathbb{Q}}, \mathfrak{n}^+_{\mathbb{Q}}, \mathfrak{b}_{\mathbb{Q}}, \mathfrak{l}_{i,\mathbb{Q}}, \mathfrak{p}_{i,\mathbb{Q}}$ (for these two sections we follow Mathieu, pages 19-20 and 24-25 in [17]). In §3 we give an explicit description of the unipotent group, $\mathbf{N}^+_{\mathbb{Q}}$, as a subset of a completion of the universal enveloping algebra of $\mathfrak{n}^+_{\mathbb{Q}}$. Finally in §4 we note that while $\mathfrak{g}_{\mathbb{Q}}$ itself is invariant under the adjoint action of the torus, when the GCM is not of finite type $\mathfrak{g}_{\mathbb{Q}}$ becomes infinite dimensional and while this process yields us a group, it is too small to be of any use, see Remark 5.32. The second step (amalgamation) was the approach championed by Jacques Tits, an exposition can be found in [16].

Finally another way of constructing Kac-Moody groups is to use integrable representations of the Kac-Moody algebra, see [6].

5.1 $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$

5.1.1 Definition

Notation 5.1. For a subalgebra $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{g}_{\mathbb{Q}}$, set:

$$\mathfrak{m}_{\mathbb{Q}}^{\pm} := \mathfrak{m}_{\mathbb{Q}} \cap \mathfrak{n}_{\mathbb{Q}}^{\pm}$$
$$\mathfrak{m}_{\mathbb{Q}}^{0} := \mathfrak{m}_{\mathbb{Q}} \cap \mathfrak{a}_{\mathbb{Q}}.$$

Definition 5.2. The *character lattice* is defined as follows:

$$\mathscr{P} := \mathrm{Hom}_{\mathbb{Z}}(\mathfrak{a}_{\mathbb{Z}}, \mathbb{Z}).$$

$\mathscr{P}$ is a lattice in $\mathfrak{a}_{\mathbb{Q}}^{*}$ which contains the root lattice, $\mathscr{Q}$. For each $i \in I$ we also set the following subset of the character lattice:

$$\mathscr{P}_i := \left\{\lambda \in \mathscr{P} : \left\langle \lambda, \alpha_i^{\vee} \right\rangle \geq 0\right\}.$$

Let $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{g}_{\mathbb{Q}}$ be a subalgebra such that:

$$\left[\mathfrak{a}_{\mathbb{Q}}, \mathfrak{m}_{\mathbb{Q}}\right] \subseteq \mathfrak{m}_{\mathbb{Q}}. \tag{$\star$}$$

Let $L(u), R(u) : \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \to \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$ be the left and right multiplications by u. Corresponding to these maps we have the left regular representation of $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$: $u \mapsto L(u)$ and the right regular representation: $u \mapsto R(\boldsymbol{\gamma}(u))$, where $\boldsymbol{\gamma}$ is the principal anti-automorphism of $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$. The transpose of these maps give us an action of $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$ on its dual:

$$\left[L^{*}(u)(\phi)\right](x) = \phi(ux)$$
$$\left[R^{*}(u)(\phi)\right](x) = \phi(x\boldsymbol{\gamma}(u))$$

For $z \in \mathfrak{a}_{\mathbb{Q}}$ let $\mathrm{ad}(z) : \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \to \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$ denote the lift of $\mathrm{ad}(z) : \mathfrak{m}_{\mathbb{Q}} \to \mathfrak{m}_{\mathbb{Q}}$. Again the transpose gives us an action of $\mathfrak{a}_{\mathbb{Q}}$ on $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$:

$$\left[\mathrm{ad}^*(z)(\phi)\right](x) = \phi(\mathrm{ad}(z)(x)).$$

Definition 5.3. $\phi \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^*$ is called *L^*-finite* (resp. *R^*-finite*) if the span of the maps $L^*(u)(\phi)$ (resp. $R^*(u)(\phi)$), as u varies over $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$, is finite dimensional in $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^*$.

Definition 5.4. Let $\mathfrak{H}^L_{\mathbb{Q}}(\mathfrak{m})$ (resp. $\mathfrak{H}^R_{\mathbb{Q}}(\mathfrak{m})$) be the set of all linear combinations of elements $\phi \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^*$ that satisfy:

(1) ϕ is L^*-finite (resp. R^*-finite).

(2) There exists $\lambda \in \mathcal{Q}$ such that $\mathrm{ad}^*(z)(\phi) = \langle \lambda, z \rangle \, \phi$ for all $z \in \mathfrak{a}_{\mathbb{Q}}$.

(3) There exists $\mu \in \mathcal{P}$ such that $L^*(z)(\phi) = \langle \mu, z \rangle \, \phi$ (resp. $R^*(z)(\phi) = \langle \mu, z \rangle \, \phi$) for all $z \in \mathfrak{m}^0_{\mathbb{Q}}$.

Lemma 5.5. $\mathfrak{H}^L_{\mathbb{Q}}(\mathfrak{m}) = \mathfrak{H}^R_{\mathbb{Q}}(\mathfrak{m})$.

Proof. Let (ϕ, λ, μ) be a triplet satisfying the conditions of Definition 5.4 with ϕ non-zero. By definition showing $\phi \in \mathfrak{H}^R_{\mathbb{Q}}(\mathfrak{m})$ will imply $\mathfrak{H}^L_{\mathbb{Q}}(\mathfrak{m}) \subseteq \mathfrak{H}^R_{\mathbb{Q}}(\mathfrak{m})$.

Set:

$$\mathfrak{I}_{\mathbb{Q}} := \left\{ u \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) : \; L^*(u)(\phi) = \phi(u \cdot) = 0 \right\}. \tag{5.6}$$

Since ϕ is L^*-finite, $\mathfrak{I}_{\mathbb{Q}}$ is a right ideal of finite co-dimension in $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$. Therefore $\mathfrak{I}_{\mathbb{Q}}$ contains, $\mathfrak{I}_{\mathbb{Q}}$, the annihilator of $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}}$, which is a two-sided ideal of of finite co-dimension (it is the largest two sided ideal contained in $\mathfrak{I}_{\mathbb{Q}}$). Now consider the following:

$$\left[R^*\big(\boldsymbol{\gamma}(\mathfrak{I}_{\mathbb{Q}})\big)(\phi)\right](x) = \phi(x\mathfrak{I}_{\mathbb{Q}}) = \phi(\mathfrak{I}_{\mathbb{Q}}) = 0.$$

Since $\mathfrak{I}_{\mathbb{Q}}$ is invariant under $\boldsymbol{\gamma}$ and has finite co-dimension ϕ is R^*-finite.

Next let $z \in \mathfrak{m}^0_{\mathbb{Q}}$, by definition:

$$\mathrm{ad}^*(z)(\phi) = \langle \lambda, z \rangle \, \phi,$$
$$L^*(z)(\phi) = \langle \mu, z \rangle \, \phi.$$

Therefore:

$$R^*(z)(\phi) = L^*(z)(\phi) - \mathrm{ad}^*(z)(\phi) = \langle \mu - \lambda,\, z\rangle\, \phi,$$

since $\mu \in \mathscr{P}, \lambda \in \mathscr{Q} \subseteq \mathscr{P}$ we have $\mu - \lambda \in \mathscr{P}$. The proof of $\mathfrak{H}^L_{\mathbb{Q}}(\mathfrak{m}) \supseteq \mathfrak{H}^R_{\mathbb{Q}}(\mathfrak{m})$ is similar. □

Notation 5.7. Based on Lemma 5.5 we set: $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}) := \mathfrak{H}^L_{\mathbb{Q}}(\mathfrak{m}) = \mathfrak{H}^R_{\mathbb{Q}}(\mathfrak{m})$.

Example 5.8. For $\mathfrak{m}_{\mathbb{Q}} = \mathfrak{a}_{\mathbb{Q}}$ the three conditions enumerated in the definition of $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{a})$ collapse to one. So $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{a})$ is the linear combination of elements $\phi \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{a})^*$ that satisfy:

$$\exists \mu \in \mathscr{P} : \quad L^*(z)(\phi) = \langle \mu,\, z\rangle\, \phi.$$

Since $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{a}) \cong \mathbb{Q}\big[\alpha_1^\vee, \cdots, \alpha_n^\vee\big]$ we see that ϕ is determined by μ up to a scalar constant, that is $\phi(z) = \langle \mu,\, z\rangle\, \phi(1)$. Now the map:

$$\phi \mapsto \phi(1)\delta_\mu,$$

where $\delta_\mu : \mathscr{P} \to \mathbb{Q}$ is the map that sends μ to 1 and is zero on the rest of the lattice, shows that $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{a}) = \mathbb{Q}[\mathscr{P}]$, where $\mathbb{Q}[\mathscr{P}]$ is the group algebra of the discrete group $\mathscr{P}$.

5.1.2 Hopf Algebra Structure

Definition 5.9. Let $\mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ be the set of all left ideals, $\mathfrak{I}_{\mathbb{Q}}$, in $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$ that satisfy:

(1) $\mathfrak{I}_{\mathbb{Q}}$ is of finite co-dimension in $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$.

(2) $\mathfrak{I}_{\mathbb{Q}}$ is stable under the adjoint action of $\mathfrak{a}_{\mathbb{Q}}$.

(3) There exists a finite subset $\Xi \subseteq \mathscr{P}$ such that their restriction to $\mathfrak{m}^0_{\mathbb{Q}}$ satisfies:

$$\forall z \in \mathfrak{m}^0_{\mathbb{Q}} : \quad \prod_{\lambda \in \Xi} z - \langle \lambda,\, z\rangle \in \mathfrak{I}_{\mathbb{Q}}.$$

Lemma 5.10. *For $\phi \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^*$ the following are equivalent:*

(1) $\phi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$.

(2) There exists a two-sided ideal $\mathfrak{J}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ *such that* $\phi(\mathfrak{J}_{\mathbb{Q}}) = 0$.

Proof. First note that if $\mathfrak{J}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ then $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{J}_{\mathbb{Q}}$ is a finite dimensional $\mathbb{Q}$-vector space and any linear map $\psi : \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \to \mathbb{Q}$ such that $\psi(\mathfrak{J}_{\mathbb{Q}}) = 0$ belongs to $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$.

For the converse, let $\phi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ be arbitrary, then we may write it as a linear combination of non-zero elements:

$$\phi = c_1\phi_1 + \cdots + c_m\phi_m,$$

where for each $k, 1 \leq k \leq m$, the triplet $(\phi_k, \lambda_k, \mu_k)$ satisfies the conditions of Definition 5.4. In the proof of Lemma 5.5 for each ϕ_k we have defined a right ideal $\mathfrak{I}_{k,\mathbb{Q}}$ and a two sided ideal $\mathfrak{J}_{k,\mathbb{Q}}$ contained in it. Now set:

$$\mathfrak{J}_{\mathbb{Q}} = \mathfrak{J}_{1,\mathbb{Q}} \cap \cdots \cap \mathfrak{J}_{m,\mathbb{Q}}.$$

Clearly $\mathfrak{J}_{\mathbb{Q}}$ is a two sided ideal of finite co-dimension and $\phi(\mathfrak{J}_{\mathbb{Q}}) = 0$.

Since $\mathfrak{J}_{\mathbb{Q}}$ is a two sided ideal it is stable under the adjoint action in particular by elements of $\mathfrak{a}_{\mathbb{Q}}$.

Finally for condition (3) of Definition 5.9 we take:

$$\varXi = \{\mu_1, \cdots, \mu_m\}. \qquad \square$$

Lemma 5.11. $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ *is a subalgebra of the commutative algebra* $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^*$.

Proof. Recall that $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^*$ is a commutative algebra with the unit map $\boldsymbol{\zeta}^*$ and product map $\boldsymbol{\nu}^*$ (see Appendix C). Since $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ is a $\mathbb{Q}$-subspace of $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^*$ we only need to show that it is closed under $\boldsymbol{\nu}^*$. In other words:

$$\boldsymbol{\nu}^*\big(\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})\big) \subset \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}).$$

Suppose $\phi, \psi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ and let $\phi\psi$ denote $\boldsymbol{\nu}^*(\phi \otimes \psi)$. By Lemma 5.10 $\phi\psi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ is equivalent to finding a two-sided ideal $\mathfrak{P}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ such that $(\phi\psi)(\mathfrak{P}_{\mathbb{Q}}) = 0$. Since $\phi, \psi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ there exist two-sided ideals $\mathfrak{J}_{\mathbb{Q}}, \mathfrak{J}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ such that $\phi(\mathfrak{I}_{\mathbb{Q}}) = \psi(\mathfrak{I}_{\mathbb{Q}}) = 0$. Let $\mathfrak{P}_{\mathbb{Q}} = \mathfrak{I}_{\mathbb{Q}}\mathfrak{I}_{\mathbb{Q}}$, then one immediately has: $(\phi\psi)(\mathfrak{P}_{\mathbb{Q}}) = 0$, all that is left to show is $\mathfrak{P}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$.

$\mathfrak{P}_{\mathbb{Q}}$ has finite co-dimension: take $\{x_1, \cdots, x_p\} \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$ to be a basis for the $\mathbb{Q}$-vector space $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}}$ and let $\{y_1, \cdots, y_q\}$ be a set that generates $\mathfrak{I}_{\mathbb{Q}}$ as an ideal. Then $t \in \mathfrak{I}_{\mathbb{Q}}$ can be written as:

$$t = \sum_{i=1}^{q} u_i y_i, \tag{5.12}$$

where $u_i \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$. For each u_i we can write:

$$u_i = \left(\sum_{j=1}^{p} c_j v_j\right) + \mathfrak{I}_{\mathbb{Q}}, \tag{5.13}$$

where $c_j \in \mathbb{Q}$. Combining (5.12) and (5.13) we arrive at the following:

$$\begin{aligned} t &= \sum_{i=1}^{q} \left(\left(\sum_{j=1}^{p} c_j v_j\right) + \mathfrak{I}_{\mathbb{Q}}\right) y_i \\ &= \sum_{i=1}^{q} \left(\sum_{j=1}^{p} c_j v_j y_i\right) + \mathfrak{I}_{\mathbb{Q}} y_i \\ &= \left(\sum_{i=1}^{q} \sum_{j=1}^{p} c_j v_j y_i\right) + \mathfrak{I}_{\mathbb{Q}}\mathfrak{I}_{\mathbb{Q}} \end{aligned}$$

which shows that $\mathfrak{I}_{\mathbb{Q}}/\mathfrak{I}_{\mathbb{Q}}\mathfrak{I}_{\mathbb{Q}}$ is finite dimensional and since:

$$\dim\left(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}}\mathfrak{I}_{\mathbb{Q}}\right) = \dim\left(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}}\right) + \dim\left(\mathfrak{I}_{\mathbb{Q}}/\mathfrak{I}_{\mathbb{Q}}\mathfrak{I}_{\mathbb{Q}}\right),$$

we see that $\mathfrak{P}_{\mathbb{Q}} = \mathfrak{I}_{\mathbb{Q}}\mathfrak{I}_{\mathbb{Q}}$ has finite co-dimension.

$\mathfrak{P}_{\mathbb{Q}}$ is stable under the adjoint action of $\mathfrak{a}_{\mathbb{Q}}$: let $a \in \mathfrak{I}_{\mathbb{Q}}, b \in \mathfrak{I}_{\mathbb{Q}}$ and $z \in \mathfrak{a}_{\mathbb{Q}}$ then:

$$\begin{aligned} [z, ab] &= zab - abz \\ &= zab - azb + azb - abz \end{aligned}$$

$$= [z,a]b + a[z,b].$$

Since $\mathfrak{I}_{\mathbb{Q}}, \mathfrak{J}_{\mathbb{Q}}$ both are stable under the adjoint action of $\mathfrak{a}_{\mathbb{Q}}$ we have: $[z,a] \in \mathfrak{I}_{\mathbb{Q}}, [z,b] \in \mathfrak{J}_{\mathbb{Q}}$.

Finally from $\mathfrak{I}_{\mathbb{Q}}, \mathfrak{J}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ we have two finite sets: $\Xi_1, \Xi_2 \subseteq \mathscr{P}$. For $\mathfrak{P}_{\mathbb{Q}}$ the union $\Xi_1 \cup \Xi_2$ satisfies the desired conditions. □

Theorem 5.14. *If the subalgebra* $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{g}_{\mathbb{Q}}$ *satisfies* $(\star)$ *then*

$$\big(\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}), \boldsymbol{\nu}^*, \boldsymbol{\zeta}^*, \boldsymbol{\mu}^*, \boldsymbol{\epsilon}^*, \boldsymbol{\gamma}^*\big)$$

is a commutative Hopf algebra over $\mathbb{Q}$.

Proof. Based on Lemma 5.11 and the fact that $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$ is itself a cocommutative Hopf algebra we only need to prove that $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ is closed under the transpose maps: $\boldsymbol{\gamma}^*, \boldsymbol{\epsilon}^*$ and $\boldsymbol{\mu}^*$. The first two are easy to verify and so turn our attention to the transpose of the product map in $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$:

$$\begin{cases} \boldsymbol{\mu}^* : \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^* \to \big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})\big)^* \\ \boldsymbol{\mu}^*(\phi)(u \otimes u') = \phi\left(\boldsymbol{\mu}(u \otimes u')\right) = \phi\left(uu'\right) \end{cases}$$

On the other hand $\phi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ and so by Lemma 5.10 there exists $\mathfrak{I}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ such that $\phi(\mathfrak{I}_{\mathbb{Q}}) = 0$. Therefore we get:

$$\boldsymbol{\mu}^*(\phi)\big(\mathfrak{I}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) + \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{I}_{\mathbb{Q}}\big) = 0.$$

Combining this with:

$$\boldsymbol{\mu}^*(\phi) \in \mathrm{HOM}_{\mathbb{Q}}\big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}),\ \mathbb{Q}\big),$$

we get:

$$\begin{aligned} \boldsymbol{\mu}^*(\phi) \in \mathrm{HOM}_{\mathbb{Q}}&\left(\frac{\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})}{\mathfrak{I}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) + \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{I}_{\mathbb{Q}}},\ \mathbb{Q}\right) \\ &= \mathrm{HOM}_{\mathbb{Q}}\big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}},\ \mathbb{Q}\big) \\ &= \mathrm{HOM}_{\mathbb{Q}}\big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}},\ \mathbb{Q}\big) \otimes_{\mathbb{Q}} \mathrm{HOM}_{\mathbb{Q}}\big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}},\ \mathbb{Q}\big) \end{aligned}$$

But since $\mathfrak{J}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ we have:

$$\textsc{Hom}_{\mathbb{Q}}\big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{J}_{\mathbb{Q}},\, \mathbb{Q}\big) \subseteq \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}).$$

Therefore:

$$\boldsymbol{\mu}^*(\phi) \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}).$$ □

Definition 5.15. Based on Theorem 5.14 for any subalgebra $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{g}_{\mathbb{Q}}$ that satisfies $(\star)$ we may define an affine group scheme over $\mathbb{Q}$:

$$\mathbf{M}_{\mathbb{Q}} = \textsc{Hom}_{\mathbb{Q}-\mathrm{alg}}(\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}),\, \mathbb{Q}).$$

5.1.3 Examples

Example 5.16. Let us return to the case when $\mathfrak{m}_{\mathbb{Q}} = \mathfrak{a}_{\mathbb{Q}}$, we have already shown that $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{a}) = \mathbb{Q}[\mathscr{P}]$. Therefore:

$$\mathbf{A}_{\mathbb{Q}} = \textsc{Hom}_{\mathbb{Q}-\mathrm{alg}}(\mathbb{Q}[\mathscr{P}],\, \mathbb{Q}) = \textsc{Hom}_{\mathbb{Z}}(\mathscr{P},\, \mathbb{Q}^{\times}).$$

So our definition agrees with that of a classical finite dimensional split torus.

Example 5.17. For each $i \in I$ define:

$$\mathfrak{l}_{i,\mathbb{Q}} = \mathfrak{n}_{-\alpha_i,\mathbb{Q}} \oplus \mathfrak{a}_{\mathbb{Q}} \oplus \mathfrak{n}_{\alpha_i,\mathbb{Q}}.$$

Then $\mathfrak{l}_{i,\mathbb{Q}}$ satisfies $(\star)$ and so we get a group $\mathbf{L}_{i,\mathbb{Q}}$, this is a finite dimensional reductive group of semi-simple rank 1 which contains $\mathbf{A}_{\mathbb{Q}}$ as a subgroup. Let $\overline{\mathbf{W}}_{i,\mathbb{Q}}$ denote the normalizer of $\mathbf{A}_{\mathbb{Q}}$ in $\mathbf{L}_{i,\mathbb{Q}}$. Then the quotient $\overline{\mathbf{W}}_{i,\mathbb{Q}}/\mathbf{A}_{\mathbb{Q}}$ is a group with two elements: $\{1, \overline{r}_i\}$ and the conjugation action of $\overline{r}_i$ on $\mathbf{A}_{\mathbb{Q}}$ is induced from the action of the fundamental reflection r_i on $\mathfrak{a}_{\mathbb{Q}}$. Let $a = \textsc{Exp}(z) \in \mathbf{A}_{\mathbb{Q}}$ then:

$$r_i(a) = \overline{r}_i a \overline{r}_i^{-1} = r_i(\textsc{Exp}(z)) = \textsc{Exp}(r_i(z)).$$

Moreover we have $\overline{r}_i^2 \in \mathbf{A}_{\mathbb{Q}}$, more precisely:

$$\begin{cases} \overline{r}_i^2 : \mathscr{P} \to \mathbb{Q}^\times \\ \overline{r}_i^2(\lambda) = (-1)^{\langle \lambda, \alpha_i^\vee \rangle} \end{cases}$$

Remark 5.18. Earlier we have defined several subalgebras of $\mathfrak{g}_{\mathbb{Q}} : \mathfrak{a}_{\mathbb{Q}}, \mathfrak{b}_{\mathbb{Q}}, \mathfrak{n}_{\mathbb{Q}}^{\pm}, \mathfrak{n}_{\pm\alpha_i,\mathbb{Q}}$ and $\mathfrak{p}_{i,\mathbb{Q}}$. Now for each $i \in I$ we define:

$$\begin{aligned} \mathfrak{n}_{i,\mathbb{Q}} &= \bigoplus_{\alpha_i \neq \alpha \in \Delta_+} \mathfrak{g}_{\mathbb{Q},\alpha}, \\ \mathfrak{b}_{\pm i,\mathbb{Q}} &= \mathfrak{a}_{\mathbb{Q}} \oplus \mathfrak{n}_{\pm\alpha_i,\mathbb{Q}}, \\ \mathfrak{c}_{i,\mathbb{Q}} &= \mathfrak{n}_{-\alpha_i,\mathbb{Q}} \oplus \mathfrak{a}_{\mathbb{Q}} \oplus \mathfrak{n}_{i,\mathbb{Q}}. \end{aligned}$$

These subalgebras of $\mathfrak{g}_{\mathbb{Q}}$ all satisfy ($\star$) and so we have the corresponding groups over $\mathbb{Q}$:

$$\mathbf{A}_{\mathbb{Q}}, \mathbf{B}_{\mathbb{Q}}, \mathbf{B}_{\pm i,\mathbb{Q}}, \mathbf{C}_{i,\mathbb{Q}}, \mathbf{L}_{i,\mathbb{Q}}, \mathbf{N}_{\mathbb{Q}}^{\pm}, \mathbf{N}_{i,\mathbb{Q}}, \mathbf{N}_{\pm\alpha_i,\mathbb{Q}}, \mathbf{P}_{i,\mathbb{Q}}.$$

5.2 Peter-Weyl Type Theorems

Definition 5.19. Given $\Lambda \in \mathscr{P}$, let $M(\Lambda)_{\mathbb{Q}}^\vee$ denote the subspace generated by weight vectors in the coinduced module:

$$\textsc{Coind}_{\mathfrak{a}_{\mathbb{Q}}}^{\mathfrak{b}_{\mathbb{Q}}}(\mathbb{Q}_\Lambda) = \textsc{Hom}_{\mathfrak{U}_{\mathbb{Q}}(\mathfrak{a})}\big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{b}), \mathbb{Q}_\Lambda\big),$$

where $\mathbb{Q}_\Lambda$ is the 1-dimensional $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{a})$-module with weight Λ. Then as $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{b})$-modules we have:

$$M(\Lambda)_{\mathbb{Q}}^\vee = M(0)_{\mathbb{Q}}^\vee \otimes_{\mathbb{Q}} \mathbb{Q}_\Lambda.$$

Therefore all these modules are isomorphic as $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)$-modules, in fact as $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)$-modules we have:

$$M(\Lambda)_{\mathbb{Q}}^\vee \cong \mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)^\vee,$$

where the latter is the restricted dual with respect to the $\mathcal{Q}_+$-grading, that is:

$$\mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)^\vee := \bigoplus_{\alpha\in\mathcal{Q}_+} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)^*_\alpha \subset \left(\bigoplus_{\alpha\in\mathcal{Q}_+} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)_\alpha\right)^* = \mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)^*.$$

Definition 5.20. Suppose $\Lambda \in \mathcal{P}_i$ then $\langle \Lambda, \alpha_i^\vee \rangle \geq 0$ and hence there exists a unique irreducible $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{l}_i)$-module of dimension $\langle \Lambda, \alpha_i^\vee \rangle + 1$, which we will denote by $\ell_i(\Lambda)_{\mathbb{Q}}$. We define $M_i(\Lambda)^\vee_{\mathbb{Q}}$ to be the subspace generated by weight vectors in the coinduced module:

$$\textsc{Coind}^{\mathfrak{p}_i,\mathbb{Q}}_{\mathfrak{l}_i,\mathbb{Q}}\big(\ell_i(\Lambda)_{\mathbb{Q}}\big) = \textsc{Hom}_{\mathfrak{U}_{\mathbb{Q}}(\mathfrak{l}_i)}\big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{p}_i), \ell_i(\Lambda)_{\mathbb{Q}}\big).$$

Lemma 5.21 ([17] page 25, Lemma 7)**.**

(1) For each $i \in I$ we have natural isomorphisms:

$$\begin{aligned} \mathbf{P}_{i,\mathbb{Q}} &= \mathbf{N}_{i,\mathbb{Q}} \times \mathbf{L}_{i,\mathbb{Q}}, \\ \mathbf{B}_{\mathbb{Q}} &= \mathbf{N}_{i,\mathbb{Q}} \times \mathbf{B}_{i,\mathbb{Q}}, \\ \mathbf{C}_{i,\mathbb{Q}} &= \mathbf{N}_{i,\mathbb{Q}} \times \mathbf{B}_{-i,\mathbb{Q}}, \\ \mathbf{B}_{\mathbb{Q}} &= \mathbf{N}^+_{\mathbb{Q}} \times \mathbf{A}_{\mathbb{Q}}. \end{aligned}$$

(2) If $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{m}'_{\mathbb{Q}}$ are two subalgebras mentioned in Remark 5.18 then there exists a natural morphism $\mathbf{M}_{\mathbb{Q}} \to \mathbf{M}'_{\mathbb{Q}}$ which is a closed immersion, in other words $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}') \to \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ is a surjection of $\mathbb{Q}$-algebras.

(3) We have the following isomorphism:

$$\mathfrak{H}_{\mathbb{Q}}(\mathfrak{a}) = \bigoplus_{\Lambda\in\mathcal{P}} \mathbb{Q}_\Lambda \otimes_{\mathbb{Q}} \mathbb{Q}^*_\Lambda \qquad \textit{As } \mathfrak{U}_{\mathbb{Q}}(\mathfrak{a}) \times \mathfrak{U}_{\mathbb{Q}}(\mathfrak{a})\textit{-modules} \tag{5.22}$$

$$\mathfrak{H}_{\mathbb{Q}}(\mathfrak{l}_i) = \bigoplus_{\Lambda\in\mathcal{P}_i} \ell_i(\Lambda)_{\mathbb{Q}} \otimes_{\mathbb{Q}} \ell_i(\Lambda)^*_{\mathbb{Q}} \qquad \textit{As } \mathfrak{U}_{\mathbb{Q}}(\mathfrak{l}_i) \times \mathfrak{U}_{\mathbb{Q}}(\mathfrak{l}_i)\textit{-modules} \tag{5.23}$$

$$\mathfrak{H}_{\mathbb{Q}}(\mathfrak{n}^+) = M(0)^\vee_{\mathbb{Q}} \qquad \textit{As } \mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)\textit{-modules} \tag{5.24}$$

$$\mathfrak{H}_{\mathbb{Q}}(\mathfrak{b}) = \bigoplus_{\Lambda\in\mathscr{P}} M(\Lambda)^{\vee}_{\mathbb{Q}} \qquad \textit{As } \mathfrak{U}_{\mathbb{Q}}(\mathfrak{b})\textit{-modules} \tag{5.25}$$

$$\mathfrak{H}_{\mathbb{Q}}(\mathfrak{p}_i) = \bigoplus_{\Lambda\in\mathscr{P}_i} \left(M_i(\Lambda)^{\vee}_{\mathbb{Q}}\right)^{\langle\Lambda,\alpha_i^{\vee}\rangle+1} \qquad \textit{As right } \mathfrak{U}_{\mathbb{Q}}(\mathfrak{p}_i)\textit{-modules} \tag{5.26}$$

Proof. (1) is easy and implies (2). For (3) we note that (5.22) and (5.23) are known from finite dimensional theory. (1) and (5.24) together imply (5.25) and (5.26). Therefore we only need to show (5.24).

We claim $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{n}^+) = \mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)^{\vee}$. First note that $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{n}^+)$ can not be any bigger than the restricted dual since any functional not in the restricted dual is not L^*-finite and can not be written as a finite linear combination of L^*-finite functionals. Pick a basis $\{u_\alpha : \alpha \in \mathcal{Q}_+\}$ consisting of the weight vectors of the adjoint action of $\mathfrak{a}_{\mathbb{Q}}$, and let $\{\phi_\alpha : \alpha \in \mathcal{Q}_+\}$ be a corresponding dual basis which spans the restricted dual. Evidently all the elements in the dual basis are L^*-finite, now by definition we have:

$$\left[\mathrm{ad}^*(z)(\phi_\alpha)\right](x) = \phi_\alpha(\mathrm{ad}(z)(x)).$$

This expression is zero unless $x \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)_\alpha$ in which case we have:

$$\phi_\alpha(\mathrm{ad}(z)(x)) = \phi_\alpha(\langle\alpha, z\rangle\, x) = \langle\alpha, z\rangle\, \phi_\alpha(x).$$

Since $\mathfrak{n}^+_{\mathbb{Q}} \cap \mathfrak{a}_{\mathbb{Q}} = \{0\}$ we have: $\phi_\alpha \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{n}^+)$. Finally we note that $\mathfrak{a}_{\mathbb{Q}}$ annihilates the unit in $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{n}^+)$, which is the map: $1^* : \mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+) \to \mathbb{Q}$, characterized by $1^*(1) = 1$ and $1^*\left(\widetilde{\mathfrak{U}}_{\mathbb{Q}}(\mathfrak{n}^+)\right) = 0$. The isomorphism between $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{n}^+)$ and $M(0)^{\vee}_{\mathbb{Q}}$ is then given by sending 1^* to $\mathbb{1}_0^*$ the dual weight vector to the highest weight $\mathbb{1}_0 \in M(0)_{\mathbb{Q}}$. □

5.3 The Unipotent Subgroup

While (5.24) gives us some information about the structure of $\mathbf{N}^+_{\mathbb{Q}}$ in this section we will give an explicit construction of this group.

Notation 5.27. In this section $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{n}^+)$ will be denoted by $\mathfrak{U}_{\mathbb{Q}}$.

Definition 5.28. Consider a completion of $\mathfrak{U}_{\mathbb{Q}}$ based on the root lattice decomposition:

$$\mathfrak{U}_{\mathbb{Q}}^c = \prod_{\alpha \in \mathcal{Q}_+} \mathfrak{U}_{\mathbb{Q},\alpha} \supset \bigoplus_{\alpha \in \mathcal{Q}_+} \mathfrak{U}_{\mathbb{Q},\alpha} = \mathfrak{U}_{\mathbb{Q}}.$$

Let $\{u_1, u_2, \cdots\}$ be a $\mathbb{Q}$-basis for $\mathfrak{U}_{\mathbb{Q}}$ with $\{u_1^*, u_2^*, \cdots\}$ as the corresponding dual basis. Then from Appendices B, C and the proof of Lemma 5.21 we have:

$$\begin{aligned} \mathfrak{U}_{\mathbb{Q}} &\cong \mathbb{Q}[u_1, u_2, \cdots], & \mathfrak{U}_{\mathbb{Q}}^c &\cong \mathbb{Q}[\![u_1, u_2, \cdots]\!], \\ \mathfrak{U}_{\mathbb{Q}}^\vee &\cong \mathbb{Q}[u_1^*, u_2^*, \cdots], & \mathfrak{U}_{\mathbb{Q}}^* &\cong \mathbb{Q}[\![u_1^*, u_2^*, \cdots]\!], \end{aligned}$$
$$\left(\mathfrak{U}_{\mathbb{Q}}^\vee\right)^* = \mathfrak{U}_{\mathbb{Q}}^c.$$

Note that $\mathfrak{U}_{\mathbb{Q}}, \mathfrak{U}_{\mathbb{Q}}^\vee$ are dense subsets of $\mathfrak{U}_{\mathbb{Q}}^c$ and $\mathfrak{U}_{\mathbb{Q}}^*$, respectively.

Definition 5.29. Let $\mathfrak{n}_{\mathbb{Q}}^c$ be the completion of $\mathfrak{n}_{\mathbb{Q}}^+$ in $\mathfrak{U}_{\mathbb{Q}}^c$ and define the exponential map $\mathrm{EXP} : \mathfrak{n}_{\mathbb{Q}}^c \to \mathfrak{U}_{\mathbb{Q}}^c$ as the formal sum:

$$\mathrm{EXP}(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Lemma 5.30. $\mathbf{N}_{\mathbb{Q}}^+$ *can be identified with* $\mathrm{EXP}\left(\mathfrak{n}_{\mathbb{Q}}^c\right) \subset \mathfrak{U}_{\mathbb{Q}}^c$.

Proof. [1] Since elements of $\mathbf{N}_{\mathbb{Q}}^+$ are $\mathbb{Q}$-algebra homomorphisms, $\mathfrak{U}_{\mathbb{Q}}^\vee \to \mathbb{Q}$, we can consider $\mathbf{N}_{\mathbb{Q}}^+$ as a subset of $\mathfrak{U}_{\mathbb{Q}}^c$.

THE MAP: Since $\mathfrak{U}_{\mathbb{Q}}^\vee$ is dense in $\mathfrak{U}_{\mathbb{Q}}^*$ we can extend any $\phi \in \mathfrak{U}_{\mathbb{Q}}^\vee$ to $\mathfrak{U}_{\mathbb{Q}}^c$ by continuity. Now for $y \in \mathfrak{n}_{\mathbb{Q}}^c$ define:

$$\begin{cases} n_{\mathrm{EXP}(y)} : \mathfrak{U}_{\mathbb{Q}}^\vee \to \mathbb{Q} \\ n_{\mathrm{EXP}(y)}(\phi) = \phi(\mathrm{EXP}(y)) \end{cases}$$

INJECTIVITY: This follows from $\mathfrak{U}_{\mathbb{Q}}^\vee$ being dense in $\mathfrak{U}_{\mathbb{Q}}^*$.

HOMOMORPHISM: We may assume y to be primitive, non-primitive elements of $\mathfrak{n}_{\mathbb{Q}}^c$ can be expressed as limits of primitive elements. Now the following calcu-

[1] This proof was communicated to me by D. H. Peterson.

lation proves the claim:

$$\boldsymbol{\nu}(\textsc{Exp}(y)) = \textsc{Exp}(y) \otimes \textsc{Exp}(y).$$

Finally we claim that $n_{\textsc{Exp}(y)}$ is a $\mathbb{Q}$-algebra homomorphism:

$$\begin{aligned} n_{\textsc{Exp}(y)}(\phi\psi) &= (\phi\psi)(\textsc{Exp}(y)) \\ &= (\phi \otimes \psi)(\boldsymbol{\nu}(\textsc{Exp}(y)) \\ &= (\phi \otimes \psi)(\textsc{Exp}(y)) \otimes \psi(\textsc{Exp}(y)) \\ &= (\phi(\textsc{Exp}(y)) \otimes (\psi(\textsc{Exp}(y)) \\ &= n_{\textsc{Exp}(y)}(\phi) n_{\textsc{Exp}(y)}(\psi) \end{aligned}$$

SURJECTIVITY: Let $n \in \mathbf{N}_{\mathbb{Q}}^{+}$ be arbitrary, we will inductively construct $y \in \mathfrak{n}_{\mathbb{Q}}^{c}$ such that $n = n_{\textsc{Exp}(y)}$, that is:

$$n(\phi) = \phi(\textsc{Exp}(y)), \qquad \forall \phi \in \mathfrak{U}_{\mathbb{Q}}^{\vee} \cong \mathbb{Q}[u_1^*, u_2^*, \cdots].$$

Let $P \in \mathbb{Q}[u_1^*, u_2^*, \cdots]$ and assume $X_k = x_1 + \cdots + x_k$ is such that

$$n(P) = P(\textsc{Exp}(X_k)), \qquad \deg(P) \leq k.$$

We would like to find x_{k+1} that satisfies:

$$n(P) = P(\textsc{Exp}(X_k + x_{k+1})), \qquad \deg(P) \leq k+1.$$

But based on the Campbell-Hausdorff formula we have:

$$P(\textsc{Exp}(X_k + x_{k+1})) = P(\textsc{Exp}(X_k))P(\textsc{Exp}(x_{k+1})), \qquad \deg(P) \leq k+1,$$

which implies that we should have:

$$n(P) = P(\textsc{Exp}(x_{k+1})), \qquad \deg(P) = k+1.$$

But this determines x_{k+1} and then we can repeat the same with $X_{k+1} = X_k + x_{k+1}$. Finally we take $y = \lim_k X_k$. □

5.4 The Split Maximal Kac-Moody Group

Identify the n copies of $\mathbf{A}_{\mathbb{Q}}$ in $\overline{\mathbf{W}}_{1,\mathbb{Q}}, \cdots, \overline{\mathbf{W}}_{n,\mathbb{Q}}$ and define $\overline{\mathbf{W}}_{\mathbb{Q}}$ as the group generated by all $\overline{\mathbf{W}}_{i,\mathbb{Q}}$ subject to one additional relation.

For all $i \neq j \in I$ where $r_i r_j \in \mathbf{W}$ is of finite order m_{ij}: $(\overline{r}_i \overline{r}_j)^{m_{ij}} = 1$.

Then we have a short exact sequence of groups:

$$\{1\} \to \mathbf{A}_{\mathbb{Q}} \to \overline{\mathbf{W}}_{\mathbb{Q}} \to \mathbf{W} \to \{1\} \text{ with } \overline{r}_i \mapsto r_i.$$

Definition 5.31. Identify all the copies of $\mathbf{B}_{\mathbb{Q}}$ in $\mathbf{P}_{1,\mathbb{Q}}, \cdots, \mathbf{P}_{n,\mathbb{Q}}$, then we define the *split maximal Kac-Moody group* as the product of $\mathbf{P}_{1,\mathbb{Q}}, \cdots \mathbf{P}_{n,\mathbb{Q}}$ and $\overline{\mathbf{W}}_{\mathbb{Q}}$ amalgamated along their intersections. The resulting group is not a group scheme, it is the direct limit of groups (see Appendix D), where each group is the $\mathbb{Q}$-points of a group scheme.

Remark 5.32 ([17] page 27). $\mathfrak{g}_{\mathbb{Q}}$ itself satisfies $(\star)$, and if $\dim(\mathfrak{g}_{\mathbb{Q}}) < \infty$ then one obtains Chevalley's simply connected group. However when $\mathfrak{g}_{\mathbb{Q}}$ is infinite dimensional the Hopf algebra $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{g})$ is too small to give us a suitable group. To see this let $\phi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{g}) \subset \mathfrak{U}_{\mathbb{Q}}(\mathfrak{g})^*$, and consider the $\mathcal{Q}$-grading:

$$\mathfrak{U}_{\mathbb{Q}}(\mathfrak{g}) = \bigoplus_{\alpha \in \mathcal{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{g})_{\alpha}.$$

Suppose ϕ is non-zero on the subspace $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{g})_{\alpha_0}$, however this would violate the L^*-finiteness of ϕ since there are infinitely many different ways of writing $\alpha_0 = \beta_0 + \gamma_0$ with $\beta_0, \gamma_0 \in \mathcal{Q}$. Therefore we denote the group defined in Definition 5.31 by $\widehat{\mathbf{G}}_{\mathbb{Q}}$ to distinguish it from $\mathbf{G}_{\mathbb{Q}}$ discussed above.

Remark 5.33. Similarly we may define $\widehat{\mathbf{G}}_{\mathcal{F}}$ where $\mathcal{F}$ is any field of characteristic zero. Set:

$$\mathfrak{H}_{\mathcal{F}}(\mathfrak{m}) := \mathcal{F} \otimes_{\mathbb{Q}} \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}),$$

where $\mathfrak{m}_{\mathcal{F}} = \mathfrak{p}_{1,\mathcal{F}}, \cdots, \mathfrak{p}_{n,\mathcal{F}}, \mathfrak{a}_{\mathcal{F}}$ then we have groups over $\mathcal{F}$: $\mathbf{P}_{1,\mathcal{F}}, \cdots, \mathbf{P}_{n,\mathcal{F}}$ and $\mathbf{A}_{\mathcal{F}}$. The definition of $\overline{\mathbf{W}}_{\mathcal{F}}$ is identical to $\overline{\mathbf{W}}_{\mathbb{Q}}$ since the quotient group $\mathbf{W}$ does not depend on $\mathcal{F}$.

Chapter 6

Groups over $\mathbb{Z}$

Starting with the group $\mathbf{M}_{\mathbb{Q}}$ defined in the previous chapter, we aim to define a group scheme over $\mathbb{Z}$: $\mathbf{M}_{\mathbb{Z}}$. In order to do so we first define a natural subring of the Hopf algebra $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}) \subset \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ (see Definition 6.1). If $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ becomes a Hopf algebra over $\mathbb{Z}$ with the maps inherited from $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ (so that we have a compatible subgroup) and is a lattice in $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ (to ensure the group is large enough) we say that $\mathfrak{m}_{\mathbb{Q}}$ is an *integral* subalgebra of $\mathfrak{g}_{\mathbb{Q}}$ and define $\mathbf{M}_{\mathbb{Z}}$ to be the spectrum of $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$.

However not every subalgebra which satisfies ($\star$) (which is needed for $\mathbf{M}_{\mathbb{Q}}$ to exist in the first place) is integral so we have to impose further conditions on $\mathfrak{m}_{\mathbb{Q}}$. In §1 we introduce $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ and the concept of an integral subalagebra, in §2 we investigate which conditions are needed for $\mathfrak{m}_{\mathbb{Q}}$ to be an integral subalgebra, in §3 we try to establish which subalgebras of $\mathfrak{g}_{\mathbb{Q}}$ are integral and finally in §4 we define $\widehat{\mathbf{G}}_{\mathbb{Z}}$ in an analogous way to $\widehat{\mathbf{G}}_{\mathbb{Q}}$ since the subalgebras used in the definition of $\widehat{\mathbf{G}}_{\mathbb{Q}}$ are integral as shown in §2.

The material of §1 and §2 are based on [17] pages 21 - 23. It should be noted that the theory of Kac-Moody groups over integers has become a very active field of research, two recent and noteworthy references are [1, 5].

6.1 $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$

Definition 6.1. Throughout this chapter $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{g}_{\mathbb{Q}}$ is a subalgebra that satisfies ($\star$) and hence $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ is a commutative Hopf algebra. Set:

$$\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) := \mathfrak{U}_{\mathbb{Z}}(\mathfrak{g}) \cap \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}),$$
$$\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}) := \left\{\phi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}) : \ \phi\left(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})\right) \subseteq \mathbb{Z}\right\}.$$

Example 6.2. Let us try and compute $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{a})$. By definition $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{a})$ is the set of linear maps in $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{a})$ that take integer values when restricted to $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$. Since elements of $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$ are integral linear combination of products of elements of the form $\binom{\alpha_i^\vee}{m}$, we conclude that $\phi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{a})$ belongs to $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{a})$ if and only if $\phi(\Pi^\vee) \subseteq \mathbb{Z}$. This combined with the map we used to show $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{a}) = \mathbb{Q}[\mathscr{P}]$ shows that $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{a}) = \mathbb{Z}[\mathscr{P}]$.

Definition 6.3. A subalgebra $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{g}_{\mathbb{Q}}$ is called *integral*, if:

(1) $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ is a commutative Hopf algebra with the maps inherited from $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$,

(2) $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ is a lattice in $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$, i.e. $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}) = \mathbb{Q} \otimes \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$.

Definition 6.4. Given an integral subalgebra, $\mathfrak{m}_{\mathbb{Q}}$, we may define a group over $\mathbb{Z}$:

$$\mathbf{M}_{\mathbb{Z}} = \mathrm{HOM}_{\mathbb{Z}-\mathrm{alg}}(\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}), \mathbb{Z}).$$

Moreover if $\mathfrak{R}$ is any commutative ring of characteristic zero with a unit, by setting $\mathfrak{H}_{\mathfrak{R}}(\mathfrak{m}) := \mathfrak{R} \otimes \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ we can define a group over $\mathfrak{R}$:

$$\mathbf{M}_{\mathfrak{R}} = \mathrm{HOM}_{\mathfrak{R}-\mathrm{alg}}\left(\mathfrak{H}_{\mathfrak{R}}(\mathfrak{m}), \mathfrak{R}\right).$$

This is compatible with our earlier definition over $\mathbb{Q}$ and any other fields of characteristic zero.

6.2 Integrality Conditions

6.2.1 Hopf Algebra

Lemma 6.5. *The following hold with no extra conditions on* $\mathfrak{m}_{\mathbb{Q}}$*:*

$$\begin{aligned} \boldsymbol{\zeta}^*(\mathbb{Z}) &\subseteq \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}), \\ \boldsymbol{\epsilon}^*(\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})) &\subseteq \mathbb{Z}, \\ \boldsymbol{\gamma}^*(\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})) &\subseteq \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}). \end{aligned}$$

Proof. The unit and antipode map conditions are automatically satisfied. The condition for the counit map follow from $\widetilde{\mathfrak{U}}_{\mathbb{Z}}(\mathfrak{m}) := \widetilde{\mathfrak{U}}_{\mathbb{Q}}(\mathfrak{m}) \cap \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})$, which then implies $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) = \mathbb{Z} \oplus \widetilde{\mathfrak{U}}_{\mathbb{Z}}(\mathfrak{m})$. □

Definition 6.6. Next we present two further conditions on $\mathfrak{m}_{\mathbb{Q}}$:

$$\boldsymbol{\nu}\left(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})\right) \subseteq \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}), \tag{†}$$

$$\forall \mathfrak{I}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m}) : \quad \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}} \text{ is a } \mathbb{Z}\text{-module of finite type.} \tag{‡}$$

And here $\mathfrak{I}_{\mathbb{Z}} = \mathfrak{I}_{\mathbb{Q}} \cap \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})$ for all $\mathfrak{I}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$.

Lemma 6.7. *If $\mathfrak{m}_{\mathbb{Q}}$ satisfies (†) then $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ is closed under multiplication.*

Proof. Let $a, b \in \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ and $u \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})$ then by definition of the transpose map we have:

$$\boldsymbol{\nu}^*(a \otimes b)(u) = (a \otimes b)(\boldsymbol{\nu}(u)).$$

but $\boldsymbol{\nu}(u) \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})$ by (†) and $a, b \in \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$, hence: $(a \otimes b)(\boldsymbol{\nu}(u)) \in \mathbb{Z}$. □

Lemma 6.8. *If $\mathfrak{m}_{\mathbb{Q}}$ satisfies (‡) then $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ is closed under co-multiplication.*

Proof. Let $\phi \in \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}) \subset \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$, since $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ is a Hopf algebra we already have:

$$\begin{aligned}\boldsymbol{\mu}^*(\phi) \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}) &\subset \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^* \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})^* \\ &\subset \big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})\big)^* \\ &= \mathrm{H{\small OM}}_{\mathbb{Q}}\big(\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}) \otimes_{\mathbb{Q}} \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}),\ \mathbb{Q}\big)\end{aligned}$$

Since

$$\boldsymbol{\mu}^*(\phi)\big(u \otimes u'\big) = \phi\big(\boldsymbol{\mu}(u \otimes u')\big) = \phi(uu'), \tag{6.9}$$

$\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})$, as a subring, is closed under multiplication and $\phi \in \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$ we have in fact:

$$\boldsymbol{\mu}^*(\phi) \in \mathrm{H{\small OM}}_{\mathbb{Z}}\big(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}),\ \mathbb{Z}\big). \tag{6.10}$$

On the other hand because $\phi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$, there exists a two sided ideal $\mathfrak{I}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ such that $\phi(\mathfrak{I}_{\mathbb{Q}}) = 0$. Using (6.9) arrive at:

$$\boldsymbol{\mu}^*(\phi)\big(\mathfrak{I}_{\mathbb{Z}} \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) + \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \otimes \mathfrak{I}_{\mathbb{Z}}\big) = 0. \tag{6.11}$$

From (6.10) and (6.11) we have:

$$\begin{aligned}\boldsymbol{\mu}^*(\phi) \in \mathrm{Hom}_{\mathbb{Z}}&\left(\frac{\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})}{\mathfrak{I}_{\mathbb{Z}} \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) + \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \otimes \mathfrak{I}_{\mathbb{Z}}}, \mathbb{Z}\right)\\ &= \mathrm{Hom}_{\mathbb{Z}}\big(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}} \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}}, \mathbb{Z}\big)\\ &= \mathrm{Hom}_{\mathbb{Z}}\big(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}}, \mathbb{Z}\big) \otimes \mathrm{Hom}_{\mathbb{Z}}\big(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}}, \mathbb{Z}\big)\end{aligned}$$

By (‡), $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}}$ is a $\mathbb{Z}$-module of finite type, it is torsion free as well since $\mathfrak{I}_{\mathbb{Q}}$ is an ideal. Therefore

$$\mathrm{Hom}_{\mathbb{Z}}\big(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}}, \mathbb{Z}\big) \subseteq \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}).$$

And from this we get

$$\boldsymbol{\mu}^*(\phi) \subseteq \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}) \otimes \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m}).$$

□

6.2.2 Lattice

Lemma 6.12. *If* $\mathfrak{m}_{\mathbb{Q}}$ *satisfies* (‡) *then* $\mathfrak{H}_{\mathbb{Q}}(\mathfrak{m}) = \mathbb{Q} \otimes \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$.

Proof. If $\phi \in \mathfrak{H}_{\mathbb{Q}}(\mathfrak{m})$ then there exists a two sided ideal $\mathfrak{I}_{\mathbb{Q}} \in \mathscr{X}_{\mathbb{Q}}(\mathfrak{m})$ such that $\phi(\mathfrak{I}_{\mathbb{Q}}) = 0$. Therefore we have a homomorphism of additive abelian groups:

$$\phi : \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}} \to \mathbb{Q}.$$

However $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}}$ is a $\mathbb{Z}$-module of finite type and so the image of ϕ is fully determined by its value on the finitely many generators of $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}}$. By taking the least common denominator of the image of this finite set of generators we see that there exists $0 \neq p \in \mathbb{Z}$ such that:

$$\phi\left(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})\right) \subseteq \tfrac{1}{p}\mathbb{Z}.$$

We therefore have $p\phi \in \mathfrak{H}_{\mathbb{Z}}(\mathfrak{m})$. □

6.3 Integral Subalgebras

Theorem 6.13. *If a subalgebra $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{g}_{\mathbb{Q}}$ satisfies* $(\star)$, $(\dagger)$ *and* $(\ddagger)$ *then it is integral.*

Proof. This follows from Lemmas 6.7, 6.5, 6.8 and 6.12. □

Using Theorem 6.13 in this section we examine whether the well known subalgebras of $\mathfrak{g}_{\mathbb{Q}}$ are integral or not. However before doing so we find an equivalent, but easier to verify, statement for $(\ddagger)$. We start with a definition:

Definition 6.14. Let $\mathcal{Y}_{\mathbb{Q}}(\mathfrak{m})$ be the set of all $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}+\mathfrak{a})$-modules, E, such that:

(1) E is finite dimensional.

(1) E is a weight module for $\mathfrak{a}_{\mathbb{Q}}$.

(2) weights of E lie in $\mathscr{P}$.

Lemma 6.15. $(\ddagger)$ *is equivalent to:*

$$\forall E \in \mathcal{Y}_{\mathbb{Q}}(\mathfrak{m}), \forall e \in E: \quad \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \cdot e \text{ is a } \mathbb{Z}\text{-module of finite type.} \qquad (\dagger\dagger)$$

Proof. If $\mathfrak{I}_{\mathbb{Q}} \in \mathcal{X}_{\mathbb{Q}}(\mathfrak{m})$ then $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}} \in \mathcal{Y}_{\mathbb{Q}}(\mathfrak{m})$. $(\dagger\dagger)$ implies that $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \cdot (x+\mathfrak{I}_{\mathbb{Q}})$ is a $\mathbb{Z}$-module of finite type for all $x \in \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})$. In particular $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \cdot (1+\mathfrak{I}_{\mathbb{Q}})$ is a $\mathbb{Z}$-module of finite type. But since $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})$ acts on $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Q}}$ by left multiplication and so $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \cdot (1+\mathfrak{I}_{\mathbb{Q}}) \cong \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m})/\mathfrak{I}_{\mathbb{Z}}$ which proves $(\ddagger)$.

Assume $(\ddagger)$ and let $E \in \mathcal{Y}_{\mathbb{Q}}(\mathfrak{m})$ be arbitrary. Now the set of elements $e \in E$ such that $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \cdot e$ is a $\mathbb{Z}$-module of finite type, is itself a $\mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}+\mathfrak{a})$-submodule of E. □

Lemma 6.16.

(1) If $\mathfrak{a}_{\mathbb{Q}} + \mathfrak{m}_{\mathbb{Q}}$ satisfies $(\dagger\dagger)$ then $\mathfrak{m}_{\mathbb{Q}}$ also satisfies $(\dagger\dagger)$.

(2) If $\mathfrak{t}_{\mathbb{Q}} = \mathfrak{a}_{\mathbb{Q}} \oplus \mathfrak{m}^{+}_{\mathbb{Q}}$ then $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{t}) = \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^{+}) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$.

(3) *If* $\mathfrak{t}_{\mathbb{Q}} = \mathfrak{a}_{\mathbb{Q}} \oplus \mathfrak{m}^+_{\mathbb{Q}}$ *then* $\mathfrak{t}_{\mathbb{Q}}$ *and* $\mathfrak{m}^+_{\mathbb{Q}}$ *both satisfy* (††).

(4) *If* $\mathfrak{a}_{\mathbb{Q}} \subseteq \mathfrak{m}_{\mathbb{Q}}$ *and* $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) = \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^-) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{t})$*, then* $\mathfrak{m}_{\mathbb{Q}}$ *satisfies* (††).

Proof. Since $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) \subseteq \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a} + \mathfrak{m})$ Lemma 6.15 implies (1) at once.

For each $\beta \in \mathcal{Q}$: $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^+)_\beta = \mathfrak{U}_{\mathbb{Q}}(\mathfrak{m}^+)_\beta \cap \mathfrak{U}_{\mathbb{Z}}(\mathfrak{n}^+)_\beta$. Since $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{n}^+)_\beta$ is a $\mathbb{Z}$-module of finite type, $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^+)_\beta$ is a direct factor of $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{n}^+)_\beta$. Hence $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^+)$ is a direct factor of $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{n}^+)$. From this and $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{b}) = \mathfrak{U}_{\mathbb{Z}}(\mathfrak{n}^+) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})$ we get (2).

Using part (1) we see that to show (3) it suffices to show that $\mathfrak{t}_{\mathbb{Q}}$ satisfies (††). So let $E \in \mathcal{Y}_{\mathbb{Q}}(\mathfrak{t})$ and $e \in E$, we can also assume that e is a weight vector and therefore: $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{a}) \cdot e = \mathbb{Z}e$. Now using part (2) we may write:

$$\mathfrak{U}_{\mathbb{Z}}(\mathfrak{t}) \cdot e = \big(\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^+) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{a})\big) \cdot e = \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^+) \cdot (\mathbb{Z}e).$$

Since E is finite dimensional there exists a finite set $\Sigma \subseteq \mathcal{Q}_+$ such that:

$$\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^+) \cdot e = \bigoplus_{\alpha \in \Sigma} \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^+)_\alpha \cdot e.$$

Since $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^+)_\alpha$ are $\mathbb{Z}$-modules of finite type this proves 3.

For (4) let $E \in \mathcal{Y}_{\mathbb{Q}}(\mathfrak{m})$ by (3), for every $e \in E$ the $\mathbb{Z}$-modules $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^-) \cdot e$ and $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{t}) \cdot e$ are of finite type. Now $\mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}) = \mathfrak{U}_{\mathbb{Z}}(\mathfrak{m}^-) \otimes \mathfrak{U}_{\mathbb{Z}}(\mathfrak{t})$ implies (4). □

Remark 6.17. The Lie algebras enumerated in Remark 5.18 are integral, we already know that all these subalgebras satisfy (⋆). For (†) we note that it is enough to show it for the generators of each subalgebra. Finally for (‡) we use the equivalent formulation (††) and Lemma 6.16. So we have the corresponding groups over $\mathbb{Z}$:

$$\mathbf{A}_{\mathbb{Z}}, \mathbf{B}_{\mathbb{Z}}, \mathbf{B}_{\pm i,\mathbb{Z}}, \mathbf{C}_{i,\mathbb{Z}}, \mathbf{L}_{i,\mathbb{Z}}, \mathbf{N}^{\pm}_{\mathbb{Z}}, \mathbf{N}_{i,\mathbb{Z}}, \mathbf{N}_{\pm\alpha_i,\mathbb{Z}}, \mathbf{P}_{i,\mathbb{Z}}.$$

Lemma 6.18.

(1) For each $i \in I$ we have the following isomorphisms:

$$\begin{aligned}\mathbf{P}_{i,\mathbb{Z}} &= \mathbf{N}_{i,\mathbb{Z}} \times \mathbf{L}_{i,\mathbb{Z}},\\ \mathbf{B}_{\mathbb{Z}} &= \mathbf{N}_{i,\mathbb{Z}} \times \mathbf{B}_{i,\mathbb{Z}},\\ \mathbf{C}_{i,\mathbb{Z}} &= \mathbf{N}_{i,\mathbb{Z}} \times \mathbf{B}_{-i,\mathbb{Z}},\\ \mathbf{B}_{\mathbb{Z}} &= \mathbf{N}^{+}_{\mathbb{Z}} \times \mathbf{A}_{\mathbb{Z}}.\end{aligned}$$

(2) If $\mathfrak{m}_{\mathbb{Q}} \subseteq \mathfrak{m}'_{\mathbb{Q}}$ are two subalgebras enumerated in Remark 5.18 then there exists a natural morphism $\mathbf{M}_{\mathbb{Z}} \to \mathbf{M}'_{\mathbb{Z}}$ which is a closed immersion.

(3) For each $i \in I$ the natural morphisms

$$\begin{aligned}\mathbf{N}_{-\alpha_i,\mathbb{Z}} \times \mathbf{B}_{\mathbb{Z}} &\to \mathbf{P}_{i,\mathbb{Z}}\\ \mathbf{N}_{\alpha_i,\mathbb{Z}} \times \mathbf{C}_{i,\mathbb{Z}} &\to \mathbf{P}_{i,\mathbb{Z}}\end{aligned}$$

are open immersions.

Proof. The only delicate point is to show (2). More precisely, to show that $\mathbf{B}_{\mathbb{Z}} \to \mathbf{P}_{i,\mathbb{Z}}$ is a closed immersion. Using isomorphism of groups $\mathbf{P}_{i,\mathbb{Z}} = \mathbf{N}_{i,\mathbb{Z}} \times \mathbf{L}_{i,\mathbb{Z}}$ and $\mathbf{B}_{\mathbb{Z}} = \mathbf{N}_{i,\mathbb{Z}} \times \mathbf{B}_{i,\mathbb{Z}}$, we reduce to show that $\mathbf{B}_{i,\mathbb{Z}} \to \mathbf{L}_{i,\mathbb{Z}}$ is a closed immersion, this is done by direct calculation. We show (3) with the same argument. □

6.4 The Arithmetic Group

Definition 6.19. Now that we have the required group schemes, we define, $\mathbf{\Gamma}$, the split maximal Kac-Moody group over $\mathbb{Z}$ as the product of $\mathbf{P}_{1,\mathbb{Z}}, \cdots, \mathbf{P}_{n,\mathbb{Z}}$ and $\overline{\mathbf{W}}_{\mathbb{Z}}$, amalgamated along their intersections. As before (see Definition 5.31) this group itself is not a group scheme over $\mathbb{Z}$.

Chapter 7

Structure Theory

In §1 we show that $\widehat{\mathbf{G}}_{\mathscr{F}}$ acts on any integrable irreducible highest weight module $L(\Lambda)_{\mathscr{F}}$. In fact we give an explicit construction for the representation map which is based on the last paragraph of page 28 in [17]. Next we see how different subgroups act and what their matrices look like in an admissible basis, most importantly we show that the Chevalley lattice is stable under $\mathbf{\Gamma}$ (see Lemma 7.9).

In §2 we use the representations to show that $\widehat{\mathbf{G}}_{\mathscr{F}}$ possesses a Tits system, this proof is based on the representation theory of $\widehat{\mathbf{G}}_{\mathscr{F}}$ and follows §5.12 in [23].

In §3 we prove an Iwasawa decomposition when $\mathscr{F} = \mathbb{R}, \mathbb{C}$. This prove uses the existence of the Tits system shown earlier, with Lemma 7.26 on the theory of Tits systems being the crucial part of the proof.

In §4 we introduce the minimal group of Kac and Peterson, we will use this group to show that the orbit of the highest weight vector is not the entire module.

7.1 Representation Theory

7.1.1 Constructing the Map

Let $L(\Lambda)_{\mathscr{F}}$ be an integrable highest weight module for $\mathfrak{g}_{\mathscr{F}}$. We wish to construct a representation:

$$\widehat{\mathbf{G}}_{\mathscr{F}} \to \mathbf{GL}(L(\Lambda)_{\mathscr{F}}).$$

By the construction of $\widehat{\mathbf{G}}_{\mathscr{F}}$ it suffices to do so for each minimal parabolic subgroup $\mathbf{P}_{i,\mathscr{F}}$. For every $i \in I$, since $\mathfrak{U}_{\mathscr{F}}(\mathfrak{p}_i)$ acts on $L(\Lambda)_{\mathscr{F}}$, every $v \in L(\Lambda)_{\mathscr{F}}$ gives rise to a map $\psi_v : \mathfrak{U}_{\mathscr{F}}(\mathfrak{p}_i) \to L(\Lambda)_{\mathscr{F}}$ defined by $\psi_v(x) = x \cdot v$. Since $L(\Lambda)_{\mathscr{F}}$ is integrable we see that all ψ_v have finite rank. But on the other hand we have a

$\mathscr{F}$-vector space isomorphism:

$$\mathrm{HOM}^{\mathrm{fin}}_{\mathscr{F}}(\mathfrak{U}_{\mathscr{F}}(\mathfrak{p}_i),\ L(\Lambda)_{\mathscr{F}}) \cong \mathfrak{U}_{\mathscr{F}}(\mathfrak{p}_i)^* \otimes_{\mathscr{F}} L(\Lambda)_{\mathscr{F}}, \tag{7.1}$$

where $\mathrm{HOM}^{\mathrm{fin}}_{\mathscr{F}}(\cdot,\cdot)$ is the set of all $\mathscr{F}$-linear maps of finite rank. So combining $v \mapsto \psi_v$ with this isomorphism gives us a map:

$$L(\Lambda)_{\mathscr{F}} \to \mathfrak{U}_{\mathscr{F}}(\mathfrak{p}_i)^* \otimes_{\mathscr{F}} L(\Lambda)_{\mathscr{F}}. \tag{7.2}$$

We claim that the image of the map in (7.2) in fact lies in $\mathfrak{H}_{\mathscr{F}}(\mathfrak{p}_i) \otimes_{\mathscr{F}} L(\Lambda)_{\mathscr{F}}$. To see this we first recall the isomorphism of (7.1) can be explicitly written as:

$$\left\{\psi : \mathfrak{U}_{\mathscr{F}}(\mathfrak{p}_i) \to L(\Lambda)_{\mathscr{F}}\right\} \mapsto \sum_{i=1}^{\infty} x_i^* \otimes \psi(x_i)$$

where $\{x_1, x_2, \cdots\}$ is a basis for $\mathfrak{U}_{\mathscr{F}}(\mathfrak{p}_i)$ of our choosing. We take it to be the basis afforded to us by the PBW Theorem with the ordering:

$$e_{-i} < \alpha_1^\vee < \cdots < \alpha_n^\vee < e_1 < \cdots < e_n < \cdots .$$

If we show that the elements of the dual basis $\{x_1^*, x_2^*, \cdots\}$ all lie in $\mathfrak{H}_{\mathscr{F}}(\mathfrak{p}_i)$, we are done. But then we may write:

$$\mathfrak{H}_{\mathscr{F}}(\mathfrak{p}_i) = \mathfrak{H}_{\mathscr{F}}(\mathfrak{n}_i) \otimes_{\mathscr{F}} \mathfrak{H}_{\mathscr{F}}(\mathfrak{l}_i)$$

and we know that $\mathfrak{H}_{\mathscr{F}}(\mathfrak{n}_i)$ and $\mathfrak{H}_{\mathscr{F}}(\mathfrak{l}_i)$ are the restricted dual of the corresponding universal enveloping algebras. Hence the same has to be true for $\mathfrak{H}_{\mathscr{F}}(\mathfrak{p}_i)$. So we have a map:

$$L(\Lambda)_{\mathscr{F}} \to \mathfrak{H}_{\mathscr{F}}(\mathfrak{p}_i) \otimes_{\mathscr{F}} L(\Lambda)_{\mathscr{F}}.$$

From the theory of affine group schemes this corresponds to a representation of the group:

$$\mathbf{P}_{i,\mathscr{F}} \to \mathbf{GL}(L(\Lambda)_{\mathscr{F}}).$$

Remark 7.3. The above construction more than giving us a representation for $\widehat{\mathbf{G}}_{\mathscr{F}}$ has the virtue that it enables us to actually compute the representation for the parabolic subgroups. Suppose we are given $p \in \mathbf{P}_{i,\mathcal{F}}$ then we may write its action based on the following sequence using the fact that $p \in \mathbf{P}_{i,\mathcal{F}}$ is in fact a homomorphism of $\mathcal{F}$-algebras: $p : \mathfrak{H}_{\mathcal{F}}(\mathfrak{p}_i) \to \mathcal{F}$ one can write:

$$L(\Lambda)_{\mathcal{F}} \longrightarrow \mathfrak{H}_{\mathcal{F}}(\mathfrak{p}_i) \otimes_{\mathcal{F}} L(\Lambda)_{\mathcal{F}} \xrightarrow{p\otimes 1} \mathcal{F} \otimes_{\mathcal{F}} L(\Lambda)_{\mathcal{F}} \xrightarrow{\cong} L(\Lambda)_{\mathcal{F}} \ ,$$

So for a given $v \in L(\Lambda)_{\mathcal{F}}$ we may write:

$$\begin{aligned} p \cdot v &= (p \otimes 1)\left(\sum_{i=1}^{\infty} m_i^* \otimes (m_i \cdot v)\right) \\ &= \sum_{i=1}^{\infty} p\big(m_i^*\big) \otimes (m_i \cdot v) \\ &= \sum_{i=1}^{\infty} p\big(m_i^*\big)(m_i \cdot v) \end{aligned}$$

where $\{m_1, m_2, \cdots\}$ is a basis of our choosing for $\mathfrak{U}_{\mathcal{F}}(\mathfrak{p}_i)$ and $\{m_1^*, m_2^*, \cdots\}$ is its dual.

7.1.2 Subgroups

Remark 7.4. Since 1^* is the unit of $\mathfrak{H}_{\mathcal{F}}(\mathfrak{p}_i)$ and p is a $\mathcal{F}$-algebra homomorphism we have: $p(1^*) = 1$ for all $p \in \mathbf{P}_{i,\mathcal{F}}$. So if we choose a basis such that $m_1 = 1$ we can write the action in a slightly more useful form:

$$p \cdot v = v + \sum_{i=2}^{\infty} p\big(m_i^*\big)(m_i \cdot v).$$

Lemma 7.5. *In any admissible basis for $L(\Lambda)_{\mathcal{F}}$ the elements of $\mathbf{A}_{\mathcal{F}}$ are represented by diagonal matrices. More precisely $a \in \mathbf{A}_{\mathcal{F}}$ acts on $L(\Lambda)_{\mathcal{F},\lambda}$ by a^{λ}.*

Proof. This follows from $\mathbf{A}_{\mathcal{F}}$ being the same as classically defined torus. □

Lemma 7.6. *In any admissible basis for $L(\Lambda)_{\mathcal{F}}$ the elements of $\mathbf{N}_{\mathcal{F}}^{+}$ are represented by upper triangular unipotent matrices. In particular: $n \cdot \mathbb{1}_{\Lambda} = \mathbb{1}_{\Lambda}$ for all $n \in \mathbf{N}_{\mathcal{F}}^{+}$.*

Proof. Let $v \in L(\Lambda)_{\mathcal{F}}$ be a weight vector arbitrary, we have:

$$n \cdot v = v + \sum_{i=2}^{\infty} n\big(m_i^*\big)(m_i \cdot v)$$

But when acting on $L(\Lambda)_{\mathcal{F}}$ any non-constant monomial in $\mathfrak{U}_{\mathcal{F}}(\mathfrak{n}^+)$ will lower depth. So the terms in the sum (which are finitely many) are vectors in $L(\Lambda)_{\mathcal{F}}$ and all of their weight components have lower depth than v. □

Corollary 7.7. $\mathbf{A}_{\mathcal{F}}$ *normalizes* $\mathbf{N}_{\mathcal{F}}^+$*, in particular* $\mathbf{B}_{\mathcal{F}} = \mathbf{N}_{\mathcal{F}}^+ \mathbf{A}_{\mathcal{F}} = \mathbf{A}_{\mathcal{F}} \mathbf{N}_{\mathcal{F}}^+$.

Proof. This follows from Lemmas 7.5 and 7.6. □

Lemma 7.8. $\mathbf{N}_{\mathcal{F}}^+$ *acts faithfully on non-trivial integrable highest weight modules.*

Proof. Suppose that there exists $n \in \mathbf{N}_{\mathcal{F}}^+$ such that $n \cdot v = v$ for all $v \in L(\Lambda)_{\mathcal{F}}$. Based on the action of n this means:

$$\forall v \in L(\Lambda)_{\mathcal{F}} : \quad \sum_{i=2}^{\infty} n\big(m_i^*\big)(m_i \cdot v) = 0.$$

Now we can find a vector $v_2 \in L(\Lambda)_{\mathcal{F}}$ such that $m_2 \cdot v_2 \neq 0$ but $m_i \cdot v_2 = 0$ for all $i > 2$. This shows that $n\big(m_2^*\big) = 0$. By repeating this argument we see that $n\big(m_i^*\big) = 0$ for all $i \geqslant 2$. □

7.1.3 The Arithmetic Subgroup and the Chevalley Lattice

Lemma 7.9. *For all* $p \in \mathbf{P}_{j,\mathbb{Z}}$ *we have:* $p \cdot L(\Lambda)_{\mathbb{Z}} \subset L(\Lambda)_{\mathbb{Z}}$.

Proof. Let $v \in L(\Lambda)_{\mathbb{Z}}$ be arbitrary, then we have:

$$p \cdot v = \sum_{i=1}^{\infty} p(m_i^*)(m_i \cdot v).$$

The series is in fact a finite sum we just need to show that each term belongs to $L(\Lambda)_{\mathbb{Z}}$. Since $m_i \in \mathfrak{U}_{\mathbb{Z}}(\mathfrak{p}_j)$ we have: $m_i \cdot v \in L(\Lambda)_{\mathbb{Z}}$, that leaves $p(m_i^*)$. However $p \in \mathbf{P}_{j,\mathbb{Z}}$ is a ring homomorphism: $p : \mathfrak{H}_{\mathbb{Z}}(\mathfrak{p}_j) \to \mathbb{Z}$ and since $\mathfrak{H}_{\mathbb{Z}}(\mathfrak{p}_j)$ is a lattice in $\mathfrak{H}_{\mathcal{F}}(\mathfrak{p}_j)$ we can always choose a basis such that $m_i^* \in \mathfrak{H}_{\mathbb{Z}}(\mathfrak{p}_j)$. □

7.2 Existence of a Tits System

Notation 7.10. For $w \in \mathbf{W}$, we use $\overline{w}$ to denote an element in $\overline{\mathbf{W}}_{\mathcal{F}}$ which lies in the coset $w\mathbf{A}_{\mathcal{F}}$ and under the quotient map $\overline{\mathbf{W}}_{\mathcal{F}} \to \mathbf{W}$ goes to w.

Definition 7.11. For any real root $\xi = w(\alpha_j)$ we define $\mathbf{N}_{\xi,\mathcal{F}} := \overline{w}\mathbf{N}_{\alpha_j,\mathcal{F}}\overline{w}^{-1}$.

Lemma 7.12. *Let* $\mathbf{Z} = \mathbf{P}_{1,\mathcal{F}} \cup \cdots \cup \mathbf{P}_{n,\mathcal{F}} \cup \overline{\mathbf{W}}_{\mathcal{F}}$ *by definition there is a canonical map* $\mathbf{Z} \to \widehat{\mathbf{G}}_{\mathcal{F}}$. *This map is injective.*

Proof. When acting on $L(\Lambda)_{\mathcal{F}}$ the image of $\mathbf{Z}$ under the canonical map acts as $\mathbf{Z}$ itself would. So if we find a module $V_{\mathcal{F}}$ on which the elements of $\mathbf{Z}$ act faithfully, we are done. Let $\{\varpi_1, \cdots, \varpi_n\}$ be a basis consisting of dominant integral weights for $\mathcal{P}$ and set:

$$V_{\mathcal{F}} := L(\varpi_1)_{\mathcal{F}} \oplus \cdots \oplus L(\varpi_n)_{\mathcal{F}}.$$

$\mathbf{A}_{\mathcal{F}}$ acts faithfully since $\{\varpi_1, \cdots, \varpi_n\}$ generate $\mathcal{P}$. By Lemma 7.8 we know that $\mathbf{N}^{+}_{\mathcal{F}}$ acts faithfully on each integrable highest weight module and therefore it acts faithfully on $V_{\mathcal{F}}$. Given an admissible basis for each module, $L(\varpi_i)_{\mathcal{F}}$, $\mathbf{A}_{\mathcal{F}}$ acts through diagonal matrices and $\mathbf{N}^{+}_{\mathcal{F}}$ acts through upper triangular unipotent matrices. Hence $\mathbf{B}_{\mathcal{F}}$ also acts faithfully. The faithfulness of the action of $\mathbf{P}_{i,\mathcal{F}}$ follows from this and the simplicity of $\mathbf{PGL}_{2,\mathcal{F}}$.

This leaves $\overline{\mathbf{W}}_{\mathcal{F}}$, suppose there exists $\overline{w} \in \overline{\mathbf{W}}_{\mathcal{F}}$ that acts trivially on $V_{\mathcal{F}}$. Since $\mathbf{A}_{\mathcal{F}}$ acts faithfully we can also assume that $w \neq 1$. Therefore there exists $\xi \in \Delta^{\mathrm{re}}_{+}$ such that $w(\xi) \in \Delta_{-}$. Since $\overline{w}$ acts trivially the action of $\overline{w}\mathbf{N}_{\xi}\overline{w}^{-1}$ is the same as $\mathbf{N}_{\xi,\mathcal{F}}$ but $\overline{w}\mathbf{N}_{\xi,\mathcal{F}}\overline{w}^{-1} = \mathbf{N}_{w(\xi),\mathcal{F}}$. However $w(\xi) \in \Delta_{-}$ which means the elements of $\mathbf{N}_{w(\xi),\mathcal{F}}$ are represented by lower triangular unipotent matrices in $\mathbf{GL}(V_{\mathcal{F}})$. This contradiction completes the proof. □

Remark 7.13. Let $\mathbf{N}^{-,\min}_{\mathcal{F}}$ denote the subgroup that is generated by the collection of the subgroups: $\{\mathbf{N}_{\xi,\mathcal{F}} : \xi \in \Delta^{\mathrm{re}}_{-}\}$. In an admissible basis for $V_{\mathcal{F}}$ elements of $\mathbf{B}_{\mathcal{F}}$ (resp. $\mathbf{N}^{-,\min}_{\mathcal{F}}$) operate through upper triangular matrices (resp. unipotent lower triangular matrices). In particular we have:

$$\mathbf{B}_{\mathcal{F}} \cap \mathbf{N}^{-,\min}_{\mathcal{F}} = \{1\}.$$

We use the superscript to emphasize the fact that $\mathbf{N}_{\mathscr{F}}^{-,\min} \neq \mathbf{N}_{\mathscr{F}}^{-}$. In fact $\mathbf{N}_{\mathscr{F}}^{-}$ is not even a subgroup of $\widehat{\mathbf{G}}_{\mathscr{F}}$.

Remark 7.14. We identify the groups $\mathbf{A}_{\mathscr{F}}, \mathbf{N}_{\mathscr{F}}^{+}, \mathbf{B}_{\mathscr{F}}, \overline{\mathbf{W}}_{\mathscr{F}}, \mathbf{P}_{i,\mathscr{F}}$ with their images in $\widehat{\mathbf{G}}_{\mathscr{F}}$.

Lemma 7.15. *$\mathbf{B}_{\mathscr{F}}$ and $\overline{\mathbf{W}}_{\mathscr{F}}$ generate $\widehat{\mathbf{G}}_{\mathscr{F}}$.*

Proof. This follows from the fact that each $\mathbf{P}_{i,\mathscr{F}}$ is generated by $\mathbf{B}_{\mathscr{F}}$ and $\overline{\mathbf{W}}_{i,\mathscr{F}}$. □

Lemma 7.16. *$\mathbf{B}_{\mathscr{F}} \cap \overline{\mathbf{W}}_{\mathscr{F}} = \mathbf{A}_{\mathscr{F}}$.*

Proof. We know $\mathbf{A}_{\mathscr{F}} \subset \mathbf{B}_{\mathscr{F}} \cap \overline{\mathbf{W}}_{\mathscr{F}}$. Let $\overline{w} \in \mathbf{B}_{\mathscr{F}} \cap \overline{\mathbf{W}}_{\mathscr{F}}$ be such that $w \neq 1$. Pick $\xi \in \Delta_{+}^{\mathrm{re}}$ such that $w(\xi) \in \Delta_{-}$. Then:

$$\overline{w}\mathbf{N}_{\xi,\mathscr{F}}\overline{w}^{-1} = \mathbf{N}_{w(\xi),\mathscr{F}} \subset \mathbf{N}_{\mathscr{F}}^{-,\min}.$$

Since $\overline{w} \in \mathbf{B}_{\mathscr{F}}$ and $\mathbf{N}_{\xi,\mathscr{F}} \subset \mathbf{B}_{\mathscr{F}}$ we have:

$$\overline{w}\mathbf{N}_{\xi,\mathscr{F}}\overline{w}^{-1} \subset \mathbf{B}_{\mathscr{F}} \cap \mathbf{N}_{\mathscr{F}}^{-,\min} = \{1\},$$

which contradicts the faithfulness of $\mathbf{N}_{\mathscr{F}}^{+} \to \widehat{\mathbf{G}}_{\mathscr{F}}$. □

Lemma 7.17. *$\overline{r}_i\mathbf{B}_{\mathscr{F}}\overline{r}_i \neq \mathbf{B}_{\mathscr{F}}$.*

Proof. This follows from: $\overline{r}_i\mathbf{N}_{\alpha_i,\mathscr{F}}\overline{r}_i = \mathbf{N}_{-\alpha_i,\mathscr{F}} \subset \mathbf{N}_{\mathscr{F}}^{-,\min}$ and $\mathbf{N}_{\mathscr{F}}^{-,\min} \cap \mathbf{B}_{\mathscr{F}} = \{1\}$. □

Lemma 7.18. *$\mathbf{P}_{i,\mathscr{F}} = \mathbf{B}_{\mathscr{F}} \cup \mathbf{B}_{\mathscr{F}}\overline{r}_i\mathbf{B}_{i,\mathscr{F}}$.*

Proof. Recall that $\mathbf{L}_{i,\mathscr{F}}$ is a finite dimensional reductive group of semi-simple rank 1 and therefore it has a Bruhat decomposition: $\mathbf{L}_{i,\mathscr{F}} = \mathbf{B}_{i,\mathscr{F}} \cup \mathbf{B}_{i,\mathscr{F}}\overline{r}_i\mathbf{B}_{i,\mathscr{F}}$. Combining this decomposition with the isomorphisms: $\mathbf{P}_{i,\mathscr{F}} = \mathbf{N}_{i,\mathscr{F}} \times \mathbf{L}_{i,\mathscr{F}}$ and $\mathbf{B}_{\mathscr{F}} = \mathbf{N}_{i,\mathscr{F}} \times \mathbf{B}_{i,\mathscr{F}}$, which we have from Lemma 5.21 completes the proof. □

Lemma 7.19. *For all $w \in \mathbf{W}$ we have: $\overline{r}_i\mathbf{B}_{\mathscr{F}}\overline{w} \subset \mathbf{B}_{\mathscr{F}}\overline{w}\mathbf{B}_{\mathscr{F}} \cup \mathbf{B}_{\mathscr{F}}\overline{r}_i\overline{w}\mathbf{B}_{\mathscr{F}}$.*

Proof. We divide the proof in two cases:

- POSITIVE CASE: suppose $\xi = w^{-1}(\alpha_i) \succ 0$ then $\overline{w}^{-1}\mathbf{N}_{\alpha_i,\mathscr{F}}\overline{w} = \mathbf{N}_{\xi,\mathscr{F}} \subset \mathbf{N}^+_{\mathscr{F}}$ and we have:

$$\begin{aligned}
\overline{r}_i\mathbf{B}_{\mathscr{F}}\overline{w} &= \overline{r}_i\mathbf{A}_{\mathscr{F}}\mathbf{N}^+_{\mathscr{F}}\overline{w} \\
&= \overline{r}_i\mathbf{A}_{\mathscr{F}}\overline{r}_i\overline{r}_i\mathbf{N}^+_{\mathscr{F}}\overline{w} \\
&= \mathbf{A}_{\mathscr{F}}\overline{r}_i\mathbf{N}_{i,\mathscr{F}}\mathbf{N}_{\alpha_i,\mathscr{F}}\overline{w} \\
&= \mathbf{A}_{\mathscr{F}}\overline{r}_i\mathbf{N}_{i,\mathscr{F}}\overline{r}_i\overline{r}_i\mathbf{N}_{\alpha_i,\mathscr{F}}\overline{w} \\
&= \mathbf{A}_{\mathscr{F}}\left(\overline{r}_i\mathbf{N}_{i,\mathscr{F}}\overline{r}_i\right)\overline{r}_i\mathbf{N}_{\alpha_i,\mathscr{F}}\overline{w} \\
&= \mathbf{A}_{\mathscr{F}}\mathbf{N}_{i,\mathscr{F}}\overline{r}_i\mathbf{N}_{\alpha_i,\mathscr{F}}\overline{w} && \overline{r}_i\mathbf{N}_{i,\mathscr{F}}\overline{r}_i \subset \mathbf{N}_{i,\mathscr{F}} \\
&= \mathbf{A}_{\mathscr{F}}\mathbf{N}_{i,\mathscr{F}}\overline{r}_i\overline{w}\overline{w}^{-1}\mathbf{N}_{\alpha_i,\mathscr{F}}\overline{w} \\
&= \mathbf{A}_{\mathscr{F}}\mathbf{N}_{i,\mathscr{F}}\overline{r}_i\overline{w}\mathbf{N}_{\xi,\mathscr{F}} \\
&\subset \mathbf{B}_{\mathscr{F}}\overline{r}_i\overline{w}\mathbf{B}_{\mathscr{F}}.
\end{aligned}$$

- NEGATIVE CASE: suppose $\xi = w^{-1}(\alpha_i) \prec 0$, then $(r_i w)^{-1}(\alpha_i) \succ 0$ and we have:

$$\begin{aligned}
\overline{r}_i\mathbf{B}_{\mathscr{F}}\overline{w} &= (\overline{r}_i\mathbf{B}_{\mathscr{F}}\overline{r}_i)\,\overline{r}_i\overline{w} \\
&\subset \mathbf{P}_{i,\mathscr{F}}\overline{r}_i\overline{w} \\
&\subset (\mathbf{B}_{\mathscr{F}} \cup \mathbf{B}_{\mathscr{F}}\overline{r}_i\mathbf{B}_{\mathscr{F}})\,\overline{r}_i\overline{w} && \text{Lemma 7.18} \\
&\subset \mathbf{B}_{\mathscr{F}}\overline{r}_i\overline{w} \cup \mathbf{B}_{\mathscr{F}}\,(\overline{r}_i\mathbf{B}_{\mathscr{F}}\overline{r}_i)\,\overline{w} \\
&\subset \mathbf{B}_{\mathscr{F}}\overline{r}_i\overline{w} \cup \mathbf{B}_{\mathscr{F}}\mathbf{B}_{\mathscr{F}}\overline{r}_i(\overline{r}_i\overline{w})\mathbf{B}_{\mathscr{F}} && \text{Positive Case} \\
&\subset \mathbf{B}_{\mathscr{F}}\overline{r}_i\overline{w} \cup \mathbf{B}_{\mathscr{F}}\overline{w}\mathbf{B}_{\mathscr{F}} \\
&\subset \mathbf{B}_{\mathscr{F}}\overline{r}_i\overline{w}\mathbf{B}_{\mathscr{F}} \cup \mathbf{B}_{\mathscr{F}}\overline{w}\mathbf{B}_{\mathscr{F}}.
\end{aligned}$$

□

Theorem 7.20. $\left(\widehat{\mathbf{G}}_{\mathscr{F}}, \mathbf{B}_{\mathscr{F}}, \overline{\mathbf{W}}_{\mathscr{F}}, \{\overline{r}_1, \cdots, \overline{r}_n\}\right)$ *is a Tits system.*

Proof. This follows from Lemmas 7.15, 7.16, 7.17 and Corollary 7.19. □

7.3 Iwasawa Decomposition

Notation 7.21. In this section $\mathscr{F} = \mathbb{R}, \mathbb{C}$.

Definition 7.22. Let $\cdot^\dagger$ denote the Hermitian conjugate with respect to positive definite inner product $\{\cdot, \cdot\}$ defined on $L(\Lambda)_{\mathscr{F}}$ in §3.4. Based on the construction of $\{\cdot, \cdot\}$ we have: $e_i^\dagger = e_{-i}$ for all $i \in I$. A linear operator $T : L(\Lambda)_{\mathscr{F}} \to L(\Lambda)_{\mathscr{F}}$ is called *unitary* if $T^\dagger = T^{-1}$. A group of linear automorphisms of $L(\Lambda)_{\mathscr{F}}$ is called unitary if all its elements are unitary operators.

Definition 7.23. Let $\mathbf{K} \subset \widehat{\mathbf{G}}_{\mathscr{F}}$ be the subgroup consisting of all the elements $g \in \widehat{\mathbf{G}}_{\mathscr{F}}$ such that $g^\dagger$ is defined and equals g^{-1}, we refer to this subgroup as the *unitary form* of $\widehat{\mathbf{G}}_{\mathscr{F}}$.

Remark 7.24. Note that $\mathbf{K}$ is non-trivial, it clearly contains $\mathbf{K}_i$, the real maximal compact subgroup of $\mathbf{L}_{i,\mathscr{F}}$ since $L(\Lambda)_{\mathscr{F}}$ decomposes as a direct sum of irreducible representations of $\mathfrak{l}_{i,\mathscr{F}}$. Moreover since $\overline{r}_i \in \mathbf{K}_i$ we see that $\mathbf{W} \subset \mathbf{K}$.

Lemma 7.25. $\mathbf{B}_{\mathscr{F}}\overline{r}_i\mathbf{B}_{\mathscr{F}} = \mathbf{B}_{\mathscr{F}}\overline{r}_i\mathbf{N}_{\alpha_i,\mathscr{F}}$.

Proof. First we note the following:

$$\overline{r}_i\mathbf{B}_{\mathscr{F}}\overline{r}_i = \overline{r}_i\left(\mathbf{N}_{i,\mathscr{F}}\mathbf{N}_{\alpha_i,\mathscr{F}}\mathbf{A}_{\mathscr{F}}\right)\overline{r}_i = \mathbf{N}_{i,\mathscr{F}}\mathbf{N}_{-\alpha_i,\mathscr{F}}\mathbf{A}_{\mathscr{F}} = \mathbf{N}_{i,\mathscr{F}}\mathbf{A}_{\mathscr{F}}\mathbf{N}_{-\alpha_i,\mathscr{F}}.$$

Then we can write:

$$\begin{aligned}\mathbf{B}_{\mathscr{F}}\overline{r}_i\mathbf{B}_{\mathscr{F}} &= \mathbf{B}_{\mathscr{F}}\overline{r}_i\mathbf{B}_{\mathscr{F}}\,(\overline{r}_i\overline{r}_i)\\ &= \mathbf{B}_{\mathscr{F}}\left(\mathbf{N}_{i,\mathscr{F}}\mathbf{A}_{\mathscr{F}}\mathbf{N}_{-\alpha_i,\mathscr{F}}\right)\overline{r}_i\\ &= \mathbf{B}_{\mathscr{F}}\mathbf{N}_{-\alpha_i,\mathscr{F}}\overline{r}_i\\ &= \mathbf{B}_{\mathscr{F}}\overline{r}_i\mathbf{N}_{\alpha_i,\mathscr{F}}\end{aligned}$$ □

Next we need a Lemma from the theory of Tits systems:

Lemma 7.26 (Proposition 3.1 [14]). *Let (G, B, N, S) be a Tits system with Weyl group $W = N/(B \cap N)$. If $w_1, w_2 \in W$ satisfy $\ell(w_1w_2) = \ell(w_1) + \ell(w_2)$, and if X_1, X_2 are subsets of G satisfying:*

(1) $Bw_1B = X_1B$ with uniqueness of expression,

(2) $Bw_1B = X_2B$ with uniqueness of expression.

Then: $Bw_1w_2B = X_1X_2B$ *with uniqueness of expression.*

Proposition 7.27. $\widehat{\mathbf{G}}_{\mathscr{F}} = \mathbf{B}_{\mathscr{F}}\mathbf{K}$.

Proof. By Bruhat decomposition we only need to prove the claim for a Bruhat cell: $\mathbf{B}_{\mathscr{F}}\overline{w}\mathbf{B}_{\mathscr{F}}$. By Lemma 7.26 it is enough to do so for the Bruhat cells corresponding to the fundamental reflections: $\mathbf{B}_{\mathscr{F}}\overline{r}_1\mathbf{B}_{\mathscr{F}}, \cdots, \mathbf{B}_{\mathscr{F}}\overline{r}_n\mathbf{B}_{\mathscr{F}}$. But using Lemma 7.25 we only need to show that $\overline{r}_i\mathbf{N}_{\alpha_i,\mathscr{F}} \subset \mathbf{B}_{\mathscr{F}}\mathbf{K}$. But in $\mathbf{L}_{i,\mathscr{F}}$ we already have that $\overline{r}_i\mathbf{N}_{\alpha_i,\mathscr{F}} \subset \mathbf{B}_{i,\mathscr{F}}\mathbf{K}_i = \mathbf{L}_{i,\mathscr{F}}$ since $\mathbf{B}_{i,\mathscr{F}} \subset \mathbf{B}_{\mathscr{F}}$ this completes the proof. □

Definition 7.28. Let $\mathbf{A}^+_{\mathscr{F}}$ (resp. $\mathbf{A}^1_{\mathscr{F}}$) be the subgroup of $\mathbf{A}_{\mathscr{F}}$ consisting of elements whose eigenvalues in $L(\Lambda)_{\mathscr{F}}$ are positive real numbers (resp. are modulus one). Then we have a *polar decomposition*: $\mathbf{A}_{\mathscr{F}} = \mathbf{A}^+_{\mathscr{F}}\mathbf{A}^1_{\mathscr{F}}$.

Lemma 7.29. $\mathbf{N}^+_{\mathscr{F}}\mathbf{A}^+_{\mathscr{F}} \cap \mathbf{K} = \{1\}$.

Proof. When acting on $L(\Lambda)_{\mathscr{F}}$ the elements of $\mathbf{N}^+_{\mathscr{F}}\mathbf{A}^+_{\mathscr{F}} \cap \mathbf{K}$ are represented by unitary upper triangular matrices in any admissible basis. Therefore they have to be diagonal so $\mathbf{N}^+_{\mathscr{F}}\mathbf{A}^+_{\mathscr{F}} \cap \mathbf{K} \subset \mathbf{A}^+_{\mathscr{F}}$ but the only unitary element of $\mathbf{A}^+_{\mathscr{F}}$ is the identity. □

Iwasawa Decomposition. $\widehat{\mathbf{G}}_{\mathscr{F}} = \mathbf{N}^+_{\mathscr{F}}\mathbf{A}^+_{\mathscr{F}}\mathbf{K}$ *with uniqueness of expression.*

Proof. The decomposition follows from $\widehat{\mathbf{G}}_{\mathscr{F}} = \mathbf{B}_{\mathscr{F}}\mathbf{K}$ and observing that:

$$\mathbf{B}_{\mathscr{F}} = \mathbf{N}^+_{\mathscr{F}}\mathbf{A}^+_{\mathscr{F}}\mathbf{A}^1_{\mathscr{F}} = \mathbf{N}^+_{\mathscr{F}}\mathbf{A}^+_{\mathscr{F}}(\mathbf{K} \cap \mathbf{B}_{\mathscr{F}}).$$

To prove the uniqueness suppose $g \in \widehat{\mathbf{G}}_{\mathscr{F}}$ has two decompositions:

$$g = nak = n'a'k'.$$

Then:

$$\begin{aligned} k'k^{-1} &= a^{-1}n^{-1}n'a' \\ &= a^{-1}n^{-1}n'\left(aa^{-1}\right)a' \\ &= \left(a^{-1}n^{-1}n'a\right)\left(a^{-1}a'\right) \\ &= n''a'' \end{aligned}$$

Therefore $k^{-1}k' \in \mathbf{N}_{\mathscr{F}}^{+}\mathbf{A}_{\mathscr{F}}^{+} \cap \mathbf{K} = \{1\}$. This implies: $k = k'$. But since we also have $\mathbf{N}_{\mathscr{F}}^{+} \cap \mathbf{A}_{\mathscr{F}}^{+} = \{1\}$ we can deduce that $a = a'$ and $n = n'$. □

Notation 7.30. For $g \in \widehat{\mathbf{G}}_{\mathscr{F}}$ we use $g_{\mathbf{N}}g_{\mathbf{A}}g_{\mathbf{K}}$ to denote its Iwasawa decomposition.

Remark 7.31. Let $\mathbf{K}'$ denote the group generated by $\mathbf{K}_1, \cdots, \mathbf{K}_n$, this is a subgroup of $\mathbf{K}$. However the proofs given in the section work with $\mathbf{K}'$ instead of $\mathbf{K}$ as well and one obtains $\widehat{\mathbf{G}}_{\mathscr{F}} = \mathbf{N}_{\mathscr{F}}^{+}\mathbf{A}_{\mathscr{F}}^{+}\mathbf{K}'$ with uniqueness of expression, which proves $\mathbf{K} = \mathbf{K}'$.

Remark 7.32. The article [8] provides a different proof of Iwasawa decomposition for split Kac-Moody groups.

7.4 The Orbit of the Highest Weight Vector

In this section we will look the orbit of $\mathbb{1}_{\Lambda} \in L(\Lambda)_{\mathscr{F}}$ under the action of $\widehat{\mathbf{G}}_{\mathscr{F}}$. But before doing so we need to introduce the minimal group:

For each $i \in I$ there exists a subgroup $\mathbf{G}_{i,\mathscr{F}} \subset \mathbf{L}_{i,\mathscr{F}}$ with Lie algebra:

$$\mathfrak{g}_{i,\mathscr{F}} = \mathfrak{n}_{-\alpha_i,\mathscr{F}} \oplus \mathscr{F}\alpha_i^{\vee} \oplus \mathfrak{n}_{\alpha_i,\mathscr{F}} \subset \mathfrak{l}_{i,\mathscr{F}}.$$

In fact we have an isomorphism $\Psi_i : \mathbf{SL}_{2,\mathscr{F}} \to \mathbf{G}_{i,\mathscr{F}}$.

Notation 7.33. For convenience we will use the following notation:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}_j := \Psi_j\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \in \mathbf{G}_{j,\mathscr{F}}.$$

Then one sees that:

$$\bar{r}_j = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}_j.$$

How the groups $\mathbf{G}_{1,\mathscr{F}} \cdots, \mathbf{G}_{n,\mathscr{F}}$ interact follows from the relations defining the Kac-Moody algebra $\mathfrak{g}_{\mathscr{F}}$ (see §2 in [14] for details):

Lemma 7.34.

(1) The torus, $\mathbf{A}_{\mathscr{F}}$, is the product of the tori in $\mathbf{G}_{1,\mathscr{F}} \cdots, \mathbf{G}_{n,\mathscr{F}}$.

(2) For $i, j \in I$ *and* $t \in \mathscr{F}^{\times}$*:*

$$\begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}_i \begin{pmatrix} a & b \\ c & d \end{pmatrix}_j \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}_i^{-1} = \begin{pmatrix} a & t^{\mathsf{A}_{ij}} b \\ t^{-\mathsf{A}_{ij}} c & d \end{pmatrix}_j .$$

(3) For $i, j \in I, i \neq j$ *and* $u, v \in \mathscr{F}$*:*

$$\begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}_i \begin{pmatrix} 1 & \\ v & 1 \end{pmatrix}_j = \begin{pmatrix} 1 & \\ v & 1 \end{pmatrix}_j \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}_i .$$

(4) For $i, j \in I$ *and* $t \in \mathscr{F}^{\times}$*:*

$$\overline{r}_j \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}_i \overline{r}_j^{-1} = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}_i \begin{pmatrix} t^{-\mathsf{A}_{ij}} & \\ & t^{\mathsf{A}_{ij}} \end{pmatrix}_j .$$

The group generated by $\mathbf{G}_{1,\mathscr{F}} \cdots, \mathbf{G}_{n,\mathscr{F}}$ was first introduced in [14]. We will refer to it as the *minimal group* and denote it by $\mathbf{G}_{\mathscr{F}}^{\mathrm{KP}}$. In [19] it is shown that the orbit of the highest weight vector in the projective space, $\mathbb{P}(L(\Lambda)_{\mathscr{F}})$, is given by quadratic equations. More precisely:

Theorem 7.35. *All* $v \in \mathbf{G}_{\mathscr{F}}^{\mathrm{KP}} \cdot \mathbb{1}_{\Lambda}$ *satisfy the following in* $L(\Lambda)_{\mathscr{F}} \otimes_{\mathscr{F}} L(\Lambda)_{\mathscr{F}}$*:*

$$(\Lambda|\Lambda)\, v \otimes v = \sum_{\alpha \in \Delta \cup \{0\}} x_{\alpha}^{(i)} \cdot v \otimes y_{\alpha}^{(i)} \cdot v,$$

where $\left\{x_{\alpha}^{(i)}\right\}$ *and* $\left\{y_{\alpha}^{(i)}\right\}$ *are dual basis for* $\mathfrak{g}_{\alpha,\mathscr{F}}$ *and* $\mathfrak{g}_{-\alpha,\mathscr{F}}$ *with respect to the standard invariant form on* $\mathfrak{g}_{\mathscr{F}}$*.*

The minimal group is a subgroup of $\widehat{\mathbf{G}}_{\mathscr{F}}$, moreover when $\mathscr{F} = \mathbb{R}, \mathbb{C}$ we have $\mathbf{A}_{\mathscr{F}}^{+}, \mathbf{K} \subset \mathbf{G}_{\mathscr{F}}^{\mathrm{KP}}$ therefore the only difference between $\mathbf{G}_{\mathscr{F}}^{\mathrm{KP}}$ and $\widehat{\mathbf{G}}_{\mathscr{F}}$ is the unipotent group, where the minimal group is missing all the positive imaginary roots. However due to the structure of the highest weight modules the action of the two unipotent groups and hence the two groups are the same.

Chapter 8

Reduction Theory

Notation 8.1. In this chapter we work only over the real numbers so we will drop all the subscripts.

Let $\mathbf{G}$ be a finite dimensional real reductive group, V a representation for $\mathbf{G}$ and a lattice $V_{\mathbb{Z}} \subset V$, reduction theory is concerned with vectors of minimal length in $g^{-1} \cdot V_{\mathbb{Z}}$ as g varies in $\mathbf{G}$. However using such an approach in the symmetrizable indefinite case immediately runs into difficulties. First since V is infinite dimensional there might not be a positive lower bound for the vectors in $g^{-1} \cdot V_{\mathbb{Z}}$, in other words one might have an infinite sequence of vectors in $g^{-1} \cdot V_{\mathbb{Z}}$ whose length approaches zero (an explicit example is given in Remark 8.3 below). So then the question becomes what is the right subset to consider, in other words before proving anything we have to find the right question first.

In §1 we introduce four different subsets that have the necessary conditions to possess a reduction theory.

In §2 we define $\widehat{\mathbf{G}}^{\mathrm{AR}}$ which is the most natural choice from a geometric point of view and contains a large subset of $\widehat{\mathbf{G}}$. However at present we don't have a reduction theory for $\widehat{\mathbf{G}}^{\mathrm{AR}}$ since it is not clear whether points on $\widehat{\mathbf{G}}^{\mathrm{AR}}$ do have minima on their $\mathbf{\Gamma}$-orbits.

In §3 we define another subset, $\widehat{\mathbf{G}}^{\mathrm{MO}}$, the elements of which all have minima on their $\mathbf{\Gamma}$-orbits. Then Borel's proof of reduction theory from the finite dimensional case ([3] 16.6) works without any changes.

In §4 we show $\widehat{\mathbf{G}}^{\mathrm{MO}}$ contains a large subset of the group: $\mathbf{N}^{+}\,\mathrm{EXP}(\mathrm{INT}(\mathbb{T}))\mathbf{K}$ (Theorem 8.23). The proof of this theorem is done in two parts: in §4.1 we give a spectral characterization of $\mathbf{N}^{+}\,\mathrm{EXP}(\mathrm{INT}(\mathbb{T}))\mathbf{K}$. Then in §4.2, using this spectral characterization, we show that $\mathbf{N}^{+}\,\mathrm{EXP}(\mathrm{INT}(\mathbb{T}))\mathbf{K}$ is indeed a subset of $\widehat{\mathbf{G}}^{\mathrm{MO}}$. The main idea of defining elements with *decay* (denoted by $\widehat{\mathbf{G}}^{\flat}$) was inspired by the proof of Lemma 17.15 in [12].

In §5 using direct calculations in the Kac-Moody group corresponding to a rank 2 GCM we show that $\widehat{\mathbf{G}}^\flat$ is not $\mathbf{\Gamma}$-invariant.

Finally in §6 we list a number of open problems.

8.1 The Four Subsets

Notation 8.2. Let ρ be the weight defined by $\langle \rho, \alpha_i^\vee \rangle = 1$ for all $i \in I$. We fix $L(\rho)$ as a representation of $\widehat{\mathbf{G}}$ where it acts from the right.

Since $L(\rho)$ is infinite dimensional having a positive lower bound does not ensure the existence of a minimum, moreover since $\mathbf{\Gamma} \cdot \mathbb{1}_\rho \subsetneq L(\rho)_\mathbb{Z}$ (see §7.4) one can consider four different $\mathbf{\Gamma}$-invariant subsets of $\widehat{\mathbf{G}}$ as candidates to work with:

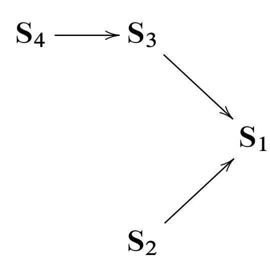
(1) $\inf_{\gamma \in \mathbf{\Gamma}} \left\| g^{-1}\gamma^{-1} \cdot \mathbb{1}_\rho \right\| > 0$.

(2) $\left\| g^{-1}\gamma^{-1} \cdot \mathbb{1}_\rho \right\|$ achieves a minimum as γ varies in $\mathbf{\Gamma}$.

(3) $\inf_{v \in L(\rho)_\mathbb{Z}} \left\| g^{-1} \cdot v \right\| > 0$.

(4) $\left\| g^{-1} \cdot v \right\|$ achieves a minimum as v varies in $L(\rho)_\mathbb{Z}$.

Let $\mathbf{S}_1, \cdots, \mathbf{S}_4$ be the corresponding subsets in $\widehat{\mathbf{G}}$. The relationship between these sets are summarized in the following diagram where arrows indicate inclusion:[2]



[2]in the following simplified form it becomes easier to order the sets: let $f : X \to \mathbb{R}_{>0}$ be a function and let $Y \subsetneq X$. Now compare the following statements: (1) f has a positive infimum on Y, (2) f has a minimum on Y, (3) f has a positive infimum on X and (4) f has a minimum on X.

8.2 The Arithmetic Set

In this section we consider the set $\mathbf{S}_3$. One can formulate the definition of $\mathbf{S}_3$ in an equivalent but more intuitive way: any element of $\widehat{\mathbf{G}}$ defines a new metric on $L(\rho)$:

$$\|v\|_g := \left\|g^{-1} \cdot v\right\|.$$

We say $g \in \widehat{\mathbf{G}}$ is an *arithmetic point* if there is positive lower bound for the length of the elements of $L(\rho)_{\mathbb{Z}}$ under the metric $\|\cdot\|_g$. The set of all arithmetic points is called the *arithmetic set*. From now we will use $\widehat{\mathbf{G}}^{\mathrm{AR}}$ to denote the arithmetic subset instead of $\mathbf{S}_3$.

Remark 8.3. This definition is based on the finite dimensional theory, in fact in that case $\widehat{\mathbf{G}}^{\mathrm{AR}}$ is the entire group (see [3] 16.2). This is not the case here, since the torus, $\mathbf{A}^+$, has non-arithmetic points: let $a \in \mathbf{A}^+$ be such that $a^{\alpha_i} < 1$ for all $i \in I$. Consider the sequence $\left\{\mathbb{1}_{\lambda_k}\right\}_{k=1}^{\infty}$, where $\mathrm{dp}(\lambda_k) \to \infty$ as $k \to \infty$, then:

$$\left\|a^{-1} \cdot \mathbb{1}_{\lambda_k}\right\| = a^{-\lambda_k} = a^{-\rho} a^{\xi_k} = a^{-\rho}\left(\left(a^{\alpha_1}\right)^{p_{1k}} \cdots \left(a^{\alpha_n}\right)^{p_{nk}}\right).$$

Since $\mathrm{ht}(\xi_k) \to \infty$ as $k \to \infty$ and $a^{\alpha_i} < 1$ for all $i \in I$ we see that:

$$\lim_{k \to \infty} \left\|a^{-1} \cdot \mathbb{1}_{\lambda_k}\right\| = 0.$$

Lemma 8.4. *Let $g \in \widehat{\mathbf{G}}$ and $v \in L(\rho)$ with the weight space decomposition:*

$$v = c_1 \mathbb{1}_{\lambda_1} + \cdots + c_k \mathbb{1}_{\lambda_k}, \quad c_j \in \mathbb{R}.$$

Then: $\left\|g^{-1} \cdot v\right\| \geq g_{\mathbf{A}}^{-\lambda_{j_0}} \left|c_{j_0}\right|$, where $\lambda_{j_0} \in \{\lambda_1, \cdots, \lambda_k\}$ is of maximal depth in that set.

Proof.

$$\begin{aligned}
\left\|g^{-1} \cdot v\right\| &= \left\|g_{\mathbf{A}}^{-1} g_{\mathbf{N}}^{-1} \cdot v\right\| && \mathbf{K} \text{ is unitary} \\
&\geq \left\|g_{\mathbf{A}}^{-1} \cdot \left(c_{j_0} \mathbb{1}_{\lambda_{j_0}}\right)\right\| && \text{choice of } \lambda_{j_0} \\
&= \left\|\left(g_{\mathbf{A}}^{-\lambda_{j_0}} c_{j_0}\right) \mathbb{1}_{\lambda_{j_0}}\right\|
\end{aligned}$$

$$= g_{\mathbf{A}}^{-\lambda_{j_0}} \left|c_{j_0}\right| \left\|\mathbb{1}_{\lambda_{j_0}}\right\| \qquad g_{\mathbf{A}}^{-\lambda_{j_0}} > 0$$
$$\geq g_{\mathbf{A}}^{-\lambda_{j_0}} \left|c_{j_0}\right| \qquad \left\|\mathbb{1}_{\lambda_{j_0}}\right\| = 1 \qquad \square$$

The following Lemma shows why the set of arithmetic points of the Torus, which we will denote by $\mathbf{A}^{\mathrm{AR}}$, is important:

Lemma 8.5. $\mathbf{N}^{+}\mathbf{A}^{\mathrm{AR}}\mathbf{K} \subset \widehat{\mathbf{G}}^{\mathrm{AR}}$.

Proof. Let $g \in \widehat{\mathbf{G}}$ be such that $g_{\mathbf{A}} \in \mathbf{A}^{\mathrm{AR}}$ we aim to show that $g \in \widehat{\mathbf{G}}^{\mathrm{AR}}$. Since the weight vectors $\left\{\mathbb{1}_\mu : \mu \in \mathcal{P}_\rho\right\}$ are a basis for $L(\rho)_{\mathbb{Z}}$ in order to show that there is a positive lower bound when g acts on $L(\rho)_{\mathbb{Z}}$ we only need to show g has a lower bound as it acts on these vectors. We have:

$$\left\|g^{-1} \cdot \mathbb{1}_\mu\right\| = \left\|g_{\mathbf{A}}^{-1} g_{\mathbf{N}}^{-1} \cdot \mathbb{1}_\mu\right\| = \left\|g_{\mathbf{A}}^{-1} \cdot \left(g_{\mathbf{N}}^{-1} \cdot \mathbb{1}_\mu\right)\right\|$$

Now let $v = g_{\mathbf{N}}^{-1} \cdot \mathbb{1}_\mu$, since elements of $\mathbf{N}^{+}$ are represented by unipotent upper triangular matrices in $\mathbf{GL}(L(\rho))$ the weight space decomposition of v can be written as follows:

$$v = \mathbb{1}_\mu + c_1 \mathbb{1}_{\lambda_1} + \cdots + c_k \mathbb{1}_{\lambda_k},$$

where μ has maximal depth among the weights $\{\mu, \lambda_1, \cdots, \lambda_k\}$. Using Lemma 8.4 with $g_{\mathbf{A}}$ and v yields:

$$\left\|g_{\mathbf{A}}^{-1} \cdot v\right\| \geq g_{\mathbf{A}}^{-\mu}.$$

In summary we have shown:

$$\left\|g^{-1} \cdot \mathbb{1}_\mu\right\| \geq \left\|g_{\mathbf{A}}^{-1} \cdot \mathbb{1}_\mu\right\| = g_{\mathbf{A}}^{-\mu},$$

However since $g_{\mathbf{A}} \in \mathbf{A}^{\mathrm{AR}}$ the right hand side has a positive lower bound. $\square$

Remark 8.6. A converse to Lemma 8.5 does not hold, in other words it is possible to have $g \in \widehat{\mathbf{G}}^{\mathrm{AR}}$ and $g_{\mathbf{A}} \notin \widehat{\mathbf{G}}^{\mathrm{AR}}$. First we need to do a calculation:

$$\begin{aligned} \left\|g^{-1} \cdot \mathbb{1}_\mu\right\|^2 &= \left\|g_{\mathbf{A}}^{-1} g_{\mathbf{N}}^{-1} \cdot \mathbb{1}_\mu\right\|^2 \\ &= \left\|g_{\mathbf{A}}^{-1} \cdot \mathbb{1}_\mu\right\|^2 + \left\|g_{\mathbf{A}}^{-1} \cdot v\right\|^2 \end{aligned}$$

$$= g_{\mathbf{A}}^{-2\mu} + \left\| g_{\mathbf{A}}^{-1} \cdot \left(\sum c_j \mathbb{1}_{\mu_j} \right) \right\|^2$$
$$= g_{\mathbf{A}}^{-2\mu} + \sum c_j^2 g_{\mathbf{A}}^{-2\mu_j}$$

$g_{\mathbf{A}} \notin \mathbf{A}^{\text{AR}}$ therefore: $g_{\mathbf{A}}^{-\mu} \to 0$ as $\text{dp}(\mu) \to \infty$. However there are at least two ways of avoiding a contradiction:

- there exists $M \in \mathbb{N}$ such that $n^{-1} \cdot \mathbb{1}_{\mu}$ has a weight component in $L(\rho)_{\mu_0}$ with $\text{dp}(\mu_0) < M$ for any weight vector $\mathbb{1}_{\mu} \in L(\rho)_{\mu}$.
- the matrix coefficients of $n^{-1} \in \mathbf{GL}(L(\rho))$ grow exponentially with depth.

Because of Lemma 8.5 next we will try to compute $\mathbf{A}^{\text{AR}}$.

Definition 8.7. We define the following subset of positive roots:

$$\Delta_{\rho}^{\text{re}} = \left\{ \xi \in \Delta_{+}^{\text{re}} : \ \rho - \xi \in \mathscr{P}_{\rho} \right\}.$$

$\Delta_{\rho}^{\text{im}}$ and Δ_{ρ} are defined similarly.

Lemma 8.8. $\Delta_{\rho} = \Delta_{+}$.

Proof. We show this by proving $\Delta_{\rho}^{\text{re}} = \Delta_{+}^{\text{re}}$ and $\Delta_{\rho}^{\text{im}} = \Delta_{+}^{\text{im}}$.

REAL ROOTS: Let $\xi = w(\alpha_i)$ be a positive real root, we define its corresponding coroot as $\xi^{\vee} = w(\alpha_i^{\vee})$. Now using Proposition 11.1 in [15] we have:

$$\Delta_{+}^{\text{re}} \backslash \Delta_{\rho}^{\text{re}} = \left\{ \xi \in \Delta_{+}^{\text{re}} : \ \langle \rho, \xi^{\vee} \rangle = 0 \right\},$$

However we see that the set is in fact empty since ρ is dominant and integral and since ξ is a positive root, $\xi^{\vee}$ will be a positive coroot.

IMAGINARY ROOTS: Using Corollary 11.9 [15] $\Delta_{\rho}^{\text{im}} = \Delta_{+}^{\text{im}}$ is equivalent to $(\rho|\xi) \neq 0$ for all $\xi = \sum p_i \alpha_i \in \Delta_{+}$, however (§5.2 [15]):

$$(\rho|\xi) = \sum_{i=1}^{n} p_i \, (\rho|\alpha_i) = \sum_{i=1}^{n} p_i \left(\tfrac{1}{2} |\alpha_i|^2 \left\langle \rho, \alpha_i^{\vee} \right\rangle \right) = \tfrac{1}{2} \sum_{i=1}^{n} p_i |\alpha_i|^2 > 0. \qquad \square$$

Lemma 8.9. $\text{EXP}(\mathbb{T}) \subseteq \mathbf{A}^{\text{AR}} \subseteq \text{EXP}(\text{CL}(\mathbb{T}))$.

Proof. Suppose $\textsc{Exp}(z) = a \in \mathbf{A}^{\mathrm{AR}}$, using Lemma 8.8 we have:

$$0 < \inf_{\mu \in \mathscr{P}_\rho} a^{-\mu} = \inf_{\xi \in \Delta_\rho} a^{-\rho} a^{\xi} = \inf_{\xi \in \Delta_\rho} a^{\xi} = \inf_{\xi \in \Delta_\rho} e^{\langle \xi, z \rangle} = \inf_{\xi \in \Delta_+} e^{\langle \xi, z \rangle}$$

Which is equivalent to:

$$\exists L \in \mathbb{R} : \ \forall \xi \in \Delta_+ : \ \langle \xi, z \rangle \geq L.$$

Now recall that if $\xi \in \Delta_+^{\mathrm{im}}$ then $n\xi \in \Delta_+^{\mathrm{im}}$ for all $n \in \mathbb{N}$. Therefore we can conclude:

$$\forall \xi \in \Delta_+^{\mathrm{im}} : \ \langle \xi, z \rangle \geq 0.$$

by Proposition 2.48 this means $z \in \textsc{Cl}(\mathbb{T})$.

Conversely let $a = \textsc{Exp}(z) \in \textsc{Exp}(\mathbb{T})$. Since the weight vectors generate the Chevalley lattice so it is enough to show:

$$0 < \inf_{\mu \in \mathscr{P}_\rho} a^{-\mu}.$$

However based on Bardy there exist $w_0 \in \mathbf{W}, z_0 \in \mathbb{D}$ such that $z = w_0(z_0)$, therefore we have:

$$a^{-\mu} = e^{\langle -\mu, z \rangle} = e^{\langle -\mu, w_0(z_0) \rangle} = e^{\left\langle w_0^{-1}(-\mu), z_0 \right\rangle} = e^{-\left\langle w_0^{-1}(\rho), z_0 \right\rangle} e^{\left\langle w_0^{-1}(\xi), z_0 \right\rangle},$$

where $\xi \in \Delta_+$. Now since $z_0 \in \mathbb{D}$ we have $\langle \alpha_i, z_0 \rangle \geq 0$ for all $i \in I$. Therefore as long as $w_0^{-1}(\xi) \in \Delta_+$ there is a lower bound for $a^{-\mu}$. Now the result follows from

$$\left| w_0^{-1} (\Delta_+) \cap \Delta_- \right| < \infty. \qquad \square$$

Corollary 8.10. $\mathbf{N}^+ \textsc{Exp}(\mathbb{T}) \mathbf{K} \subseteq \widehat{\mathbf{G}}^{\mathrm{AR}}$.

Proof. This follows from Lemma 8.5 and Lemma 8.9. $\square$

8.3 Points with Minima

The problem with the arithmetic set is that in order to prove the reduction theorem one needs to first show the *existence of minima*. That is, for a given point g the function Φ achieves a minimum on the set $\mathbf{\Gamma} g$. However since $L(\rho)$ is infinite dimensional it is not obvious that an arithmetic point would achieve its minimum, we might have a situation where the infimum is not in fact a minimum. Therefore we consider the subset $\mathbf{S}_2$ instead.

Definition 8.11. First we define a function $\Phi : \widehat{\mathbf{G}} \to \mathbb{R}_{>0}$ given by: $\Phi(g) = \left\| g^{-1} \cdot \mathbb{1}_\rho \right\|$. Then $\mathbf{S}_2$ is the set of all elements $g \in \widehat{\mathbf{G}}$, such that Φ achieves a positive minimum when considered as a function on the orbit: $\mathbf{\Gamma} g$. More formally:

$$\mathbf{S}_2 = \left\{ g \in \widehat{\mathbf{G}} : \exists \gamma_0 \in \mathbf{\Gamma}, \forall \gamma \in \mathbf{\Gamma} : \Phi(\gamma_0 g) \leq \Phi(\gamma g) \right\}.$$

From now on we will use $\widehat{\mathbf{G}}^{\mathrm{MO}}$ to denote $\mathbf{S}_2$.

Remark 8.12. By design, $\widehat{\mathbf{G}}^{\mathrm{MO}}$ is a $\mathbf{\Gamma}$-invariant subset of $\widehat{\mathbf{G}}$. In fact if one defines:

$$\mathbf{\Omega} = \left\{ g \in \widehat{\mathbf{G}} : \forall \gamma \in \mathbf{\Gamma} : \Phi(g) \leq \Phi(\gamma g) \right\},$$

then: $\widehat{\mathbf{G}}^{\mathrm{MO}} = \mathbf{\Gamma} \mathbf{\Omega}$.

Remark 8.13. It is not obvious from the definition that $\widehat{\mathbf{G}}^{\mathrm{MO}}$ is in fact non-empty, below we will show that it does contain a large subset of $\widehat{\mathbf{G}}$ (see 8.23).

Definition 8.14. For $\sigma > 0$ define:

$$\mathbf{A}^\sigma = \left\{ a \in \mathbf{A}^+ : \forall i \in I : a^{\alpha_i} \geq \sigma \right\}.$$

We also use $\widehat{\mathbf{G}}^\sigma$ to denote the set $\mathbf{N}^+ \mathbf{A}^\sigma \mathbf{K}$.

Reduction Theorem. *There exists a real constant $\sigma > 0$ such that for any $g \in \widehat{\mathbf{G}}^{\mathrm{MO}}$ there exists $\gamma \in \mathbf{\Gamma}$ with $\gamma g \in \widehat{\mathbf{G}}^\sigma$.*

8.3.1 Proof of Reduction Theorem

Remark 8.15. Since $\widehat{\mathbf{G}}^{\mathrm{MO}} = \mathbf{\Gamma} \mathbf{\Omega}$, it is enough to show that $\mathbf{\Omega} \subset \widehat{\mathbf{G}}^\sigma$ for some real constant $\sigma > 0$.

Remark 8.16. For each $i \in I$ we can write $\widehat{\mathbf{G}} = \mathbf{N}_i \mathbf{A}_i \mathbf{L}_i \mathbf{K}$, where $\mathbf{A}_i = \mathrm{KER}(\alpha_i) \subset \mathbf{A}$. In particular any $g \in \widehat{\mathbf{G}}$ has a decomposition:

$$g = n_g a_g l_g k_g,$$

where $n_g \in \mathbf{N}_i, a_g \in \mathbf{A}_i, l_g \in \mathbf{L}_i$ and $k_g \in \mathbf{K}$.

Lemma 8.17. *Φ is left-invariant under $\mathbf{N}^+$ and right-invariant under $\mathbf{K}$, in fact we have: $\Phi(g) = g_{\mathbf{A}}^{-\rho}$.*

Proof. Using Iwasawa decomposition, $\mathbf{N}^+ \cdot \mathbb{1}_\rho = \mathbb{1}_\rho$ and the fact that $\mathbf{K}$ is unitary with respect to $\|\cdot\|$ we see that $\Phi(g) = \Phi(\mathbf{N}^+ g) = \Phi(g\mathbf{K})$ and the first assertion is proven. Now we can write:

$$\Phi(g) = \Phi(g_{\mathbf{A}}) = \left\| g_{\mathbf{A}}^{-1} \cdot \mathbb{1}_\rho \right\| = \left\| g_{\mathbf{A}}^{-\rho} \mathbb{1}_\rho \right\| = g_{\mathbf{A}}^{-\rho} \left\| \mathbb{1}_\rho \right\|.$$

Now recall that $\|\cdot\|$ is normalized such that $\left\| \mathbb{1}_\rho \right\| = 1$. □

Lemma 8.18. $\Phi(g) = \Phi(a_g)\Phi(l_g)$.

Proof. Using the decomposition from Remark 8.16 and then applying Lemma 8.17 we see that $\Phi(g) = \Phi(a_g l_g)$, now we use the definition of Φ:

$$\Phi(a_g l_g) = \left\| l_g^{-1} a_g^{-1} \cdot \mathbb{1}_\rho \right\| = a_g^{-\rho} \left\| l_g^{-1} \cdot \mathbb{1}_\rho \right\| = \Phi(a_g)\Phi(l_g). \qquad \square$$

Lemma 8.19. *If $\gamma \in \mathbf{\Gamma} \cap \mathbf{L}_i$ then: $\Phi(\gamma g) = \Phi(a_g)\Phi(\gamma l_g)$.*

Proof. $\gamma \in \mathbf{L}_i$ therefore it normalizes $\mathbf{N}_i$ and commutes with $\mathbf{A}_i$:

$$\Phi(\gamma g) = \Phi\big(\gamma n_g a_g l_g k_g\big) = \Phi\big(n'_g a_g \gamma l_g k_g\big) = \Phi(a_g \gamma l_g),$$

now we use the previous Lemma. □

Lemma 8.20. *If $g \in \mathbf{\Omega}$ then: $\Phi(l_g) \le \inf_{\gamma \in \mathbf{\Gamma} \cap \mathbf{L}_i} \Phi(\gamma l_g)$.*

Proof.

$$\begin{aligned}
\Phi(a_g)\Phi(l_g) &= \Phi(g) && \text{Lemma 8.18} \\
&\leq \inf_{\gamma \in \mathbf{\Gamma}} \Phi(\gamma g) && g \in \mathbf{\Omega} \\
&\leq \inf_{\gamma \in \mathbf{\Gamma} \cap \mathbf{L}_i} \Phi(\gamma g) && \mathbf{\Gamma} \cap \mathbf{L}_i \subset \mathbf{\Gamma} \\
&= \inf_{\gamma \in \mathbf{\Gamma} \cap \mathbf{L}_i} \Phi(a_g)\Phi(\gamma l_g) && \text{Lemma 8.19} \\
&= \Phi(a_g) \inf_{\gamma \in \mathbf{\Gamma} \cap \mathbf{L}_i} \Phi(\gamma l_g)
\end{aligned}$$

Cancelling $\Phi(a_g)$ from both sides gives us the result. □

Lemma 8.21. *There is a real constant, $\varepsilon > 0$, such that:* $\inf_{\gamma \in \mathbf{\Gamma} \cap \mathbf{L}_i} \Phi(\gamma l_g) \leq \varepsilon$.

Proof. This follows from $\mathbf{L}_i$ being a finite dimensional reductive group of semi-simple rank 1. □

Lemma 8.22. *If $g \in \mathbf{\Omega}$ then for all $i \in I$ we have:*

$$l_{g,\mathbf{A}}^{\alpha_i} \geq \varepsilon^{-2}.$$

Proof. Based on Lemma 8.21 and Lemma 8.20 we have:

$$\varepsilon \geq \inf_{\gamma \in \mathbf{\Gamma} \cap \mathbf{L}_i} \Phi(\gamma l_g) \geq \Phi(l_g) = l_{g,\mathbf{A}}^{-\rho}.$$

Now note that $\rho|_{\mathbf{L}_i} = \frac{1}{2}\alpha_i$. □

Finally Lemma 8.22 combined with the following observation completes the proof:

$$g_{\mathbf{A}}^{\alpha_i} = \left(l_{g,\mathbf{A}} a_g\right)^{\alpha_i} = l_{g,\mathbf{A}}^{\alpha_i}.$$

8.4 A Subset of $\widehat{\mathbf{G}}^{\mathrm{MO}}$

In this section we prove the following:

Theorem 8.23. $\mathbf{N}^+ \,\mathrm{EXP}(\mathrm{INT}(\mathbb{T}))\mathbf{K} \subset \widehat{\mathbf{G}}^{\mathrm{MO}}$.

8.4.1 A Spectral Characterization of $\mathrm{INT}(\mathbb{T})$

Definition 8.24. For $a \in \mathbf{A}^+$ and $C > 0$ define:

$$\mathscr{P}_\rho(a, C) := \left\{\mu \in \mathscr{P}_\rho : \; a^\mu \geq C\right\}.$$

Definition 8.25. We say $a \in \mathbf{A}^+$ *decays in* $L(\rho)$ if $\mathscr{P}_\rho(a, C)$ is finite for all $C > 0$. Notice that if a decays in $L(\rho)$ then $a^\mu \to 0$ as $\mathrm{dp}(\mu) \to \infty$ for $\mu \in \mathscr{P}_\rho$. The subset of $\mathbf{A}^+$ consisting of all such elements is denoted by $\mathbf{A}^\flat$.

Lemma 8.26. $\mathbf{A}^\flat \subset \mathrm{EXP}(\mathrm{INT}(\mathbb{T}))$

Proof. Let $\mathrm{EXP}(z) = a \in \mathbf{A}^\flat$, then one has the following:

$$\begin{aligned}
\forall C > 0 : \; \left|\mathscr{P}_\rho(a, C)\right| < \infty &\leftrightarrow \forall C > 0 : \; \left|\left\{\lambda \in \mathscr{P}_\rho : \; a^\lambda \geq C\right\}\right| < \infty \\
&\leftrightarrow \forall C > 0 : \; \left|\left\{\xi \in \Delta_\rho : \; a^{\rho-\xi} \geq C\right\}\right| < \infty \\
&\leftrightarrow \forall C > 0 : \; \left|\left\{\xi \in \Delta_\rho : \; a^{\xi} \leq a^\rho C^{-1}\right\}\right| < \infty \\
&\leftrightarrow \forall C' > 0 : \; \left|\left\{\xi \in \Delta_\rho : \; a^{\xi} \leq C'\right\}\right| < \infty \\
&\leftrightarrow \forall C' > 0 : \; \left|\left\{\xi \in \Delta_\rho : \; e^{\langle \xi, z\rangle} \leq C'\right\}\right| < \infty \\
&\leftrightarrow \forall K \in \mathbb{R} : \; \left|\left\{\xi \in \Delta_\rho : \; \langle \xi, z\rangle \leq K\right\}\right| < \infty
\end{aligned}$$

Taking $K = 0$ and using Lemma 8.8 implies $z \in \mathrm{INT}(\mathbb{T})$. □

Lemma 8.27. $\mathbf{A}^\flat$ *is invariant under the action of* $\mathbf{W}$.

Proof. Let $\mathrm{EXP}(z) = a \in \mathbf{A}^\flat$ and take $C > 0$ and $w \in \mathbf{W}$ to be arbitrary:

$$\begin{aligned}
\mathscr{P}_\rho\left(waw^{-1}, C\right) &= \left\{\lambda \in \mathscr{P}_\rho : \; \left(waw^{-1}\right)^\lambda \geq C\right\} \\
&= \left\{\lambda \in \mathscr{P}_\rho : \; \mathrm{EXP}\left(w(z)\right)^\lambda \geq C\right\} \\
&= \left\{\lambda \in \mathscr{P}_\rho : \; e^{\langle \lambda, w(z)\rangle} \geq C\right\} \\
&= \left\{\lambda \in \mathscr{P}_\rho : \; e^{\left\langle w^{-1}(\lambda), z\right\rangle} \geq C\right\} \\
&= \left\{\lambda \in \mathscr{P}_\rho : \; a^{w^{-1}(\lambda)} \geq C\right\}
\end{aligned}$$

And this last set is finite because $a \in \mathbf{A}^\flat$ and $\mathbf{W}$ permutes the set of weights. $\square$

Lemma 8.28. $\mathrm{E{\scriptsize XP}}\,(\mathbb{D}_{\mathrm{fin}}) \subset \mathbf{A}^\flat$.

Proof. Let $z \in \mathbb{D}_{\mathrm{fin}}$ then by definition $z \in \mathbb{F}_J$, where $J \subset I$ is an arbitrary subset of finite type. Take M_z to be the minimum of the set $\{\langle \alpha_i, z\rangle : i \notin J\}$, note that M_z is a positive real number. For any $\xi \in \Delta_+^{\mathrm{re}}$ one sees that $\langle \xi, z\rangle \geq \mathrm{ht}_J(\xi) M_z$, where ht_J is defined as follows:

$$\mathrm{ht}_J \left(\sum\nolimits_{i \in I} p_i \alpha_i\right) = \sum\nolimits_{i \notin J} p_i.$$

Therefore for $\lambda \in \mathscr{P}_\rho$ one has:

$$\langle \lambda, z\rangle = \langle \rho - \xi, z\rangle = \langle \rho, z\rangle - \langle \xi, z\rangle \leq \langle \rho, z\rangle - \mathrm{ht}_J(\xi) M_z.$$

And so for $a = \mathrm{E{\scriptsize XP}}(z)$ we can write:

$$a^\lambda = e^{\langle \lambda, z\rangle} \leq e^{\langle \rho, z\rangle} e^{-\mathrm{ht}_J(\xi) M_z}.$$

Let $C > 0$ be arbitrary, then:

$$C \leq a^\lambda \leq e^{\langle \rho, z\rangle} e^{-\mathrm{ht}_J(\xi) M_z}.$$

which implies $\mathrm{ht}_J(\xi)$ has to bounded. Therefore:

$$\mathscr{P}_\rho(a, C) = \{\xi \in \Delta_+ : \mathrm{ht}_J(\xi) \text{ is bounded}\}$$

So we can partition $\mathscr{P}_\rho(a, C)$ into finitely many subsets in each of which $p_j, j \notin J$ are fixed while the rest of coefficients vary. However J is of finite type, hence each partition itself has to be finite, hence $a \in \mathbf{A}^\flat$. $\square$

Proposition 8.29. $\mathbf{A}^\flat = \mathrm{E{\scriptsize XP}}(\mathrm{I{\scriptsize NT}}(\mathbb{T}))$.

Proof. Using Lemma 8.27 and 8.28 we have: $\mathrm{E{\scriptsize XP}}(\mathrm{I{\scriptsize NT}}(\mathbb{T})) = \mathrm{E{\scriptsize XP}}(\mathbf{W} \cdot \mathbb{D}_{\mathrm{fin}}) \subset \mathbf{A}^\flat$. This combined with Lemma 8.26 shows $\mathrm{E{\scriptsize XP}}(\mathrm{I{\scriptsize NT}}(\mathbb{T})) = \mathbf{A}^\flat$. $\square$

8.4.2 Decay and Minima

Definition 8.30. Set $\widehat{\mathbf{G}}^\flat := \mathbf{N}^+\mathbf{A}^\flat\mathbf{K}$, then the statement of Theorem 8.23 can be rewritten as $\widehat{\mathbf{G}}^\flat \subset \widehat{\mathbf{G}}^{\mathrm{MO}}$. Note that this is a strict inclusion since $1 \in \widehat{\mathbf{G}}^{\mathrm{MO}}$ but $1 \notin \widehat{\mathbf{G}}^\flat$.

Definition 8.31. For $a \in \mathbf{A}^+$ and $C > 0$, let $\overline{\mathscr{P}_\rho(a, C)}$ be a subset of $\mathscr{P}_\rho$ defined as follows: if $\lambda \in \mathscr{P}_\rho(a, C)$ then λ and all weights in $\mathscr{P}_\rho$ of lower depth belong to $\overline{\mathscr{P}_\rho(a, C)}$.

Definition 8.32. For $a \in \mathbf{A}^+$ and $C > 0$ define:

$$L(\rho; a, C) := \bigoplus_{\mu \in \overline{\mathscr{P}_\rho(a, C)}} L(\rho)_\mu,$$

$$\boldsymbol{\Gamma}(a, C) := \left\{\gamma \in \boldsymbol{\Gamma} : \gamma^{-1} \cdot \mathbb{1}_\rho \in L(\rho; a, C)\right\}.$$

Remark 8.33. $L(\rho; a, C)$ is a $\mathbf{N}^+\mathbf{A}^+$-stable subspace of $L(\rho)$ for all $a \in \mathbf{A}^+$ and $C > 0$.

Lemma 8.34. *If $a \in \mathbf{A}^\flat$ then for all $C > 0$, $L(\rho; a, C)$ is finite dimensional.*

Proof. $a \in \mathbf{A}^\flat$ so $\mathscr{P}_\rho(a, C)$ is finite for all $C > 0$. From the structure of the highest weight module it follows that $\overline{\mathscr{P}_\rho(a, C)}$ is finite for all $C > 0$ as well. Since the weight spaces of $L(\rho)$ are finite dimensional, $L(\rho; a, C)$ is finite dimensional for all $C > 0$. □

Lemma 8.35. *If $g \in \widehat{\mathbf{G}}^\flat$, for all $C > 0, \Phi$ achieves a minimum on the set:* $\boldsymbol{\Gamma}(g_{\mathbf{A}}, C)g$.

Proof. Let $g \in \widehat{\mathbf{G}}^\flat$, since Φ is right invariant under $\mathbf{K}$ we may assume that $g \in \mathbf{N}^+\mathbf{A}^+$. Take $C > 0$ to be arbitrary and consider the following set:

$$S = \left\{\gamma^{-1} \cdot \mathbb{1}_\rho : \gamma \in \boldsymbol{\Gamma}(g_{\mathbf{A}}, C)\right\} \subset L(\rho)_{\mathbb{Z}} \cap L(\rho; g_{\mathbf{A}}, C).$$

To prove the Lemma we need to show that $g^{-1}(S)$ has an element of minimal length. However $L(\rho; g_{\mathbf{A}}, C)$ is $\mathbf{N}^+\mathbf{A}^+$-stable and therefore $g^{-1}(S)$ still lies in

$L(\rho;\, g_{\mathbf{A}},\, C)$. Since $g \in \widehat{\mathbf{G}}^{\flat}$, by Lemma 8.34 $L(\rho;\, g_{\mathbf{A}},\, C)$ is finite dimensional. Therefore $g^{-1}(S)$, as a discrete subset of the finite dimensional subspace, has an element of minimal length. □

Lemma 8.36. *Let $g \in \widehat{\mathbf{G}}, v \in L(\rho)_{\mathbb{Z}}$ and $C > 0$. If $v \notin L(\rho;\, g_{\mathbf{A}},\, C)$ then: $\left\|g^{-1} \cdot v\right\| > C^{-1}$.*

Proof. Since $v \notin L(\rho;\, g_{\mathbf{A}},\, C)$, based on the construction of $\overline{\mathscr{P}_\rho(g_{\mathbf{A}},\, C)}$ we see that among all the weights of maximal depth that appear in the weight decomposition of v, there is at least one weight, λ, such that $\lambda \notin \overline{\mathscr{P}_\rho(g_{\mathbf{A}},\, C)}$. Therefore $\lambda \notin \mathscr{P}_\rho(g_{\mathbf{A}},\, C)$ as well and we have:

$$g_{\mathbf{A}}^{\lambda} < C. \tag{8.37}$$

On the other hand Lemma 8.4 gives us:

$$\left\|g^{-1} \cdot v\right\| \geq g_{\mathbf{A}}^{-\lambda} \left|c_\lambda\right|.$$

Since $v \in L(\rho)_{\mathbb{Z}}$ we have: $|c_\lambda| \geq 1$, which gives us:

$$\left\|g^{-1} \cdot v\right\| \geq g_{\mathbf{A}}^{-\lambda}. \tag{8.38}$$

Combining (8.38) and (8.37) gives the desired result. □

Corollary 8.39. *Let $g \in \widehat{\mathbf{G}}, \gamma \in \mathbf{\Gamma}$ and $C > 0$. If $\gamma \notin \mathbf{\Gamma}(g_{\mathbf{A}},\, C)$ then: $\Phi(\gamma g) > C^{-1}$.*

Corollary 8.40. *Let $g \in \widehat{\mathbf{G}}, \gamma \in \mathbf{\Gamma}$. Then $\gamma \in \mathbf{\Gamma}\left(g_{\mathbf{A}},\, \Phi(\gamma g)^{-1}\right)$.*

Let $g \in \widehat{\mathbf{G}}^{\flat}$, by Lemma 8.35 Φ achieves its minimum in $\mathbf{\Gamma}(g_{\mathbf{A}}, C)$ for all $C > 0$. So pick an arbitrary element $\theta \in \mathbf{\Gamma}$ and let γ_0 denote the minimum in $\mathbf{\Gamma}\left(g_{\mathbf{A}},\, \Phi(\theta g)^{-1}\right)$. We claim γ_0 is the minimum on the entire orbit: $\mathbf{\Gamma} g$. Suppose $\gamma \in \mathbf{\Gamma}$ is arbitrary, there are two cases:

- $\gamma \in \mathbf{\Gamma}\left(g_{\mathbf{A}},\, \Phi(\theta g)^{-1}\right)$: then $\Phi(\gamma_0 g) \leq \Phi(\gamma g)$ follows from our choice of γ_0.

- $\gamma \notin \mathbf{\Gamma}\big(g_{\mathbf{A}},\, \Phi(\theta g)^{-1}\big)$: then $\Phi(\theta g) < \Phi(\gamma g)$ holds by Lemma 8.39. But from Corollary 8.40 we have $\theta \in \mathbf{\Gamma}\big(g_{\mathbf{A}},\, \Phi(\theta g)^{-1}\big)$ and therefore: $\Phi(\gamma_0 g) \leq \Phi(\theta g)$.

Thus we have a *global* minimum on $\mathbf{\Gamma} g$ and therefore $g \in \widehat{\mathbf{G}}^{\mathrm{MO}}$. This completes the proof of Theorem 8.23.

8.5 $\widehat{\mathbf{G}}^\flat$ is not $\mathbf{\Gamma}$-invariant

In this section we will show using indefinite GCMs of rank 2 that $\widehat{\mathbf{G}}^\flat$ is not in fact $\mathbf{\Gamma}$-invariant. But first we need to carry on a few calculations in $\mathbf{SL}_2$.

8.5.1 Iwasawa Decomposition in $\mathbf{SL}_2$

Definition 8.41. For $u \in \mathbb{R}$ and $t \in \mathbb{R}_{>0}$ define:

$$\chi_-(u) = \begin{pmatrix} 1 & \\ u & 1 \end{pmatrix}, \quad \eta(t) = \begin{pmatrix} t & \\ & t^{-1} \end{pmatrix}, \quad \chi_+(u) = \begin{pmatrix} 1 & u \\ & 1 \end{pmatrix}.$$

Lemma 8.42. *The Iwasawa decomposition of $\mathbf{SL}_2$ can be explicitly written as follows:*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi_+\Big(\frac{ac + bd}{\sigma^2}\Big)\eta(\sigma^{-1})\begin{pmatrix} d\sigma^{-1} & -c\sigma^{-1} \\ c\sigma^{-1} & d\sigma^{-1} \end{pmatrix},$$

where $\sigma = \sqrt{c^2 + d^2}$.

Lemma 8.43. *Let r be the non-trivial element of the Weyl group in $\mathbf{SL}_2$:*

$$r = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}.$$

Then its action on $\mathbf{SL}_2/\mathbf{SO}_2$ is given by:

$$r \cdot \chi_+(u)\eta(t) = \chi_+\Big(-\frac{u}{u^2 + t^4}\Big)\eta\Big(\frac{t}{\sqrt{u^2 + t^4}}\Big).$$

Proof. We use the Iwasawa decomposition:

$$\begin{aligned} r \cdot \chi_+(u)\eta(t) &= \begin{pmatrix} & t^{-1} \\ -t & -t^{-1}u \end{pmatrix} \\ &= \chi_+\Big(-\frac{u}{u^2+t^4}\Big)\eta\Big(\frac{t}{\sqrt{u^2+t^4}}\Big) \begin{pmatrix} -\frac{u}{\sqrt{u^2+t^4}} & \frac{t^2}{\sqrt{u^2+t^4}} \\ -\frac{t^2}{\sqrt{u^2+t^4}} & -\frac{u}{\sqrt{u^2+t^4}} \end{pmatrix} \end{aligned} \qquad \square$$

8.5.2 The Counterexample

Let $\widehat{\mathbf{G}}$ be the group corresponding to the rank 2 GCM

$$\mathsf{A} = \big(\mathsf{A}_{ij}\big) = \begin{pmatrix} 2 & -n \\ -m & 2 \end{pmatrix}, \qquad m, n \in \mathbb{N},\ n \geq m.$$

Let $na = \chi_1(u)\eta_1(t_1)\eta_2(t_2)$ and consider it as an element of $\widehat{\mathbf{G}}/\mathbf{K} = \mathbf{N}^+\mathbf{A}^+$. We will calculate $\overline{r}_1 \cdot na \in \widehat{\mathbf{G}}/\mathbf{K}$ to show that $\widehat{\mathbf{G}}^\flat$ is not $\boldsymbol{\Gamma}$-invariant. Since the minimal group, $\mathbf{G}^{\text{KP}}$ (see §7.4) is a subgroup of $\widehat{\mathbf{G}}$ and $\overline{r}_1 \cdot na \in \mathbf{G}^{\text{KP}}$ we will do the calculations in the minimal group where we have concrete generators and relations for the whole group. First we introduce the following notation to simplify our task:

$$\begin{aligned} \sigma &= \sqrt{u^2 + t_1^4}, \\ \psi &= \sqrt{u^2 + t_1^4 t_2^{-2m}}. \end{aligned}$$

Now using Lemmas 8.42, 8.43 and 7.34 we have:

$$\begin{aligned} \overline{r}_1 \cdot na &= \overline{r}_1 \chi_1(u)\eta_1(t_1)\eta_2(t_2) \\ &= \left\{ \chi_1\Big(-\frac{u}{\sigma^2}\Big)\eta_1\Big(\frac{t_1}{\sigma}\Big) \begin{pmatrix} -u\sigma^{-1} & t_1^2\sigma^{-1} \\ -t_1^2\sigma^{-1} & -u\sigma^{-1} \end{pmatrix}_1 \right\} \eta_2(t_2) \\ &= \chi_1\Big(-\frac{u}{\sigma^2}\Big)\eta_1\Big(\frac{t_1}{\sigma}\Big)\eta_2(t_2) \left\{ \eta_2(t_2)^{-1} \begin{pmatrix} -u\sigma^{-1} & t_1^2\sigma^{-1} \\ -t_1^2\sigma^{-1} & -u\sigma^{-1} \end{pmatrix}_1 \eta_2(t_2) \right\} \\ &= \chi_1\Big(-\frac{u}{\sigma^2}\Big)\eta_1\Big(\frac{t_1}{\sigma}\Big)\eta_2(t_2) \begin{pmatrix} -u\sigma^{-1} & t_2^m\big(t_1^2\sigma^{-1}\big) \\ t_2^{-m}\big(-t_1^2\sigma^{-1}\big) & -u\sigma^{-1} \end{pmatrix}_1 \end{aligned}$$

$$
\begin{aligned}
&= \chi_1\Big(-\frac{u}{\sigma^2}\Big)\eta_1\Big(\frac{t_1}{\sigma}\Big)\eta_2(t_2)\times \\
&\quad\times\left\{\chi_1\left(\frac{ut_1^2}{\psi^2}\left(t_2^{-m}-t_2^{m}\right)\right)\eta_1\left(\frac{\sigma}{\psi}\right)\begin{pmatrix} -u\psi^{-1} & t_1^2t_2^m\psi^{-1} \\ -t_1^2t_2^{-m}\psi^{-1} & -u\psi^{-1}\end{pmatrix}_1\right\} \\
&= \chi_1\Big(-\frac{u}{\sigma^2}\Big)\eta_1\Big(\frac{t_1}{\sigma}\Big)\eta_2(t_2)\chi_1\left(\frac{ut_1^2}{\psi^2}(t_2^{-m}-t_2^{m})\right)\eta_1\left(\frac{\sigma}{\psi}\right) \\
&= \chi_1\Big(-\frac{u}{\sigma^2}\Big)\eta_1\Big(\frac{t_1}{\sigma}\Big)\chi_1\left(t_2^{-m}\left[\frac{ut_1^2}{\psi^2}(t_2^{-m}-t_2^{m})\right]\right)\eta_2(t_2)\eta_1\left(\frac{\sigma}{\psi}\right) \\
&= \chi_1\Big(-\frac{u}{\sigma^2}\Big)\chi_1\left(\frac{t_1^2}{\sigma^2}\left[t_2^{-m}\frac{ut_1^2}{\psi^2}(t_2^{-m}-t_2^{m})\right]\right)\eta_1\Big(\frac{t_1}{\sigma}\Big)\eta_2(t_2)\eta_1\left(\frac{\sigma}{\psi}\right) \\
&= \chi_1\left(-\frac{u}{\psi^2}\right)\eta_1\left(\frac{t_1}{\psi}\right)\eta_2(t_2)
\end{aligned}
$$

Now in this calculation lets consider what has happened at the level of $\mathbf{A}^+$-component. The initial point $(t_1,\, t_2)$ has been transformed into:

$$\left(\frac{t_1}{\sqrt{u^2+t_1^4t_2^{-2m}}},\; t_2\right).$$

First note that if one takes $u=0$ then one recovers the simple action of the fundamental reflection on $\mathbf{A}^+$ but if $u\to\infty$ the first component approaches 0. Hence even if $a\in \mathrm{Exp}(\mathrm{Int}(\mathbb{T}))$ by taking u large enough in $\chi_1(u)\in \mathbf{N}_{\alpha_1}$ we can make sure that $r_1\cdot na \notin \mathbf{N}^+\,\mathrm{Exp}(\mathrm{Int}(\mathbb{T}))$. In other words $\widehat{\mathbf{G}}^\flat$ is not $\mathbf{\Gamma}$-invariant.

8.6 Open Problems

Question 8.44. *Find a relationship between* $\widehat{\mathbf{G}}^{\mathrm{AR}}$ *and* $\widehat{\mathbf{G}}^{\mathrm{MO}}$.

Question 8.45. *Compute* $\mathbf{A}^{\mathrm{AR}}$ *and* $\mathbf{A}^{\mathrm{MO}}$ *(points in* $\mathbf{A}^+$ *which also belong to* $\widehat{\mathbf{G}}^{\mathrm{MO}}$*).*

Question 8.46. *Compute the projection of* $\widehat{\mathbf{G}}^{\mathrm{AR}}$ *and* $\widehat{\mathbf{G}}^{\mathrm{MO}}$ *onto* $\mathbf{A}^+$ *(This is a far harder problem than computing* $\mathbf{A}^{\mathrm{AR}}$ *and* $\mathbf{A}^{\mathrm{MO}}$*).*

Question 8.47. *Is* $\mathbf{N}^+/(\mathbf{N}^+\cap\mathbf{\Gamma})$ *a projective limit of finite dimensional compact spaces and hence compact?*

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Index of Notation

Appendix A

Formulas in Associative Algebras

Let $\mathscr{F}$ be a field of characteristic zero. In this appendix we present a number of formulas that hold in any associative $\mathscr{F}$-algebra $\mathfrak{A}_{\mathscr{F}}$ with a unit.

Notation A.1. For $x \in \mathfrak{A}_{\mathscr{F}}$ and $n \in \mathbb{N}$ we set:

$$x^{[n]} = \frac{x^n}{n!},$$
$$\binom{x}{n} = \frac{x(x-1)\cdots(x-n+1)}{n!}.$$

Also we may denote $\mathrm{ad}(x)(y)$ by $[x, y]$ when the latter is more convenient.

Lemma A.2 ([4] page 178)**.** *If $x, y \in \mathfrak{A}_{\mathscr{F}}$ and $n \in \mathbb{N}$:*

$$\frac{1}{n!}\,\mathrm{ad}(x)^n(y) = \sum_{p+q=n} (-1)^q x^{[p]} y x^{[q]}$$

Lemma A.3 ([4] page 178)**.** *Suppose $z, x \in \mathfrak{A}_{\mathscr{F}}$ such that $[z, x] = cx$ for $c \in \mathscr{F}$. Then for all $n \in \mathbb{N}$, and all $P \in \mathscr{F}[X]$, we have:*

$$P(z)x^{[n]} = x^{[n]} P(z + nc).$$

Definition A.4. (x, y, z) with $x, y, z \in \mathfrak{A}_{\mathscr{F}}$ is called a *S-triple* if:

$$[x, y] = z, \qquad [z, x] = 2x, \qquad [z, y] = -2y.$$

Lemma A.5 ([4] page 70)**.** *If (x, y, z) is a S-triple in $\mathfrak{A}_{\mathscr{F}}$ then we have the follow-*

ing:

$$\begin{aligned}
\left[z, x^n\right] &= 2nx^n \\
\left[z, y^n\right] &= -2ny^n \\
\left[y, x^n\right] &= nx^{n-1}(-z-n+1) = n(-z+n-1)x^{n-1} \\
\left[x, y^n\right] &= ny^{n-1}(z-n+1) = n(z+n-1)y^{n-1}
\end{aligned}$$

Lemma A.6 ([24] page 9)**.** *If (x, y, z) is a S-triple in $\mathfrak{A}_{\mathscr{F}}$ for $p, q \in \mathbb{N}$, we have:*

$$x^{[p]}y^{[q]} = \sum_{r=0}^{\min(p,q)} y^{[q-r]} \binom{z-q-p+2r}{r} x^{[p-r]} \tag{A.7}$$

Appendix B

Lie Algebras

Let $\mathfrak{R}$ be a commutative ring with a unit element.

Definition B.1. An $\mathfrak{R}$-module $\mathfrak{L}_{\mathfrak{R}}$, is called an $\mathfrak{R}$-*algebra* if one has a $\mathfrak{R}$-homomorphism: $\boldsymbol{\mu} : \mathfrak{L}_{\mathfrak{R}} \otimes_{\mathfrak{R}} \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{L}_{\mathfrak{R}}$.

Definition B.2. Given an $\mathfrak{R}$-algebra $\mathfrak{L}_{\mathfrak{R}}$ we define its *opposite algebra*, $\mathfrak{L}_{\mathfrak{R}}^{\mathrm{op}}$ to be the identical to $\mathfrak{L}_{\mathfrak{R}}$ as an $\mathfrak{R}$-module but with $\boldsymbol{\mu}^{\mathrm{op}}(x \otimes y) = \boldsymbol{\mu}(y \otimes x)$.

Definition B.3. An $\mathfrak{R}$-linear map $D : \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{L}_{\mathfrak{R}}$ is called a *derivation* if it satisfies Leibniz's law: $D(xy) = D(x)y + xD(y)$.

Lemma B.4. *If D is a derivation and $x_1, x_2, \cdots, x_k \in \mathfrak{L}_{\mathfrak{R}}$ then:*

$$D(x_1 x_2 \cdots x_k) = \sum_{i=1}^{k} x_1 \cdots x_{i-1} D(x_i) x_{i+1} \cdots x_k.$$

Definition B.5. For $x \in \mathfrak{L}_{\mathfrak{R}}$ define $\mathrm{ad}(x) : \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{L}_{\mathfrak{R}}$ by $\mathrm{ad}(x)(y) = xy - yx$.

Lemma B.6. $\mathrm{ad}(x)$ *is a derivation for all $x \in \mathfrak{L}_{\mathfrak{R}}$.*

Definition B.7. A $\mathfrak{R}$-*Lie algebra* is an $\mathfrak{R}$-algebra with the following properties:

1) The map $\boldsymbol{\mu} : \mathfrak{L}_{\mathfrak{R}} \otimes_{\mathfrak{R}} \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{L}_{\mathfrak{R}}$ admits a factorization:

$$\mathfrak{L}_{\mathfrak{R}} \otimes_{\mathfrak{R}} \mathfrak{L}_{\mathfrak{R}} \to \bigwedge^2 \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{L}_{\mathfrak{R}},$$

lets denote the image of $x \otimes y$ under this map by $[x, y]$ then condition becomes:

$$[x, x] = 0, \qquad \forall x \in \mathfrak{L}_{\mathfrak{R}}.$$

2) We have *Jacobi's identity*:

$$[[x,y],z]+[[y,z],x]+[[z,x],y]=0$$

Definition B.8. Given a $\mathfrak{R}$-*Lie algebra* $\mathfrak{L}_{\mathfrak{R}}$ we define its *opposite Lie algebra* with the following bracket:

$$[x,y]^{\mathrm{op}} := -[x,y].$$

Definition B.9. Let $\mathfrak{L}_{\mathfrak{R}}, \mathfrak{L}'_{\mathfrak{R}}$ be two $\mathfrak{R}$-Lie algebras, a map $\varphi : \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{L}'_{\mathfrak{R}}$ is called a *Lie homomorphism* if φ is $\mathfrak{R}$-linear and satisfies: $\varphi([x,y]) = [\varphi(x), \varphi(y)]$, for all $x, y \in \mathfrak{L}_{\mathfrak{R}}$.

Example B.10. Let $\mathfrak{L}_{\mathfrak{R}}$ be an arbitrary $\mathfrak{R}$-algebra. One can equip $\mathfrak{L}_{\mathfrak{R}}$ with a Lie algebra structure by defining: $[x,y]=0$ for all $x, y \in \mathfrak{L}_{\mathfrak{R}}$. Such a Lie algebra is called *commutative*.

Example B.11. The set $\mathrm{DER}(\mathfrak{L}_{\mathfrak{R}})$ of all derivations of an $\mathfrak{R}$-algebra $\mathfrak{L}_{\mathfrak{R}}$ is a Lie algebra with the product: $[D, D'] = DD' - D'D$.

Example B.12. Let $\mathfrak{L}_{\mathfrak{R}}$ be an $\mathfrak{R}$-algebra then $[x,y]=xy-yx$ turns $\mathfrak{L}_{\mathfrak{R}}$ into an $\mathfrak{R}$-Lie algebra. We use $\mathrm{LIE}(\mathfrak{L}_{\mathfrak{R}})$ when want to emphasize the Lie algebra structure on $\mathfrak{L}_{\mathfrak{R}}$. Note that the underlying sets of $\mathfrak{L}_{\mathfrak{R}}$ and $\mathrm{LIE}(\mathfrak{L}_{\mathfrak{R}})$ are identical.

Theorem B.13. *Let $\mathfrak{L}_{\mathfrak{R}}$ be an $\mathfrak{R}$-Lie algebra. For any $x \in \mathfrak{L}_{\mathfrak{R}}$ define a map* $\mathrm{ad}(x) : \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{L}_{\mathfrak{R}}$ *by* $\mathrm{ad}(x)(y) = [x,y]$*, then:*

1) $\mathrm{ad}(x)$ *is a derivation of* $\mathfrak{L}_{\mathfrak{R}}$.

2) The map $x \mapsto \mathrm{ad}(x)$ *is a Lie homomorphism of* $\mathfrak{L}_{\mathfrak{R}}$ *into* $\mathrm{DER}(\mathfrak{L}_{\mathfrak{R}})$.

Definition B.14. A *universal enveloping algebra* of an $\mathfrak{R}$-Lie algebra $\mathfrak{L}_{\mathfrak{R}}$ is a pair $(\varepsilon, \mathfrak{U}_{\mathfrak{R}})$, where:

- $\mathfrak{U}_{\mathfrak{R}}$ is an associative $\mathfrak{R}$-algebra with a unit.
- $\varepsilon : \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{U}_{\mathfrak{R}}$ is a Lie algebra homomorphism.
- $\mathrm{HOM}_{\mathfrak{R}-\mathrm{lie}}(\mathfrak{L}_{\mathfrak{R}}, \mathrm{LIE}(\mathfrak{T}_{\mathfrak{R}})) \cong \mathrm{HOM}_{\mathfrak{R}-\mathrm{alg}}(\mathfrak{U}_{\mathfrak{R}}, \mathfrak{T}_{\mathfrak{R}})$.

Any Lie algebra possesses a universal enveloping algebra, moreover we have the following functorial properties:

$$\mathfrak{U}_{\mathfrak{R}}(\mathfrak{L} \oplus \mathfrak{L}') = \mathfrak{U}_{\mathfrak{R}}(\mathfrak{L}) \otimes_{\mathfrak{R}} \mathfrak{U}_{\mathfrak{R}}(\mathfrak{L}').$$

If we have a $\mathfrak{R}$-Lie algebra homomorphism: $\Psi : \mathfrak{L}_{\mathfrak{R}} \to \mathfrak{L}'_{\mathfrak{R}}$ then by universal property:

$$\mathfrak{U}(\Psi) : \mathfrak{U}_{\mathfrak{R}}(\mathfrak{L}) \to \mathfrak{U}_{\mathfrak{R}}(\mathfrak{L}').$$

is a homomorphism of $\mathfrak{R}$-algebras.

PBW Theorem ([24] page 8)**.** *Let $\mathfrak{L}_{\mathfrak{R}}$ be an $\mathfrak{R}$-Lie algebra and $(\varepsilon, \mathfrak{U}_{\mathfrak{R}})$ its universal enveloping algebra. Then:*

- *ε is injective.*
- *If $\mathfrak{L}_{\mathfrak{R}}$ is identified with its image in $\mathfrak{U}_{\mathfrak{R}}$ and if $\{x_1, x_2, \cdots\}$ is an ordered $\mathfrak{R}$-basis for $\mathfrak{L}_{\mathfrak{R}}$, then all monomials*

  $$x_{i_1}^{a_{i_1}} x_{i_2}^{a_{i_2}} \cdots x_{i_k}^{a_{i_k}},$$

  in which $i_1 < \cdots < i_k$ and $a_{i_1}, \cdots, a_{i_k} \in \mathbb{N}$, form an $\mathfrak{R}$-basis for $\mathfrak{U}_{\mathfrak{R}}$.

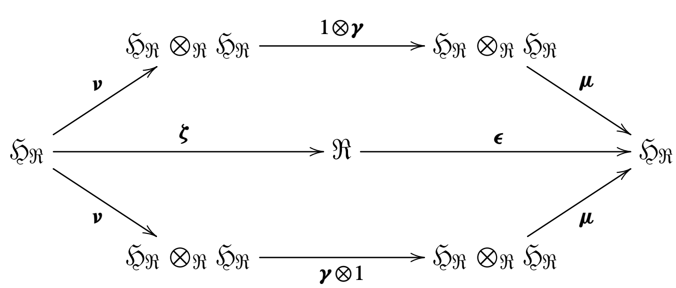
Appendix C

Hopf Algebra

C.1 Definition

Let $\mathfrak{R}$ be a commutative ring with a unit. A *Hopf algebra* over $\mathfrak{R}$ is a 6-tuple, $(\mathfrak{H}_{\mathfrak{R}}, \boldsymbol{\mu}, \boldsymbol{\epsilon}, \boldsymbol{\nu}, \boldsymbol{\zeta}, \boldsymbol{\gamma})$ such that:

- $\mathfrak{H}_{\mathfrak{R}}$ is an $\mathfrak{R}$-algebra with $\boldsymbol{\mu}, \boldsymbol{\epsilon}$ defining product and unit.
- $\mathfrak{H}_{\mathfrak{R}}$ is an $\mathfrak{R}$-coalgebra with $\boldsymbol{\nu}, \boldsymbol{\zeta}$ defining co-product and co-unit.
- These two structures are compatible, i.e. $\boldsymbol{\nu}, \boldsymbol{\zeta}$ are $\mathfrak{R}$-algebra homomorphisms.
- $\boldsymbol{\gamma} : \mathfrak{H}_{\mathfrak{R}} \longrightarrow \mathfrak{H}_{\mathfrak{R}}$ is an $\mathfrak{R}$-algebra homomorphism, usually called *antipode*, such that the following diagram commutes:



Given a Hopf algebra $\mathfrak{H}_{\mathfrak{R}}$ over $\mathfrak{R}$ we can equip $\mathrm{Hom}_{\mathfrak{R}-\mathrm{alg}}(\mathfrak{H}_{\mathfrak{R}}, \mathfrak{R})$ with the structure of a group scheme over $\mathfrak{R}$.

C.2 Enveloping Algebra

Let $\mathfrak{g}_{\mathfrak{R}}$ be a $\mathfrak{R}$-Lie algebra and consider the following Lie algebra homomorphism:

$$\begin{cases} \mathfrak{g}_{\mathfrak{R}} \to \{0\} \\ x \mapsto 0 \end{cases}, \begin{cases} \mathfrak{g}_{\mathfrak{R}} \to \mathfrak{g}_{\mathfrak{R}} \oplus \mathfrak{g}_{\mathfrak{R}} \\ x \mapsto (x, x) \end{cases}, \begin{cases} \mathfrak{g}_{\mathfrak{R}} \to \mathfrak{g}_{\mathfrak{R}}^{\mathrm{op}} \\ x \mapsto -x \end{cases}.$$

Then we have the corresponding maps for enveloping algebras:

$$\begin{aligned} \boldsymbol{\zeta} &: \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}) \to \mathfrak{U}_{\mathfrak{R}}(\{0\}) = \mathfrak{R}, \\ \boldsymbol{\nu} &: \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}) \to \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g} \oplus \mathfrak{g}) = \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}) \otimes_{\mathfrak{R}} \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}), \\ \boldsymbol{\gamma} &: \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}) \to \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}^{\mathrm{op}}) = \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^{\mathrm{op}}. \end{aligned}$$

Consider $\mathfrak{g}_{\mathfrak{R}}$ as a subset of $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})$ then for $x \in \mathfrak{g}_{\mathfrak{R}}$ then: $\boldsymbol{\zeta}(x) = 0$, $\boldsymbol{\nu}(x) = 1 \otimes x + x \otimes 1$ and $\boldsymbol{\gamma}(x) = -x$.

Definition C.1. An element $x \in \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})$ is called *primitive* if: $\boldsymbol{\nu}(x) = x \otimes 1 + 1 \otimes x$.

Lemma C.2. $\left(\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}), \boldsymbol{\mu}, \boldsymbol{\epsilon}, \boldsymbol{\nu}, \boldsymbol{\zeta}, \boldsymbol{\gamma}\right)$ *is a co-commutative Hopf algebra, where:*

- $\boldsymbol{\mu}$ *is the multiplication,*
- $\boldsymbol{\epsilon}$ *is the unit,*
- $\boldsymbol{\nu}$ *is the co-multiplication,*
- $\boldsymbol{\zeta}$ *is the co-unit,*
- $\boldsymbol{\gamma}$ *is the antipode.*

C.3 The Dual of the Enveloping Algebra

The material in the section follows [9] §2.7.8.

Consider the transpose of the co-product:

$$\boldsymbol{\nu}^* : \left(\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}) \otimes_{\mathfrak{R}} \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})\right)^* \to \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$$

combined with the restriction:

$$\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^* \otimes_{\mathfrak{R}} \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^* \subset \big(\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}) \otimes_{\mathfrak{R}} \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})\big)^*$$

we get a linear map: $\boldsymbol{\nu}^* : \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^* \otimes_{\mathfrak{R}} \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^* \to \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$. More explicitly for $f, g \in \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$ the product is give by:

$$(fg)(u) = \boldsymbol{\nu}^*(f \otimes g)(u) = (f \otimes g)(\boldsymbol{\nu}(u)).$$

The vector space $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$ is thus equipped with the structure of an algebra.

Notation C.3. Let $\mathbb{Z}_+^\infty$ be the set of all infinite sequences of non-negative integers and for $\nu \in \mathbb{Z}_+^\infty$ define:

$$\begin{aligned} x^\nu &= \prod_{n=1}^{\infty} x_n^{\nu_n} \in \mathfrak{R}[\![x_1, x_2, \cdots]\!] \\ e_\nu &= \prod_{n=1}^{\infty} e_n^{[\nu_n]} \in \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g}). \end{aligned}$$

Lemma C.4. *For $\nu \in \mathbb{Z}_+^\infty$ we have:*

$$\boldsymbol{\nu}(e_\nu) = \sum_{\lambda+\mu=\nu} e_\lambda \otimes e_\mu.$$

Proof. We have:

$$\begin{aligned} \boldsymbol{\nu}\left(e_1^{[\nu_1]}\right) &= \frac{1}{\nu_1!}\boldsymbol{\nu}\left(e_1^{\nu_1}\right) = \frac{1}{\nu_1!}\boldsymbol{\nu}(e_1)^{\nu_1} \\ &= \frac{1}{\nu_1!}\left(e_1 \otimes 1 + 1 \otimes e_1\right)^{\nu_1} \\ &= \sum_{\lambda_1+\mu_1=\nu_1} \frac{1}{\lambda_1!\mu_1!} e_1^{\lambda_1} \otimes e_1^{\mu_1} \end{aligned}$$

Hence:

$$\boldsymbol{\nu}(e_\nu) = \boldsymbol{\nu}\left(\prod_{i=1}^{n} e_i^{[\nu_i]}\right)$$

$$
\begin{aligned}
&= \prod_{i=1}^{n} \boldsymbol{\nu}\left(e_i^{[\nu_i]}\right) \\
&= \prod_{i=1}^{n} \sum_{\lambda_i+\mu_i=\nu_i} \frac{1}{\lambda_i!\mu_i!} e_i^{\lambda_1} \otimes e_i^{\mu_1} \\
&= \sum_{\substack{\lambda_1+\mu_1=\nu_1 \\ \cdots \\ \lambda_n+\mu_n=\nu_n}} \frac{1}{\lambda_1!\mu_1!\cdots\lambda_n!\mu_n!} e_1^{\lambda_1}\cdots e_n^{\lambda_n} \otimes e_n^{\mu_1}\cdots e_n^{\mu_n} \\
&= \sum_{\lambda+\mu=\nu} e_\lambda \otimes e_\mu
\end{aligned}
$$

□

Theorem C.5. *There is an isomorphism of the $\mathfrak{R}$-algebras: $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^* \cong \mathfrak{R}[\![x_1, x_2, \cdots]\!]$ given by the map:*

$$
f \mapsto s_f := \sum_\nu f(e_\nu)x^\nu.
$$

Proof. Since $\{e_\nu : \nu \in \mathbb{Z}_+^\infty\}$ is a basis for $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})$ the mapping $f \mapsto s_f$ is bijective (in the same way that the dual of an infinite direct sum is an infinite direct product). Moreover if $f, g \in \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$, then:

$$
\begin{aligned}
s_{fg} &= \sum_{\nu\in\mathbb{Z}_+^\infty} (fg)(e_\nu) \\
&= \sum_{\nu\in\mathbb{Z}_+^\infty} (f \otimes g)(\boldsymbol{\nu}(e_\nu)) \\
&= \sum_{\nu\in\mathbb{Z}_+^\infty} (f \otimes g)\left(\sum_{\lambda+\mu=\nu} e_\lambda \otimes e_\mu\right) \\
&= \sum_{\nu\in\mathbb{Z}_+^\infty} \sum_{\lambda+\mu=\nu} f(e_\lambda) \otimes g(e_\mu) \\
&= \sum_{\lambda,\mu\in\mathbb{Z}_+^\infty} f(e_\lambda)g(e_\mu)x^{\lambda+\mu} = s_f s_g.
\end{aligned}
$$

□

In particular, the algebra $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$ is associative and commutative. Its unity, 1^*, is a linear form such that $\mathrm{Ker}(1^*) = \mathfrak{A}_{\mathfrak{R}}(\mathfrak{g})$ and $1^*(1) = 1$. The unit map then is the transpose co-unit map in $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})$, that is we have: $\boldsymbol{\zeta}^* : \mathfrak{R} \to \mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$, where

$\zeta^*(1) = 1^*$. Moreover the transpose of the principal anti-automorphism of $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})$ is an automorphism of $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$ and is called the *principal automorphism of* $\mathfrak{U}_{\mathfrak{R}}(\mathfrak{g})^*$.

Appendix D

Amalgams and Tits Systems

The material of this appendix follows [21].

D.1 Direct Limits

Let $(G_i)_{i \in I}$ be a family of groups such that for each pair we have a homomorphism $f_{ij} : G_i \to G_j$. A group G and a family of homomorphisms $(g_i : G_i \to G)_{i \in I}$ is called the *direct limit* of the family $(G_i)_{i \in I}$ relative to (f_{ij}) if:

(1) $g_j \circ f_{ij} = g_i$ for all $i, j \in I$.

(2) If H is a group with a family of homomorphisms $(h_i : G_i \to H)_{i \in I}$ such that $h_j \circ f_{ij} = h_i$ for all $i, j \in I$. Then there is exactly one homomorphism $\phi : G \to H$ such that $h_i = \phi \circ g_i$

Proposition D.1. *The pair consisting of G and the family of homomorphisms* $(g_i : G_i \to G)_{i \in I}$ *exists and is unique up to unique isomorphism.*

D.2 Tits Systems

A *Tits System* is a 4-tuple (G, B, N, S) where G is a group, B and N are subgroups of G, and S a subset of $W = N/(B \cap N)$ satisfying the following axioms:

(1) The set $B \cup N$ generates G and $B \cap N$ is a normal subgroup of N.

(2) The set S generates $W = N/(B \cap N)$ and consists of elements of order 2.

(3) $BsB\ BwB \subset BwB \cup BswB$ for $s \in S$ and $w \in W$.

(4) For each $s \in S$ one has $sBs^{-1} \neq B$.

The group W is called the *Weyl group* of (G, B, N, S); the pair (W, S) is a Coxeter system, that is S and the relations:

$$(st)^{m_{st}} = 1, \qquad s, t \in S \text{ and } m_{st} < \infty,$$

is a presentation for W. The group G is the disjoint union of double cosets $BwB, w \in W$. This is called the *Bruhat decomposition*.

If $S' \subset S$, let $W_{S'}$ be the subgroup of W generated by elements of S'; then $P_{S'} = BW_{S'}B$. Then $S' \mapsto P_{S'}$ is a bijection of the set of subsets of S onto the set of subgroups of G containing B. $P_{S'}$ is then called the *standard Parabolic subgroup* of type S'.

D.3 Tits's Theorem

Let G be a group, and let $(G_i)_{i \in I}$ be a family of subgroups of G. We say G is the *product of G_i amalgamated along their intersections* if G is the direct limit of the system formed by the G_i, the $G_i \cap G_j$ and the inclusions:

$$G_i \cap G_j \subset G_i, \qquad G_i \cap G_j \subset G_j.$$

Theorem D.2 ([25]). *Let (G, B, N, S) be a Tits system; Then G is the product of N and $(P_{\{s\}})_{s \in S}$ amalgamated along their intersections.*