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# The Local Structure of Characters

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Let $G$ be a connected real semisimple Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be the Cartan decomposition and $K$ the maximal compact subgroup with Lie algebra $\mathfrak{k}$. Let $\Theta$ be the character of an irreducible representation. Then $\Theta$ has an asymptotic expansion at zero (in the sense of Taylor series). As consequences of this expansion we obtain results about the asymptotic directions in which the $K$-types occur and about the Gelfand-Kirillov dimension of the representation.

## 1. INTRODUCTION

Let $G$ be a connected semisimple Lie group with finite center. Let $\pi$ be an irreducible admissible representation of $G$ on a Hilbert space $\mathscr{H}$, and $\Theta_\pi$ its distribution character: if $f \in C_c^\infty(G)$,

$$\Theta_\pi(f) = \operatorname{Tr}\left(\int_G f(g)\,\pi(g)\,dg\right).$$

Harish-Chandra has made an exhaustive analysis of the analytic properties of the distribution $\Theta_\pi$ [5, 6]; in particular it is given by integration against a function which is locally $L^1$. Our goal is to relate the singularities of $\Theta_\pi$ at the identity to the structure of $\pi$. In section 2 and the first part of section 3, we apply Harish-Chandra's results in a simple way to get the following theorem.

THEOREM 1.1. *Let $\theta_\pi$ be the lift of $\Theta_\pi$ to a neighborhood of the identity on $\mathfrak{g} = \mathrm{Lie}(G)$. If $f \in C_c^\infty(\mathfrak{g})$ and $t > 0$, define*

$$f_t(X) = t^{-\dim \mathfrak{g}} f(t^{-1}X).$$

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*Then there are an integer r and tempered distributions* $\{D_i\}_{i=-r}^{\infty}$ *on* $\mathfrak{g}$, *such that for* $f \in C_c^\infty(\mathfrak{g})$,

$$\theta_\pi(f_t) \sim \sum_{i=-r}^{\infty} t^i D_i(f)$$

*as* $t \to 0^+$. (*This asymptotic expansion should be understood in the same sense as a Taylor series for a smooth function of t.*) *Furthermore, the support of* $\hat{D}_i$ *is a union of nilpotent orbits in* $\mathfrak{g}^*$.

We define $AS(\Theta_\pi) \subseteq \mathfrak{g}^*$ to be the union of the supports of the $\hat{D}_i$; $AS(\Theta_\pi)$ is a union of nilpotent orbits. (Probably $AS(\Theta_\pi)$ coincides with the wavefront set of $\Theta_\pi$ at the identity, but we have been unable to prove this.) Let $K \subseteq G$ be a maximal compact subgroup. Then $\pi|_K$ has a distribution character $\Theta_\pi^K$. Following Kashiwara and Vergne [8], we identify $\hat{K}$ with a subset of the orbits of $K$ in the dual $\mathfrak{k}^*$ of the Lie algebra of $K$ (via the highest weight theory); and we define ([8], Definition 6.1(b))

$$AS(\Theta_\pi^K) = \{\mu \in \mathfrak{k}^* \mid \mu = \lim_{n\to\infty} t_n\mu_n \text{, with } \mu_n \in \mathfrak{k}^*,$$
$$t_n \in \mathbb{R}^+,\ \mu_n \text{ occurs in } \pi|_K \text{, and } t_n \to 0\}.$$

THEOREM 1.2 (see Theorem 3.5 and its proof). $\theta_\pi^K$ *has an asymptotic expansion* $\sum t^j E_j$ *in tempered distributions on* $\mathfrak{k}$, *computable in terms of the* $D_i$. *In particular*

$$AS(\Theta_\pi^K) = \bigcup_j \operatorname{supp} \hat{E}_j \subseteq \{\mu \in \mathfrak{k}^* \mid \exists\lambda \in AS(\Theta_\pi) \textit{ with } \lambda|_{\mathfrak{k}} = \mu\}.$$

*If* (*for example*) $\pi$ *is a discrete series representation, then equality holds.*

The last assertion (together with Proposition 3.7, which computes $AS(\Theta_\pi)$ when $\pi$ is in the discrete series) proves a conjecture of Kashiwara and Vergne ([8], end of Example 6.3). The rest of section 3 is largely devoted to showing that these expansions behave well under parabolic induction (Theorem 3.5).

In section 4 we relate $AS(\Theta_\pi)$ to primitive ideals. Among other things, we show

THEOREM 1.3. *The dimension d of* $AS(\Theta_\pi)$ *is equal to the Gelfand-Kirillov dimension of* $U(\mathfrak{g})$ *modulo the annihilator of* $\pi$. *Furthermore, if*

$$\theta_\pi \sim \sum_{i=-r}^{\infty} t^i D_i$$

*and* $D_{-r} \neq 0$, *then* $d = 2r$.

Thus the Gelfand-Kirillov dimension of the annihilator of $\pi$ measures the singularity of $\Theta_\pi$ at the identity; this may be regarded as related to the formula

$$\Theta_\pi(1) = \dim \pi$$

for finite dimensional representations.

Finally, we study the behavior of the asymptotic expansions under coherent continuation (Theorem 4.7). This leads to a proof of Conjecture 5.1 of [11], describing the asymptotic destribution of eigenvalues of the Casimir operator for $K$ in $\pi$ (Corollary 4.8).

## 2. Asymptotic Expansions of Distributions

Let $\Omega$ be a neighborhood of zero in $\mathbb{R}^n$, and $\theta$ a distribution on $C_c^\infty(\Omega)$. For $f \in C_c^\infty(\mathbb{R}^n)$ and $t > 0$, we define $f_t \in C_c^\infty(\mathbb{R}^n)$ by

$$f_t(x) = t^{-n} f(t^{-1}x).$$

We say that $\theta$ *admits an asymptotic expansion at* 0 if there is an integer $r$ and a family of distributions $\{D_i \mid r \leqslant i < \infty\}$ on $\mathbb{R}^n$, such that for $f \in C_c^\infty(\mathbb{R}^n)$

$$\theta(f_t) \sim \sum_{i=r}^{\infty} t^i D_i(f) \qquad \text{as} \quad t \to 0^+$$

in the following sense. For each positive integer $N$ and compact set $K$ there is a constant $C = C_{N,K} > 0$, a positive integer $k = k_{N,K}$, and a number $\epsilon = \epsilon_{N,K} > 0$, such that if $\operatorname{supp} f \subseteq K$, and $0 < t \leqslant \epsilon$, then $\operatorname{supp} f_t \subseteq \Omega$ (so that $\theta(f_t)$ is defined) and

$$\left| \theta(f_t) - \sum_{i=r}^{N} t^i D_i(f) \right| \leqslant C \cdot \sup_{|\alpha| \leqslant k} | D^\alpha f | \, t^{N+1}.$$

(Here $\alpha = (\alpha_1, ..., \alpha_n)$ is a multi-index, $|\alpha| = \sum \alpha_i$, and

$$D^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$$

as usual.) Obviously the $D_i$ are unique if they exist. If $D_r \neq 0$, we call $r$ the *order* of $\theta$ at 0. We write $\theta \sim \sum_{i=r}^\infty t^i D_i$.

Lemma 2.1. *Let $\theta$ be a distribution on a neighborhood $\Omega$ of* 0 *in $\mathbb{R}^n$, admitting an asymptotic expansion $\theta \sim \sum t^i D_i$. Then $D_i$ is homogeneous of degree $i$; that is, if $f \in C_c^\infty(\mathbb{R}^n)$ and $s > 0$,*

$$D_i(f_s) = s^i D_i(f).$$

The trivial proof is left to the reader.

We will be interested in the Fourier transforms of asymptotic expansions; so we need

LEMMA 2.2. *Every homogeneous distribution on $\mathbb{R}^n$ is tempered.*

*Proof.* Let $D$ be a distribution on $C_c^\infty(\mathbb{R}^n)$, homogeneous of degree $i$. We must show that there are integers $k_1$, $k \geqslant 0$, and a constant $C > 0$, such that if $f \in C_c^\infty(\mathbb{R}^n)$

$$| D(f)| \leqslant C \cdot \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha| \leqslant k_2}} (1 + | x |)^{k_1} | D^\alpha f |.$$

To see this, first choose $C_1 > 0$ and an integer $k_2 \geqslant 0$ so that if $f$ is supported in the ball of radius 2 about 0, then

$$| D(f)| \leqslant C_1 \sup_{|\alpha| \leqslant k_2} \sup | D^\alpha f |.$$

Put $k_1 = \max(0, i + k_2 + n + 1\}$. To get the desired estimate, we need a special partition of unity. Choose a positive smooth function $\varphi$ on say $(-\infty, 1.1)$ satisfying

(a) $\varphi(x) = 0$ for $x \leqslant .6$, and $\varphi(x) \leqslant 1$ always

(b) $\varphi(x) = 1$ for $.9 \leqslant x < 1.1$.

Extending $\varphi$ to all of $\mathbb{R}$ by defining

(c) $\varphi(x) = 1 - \varphi(x/2)$ for $1.1 \leqslant x \leqslant 1.8$

(d) $\varphi(x) = 0$ for $x > 1.8$.

Put $\varphi_j(x) = \varphi(2^{-j}x)$. Then $\varphi_j \in C_c^\infty(\mathbb{R})$, supp $\varphi_j \subseteq (2^{j-1}, 2^{j+1})$, and $\sum_{j=0}^\infty \varphi_j(x) = 1$ for $x \geqslant .9$. Set

$$D_0(f) = D\left(f\left(1 - \sum_{j=0}^\infty \varphi_j(| x |)\right)\right)$$

$$D_1(f) = D\left(f \cdot \sum_{j=0}^\infty \varphi_j(| x |)\right).$$

Obviously $D_0$ satisfies an estimate of the desired sort; we need only show that $D_1$ does. Put $f^j(x) = f(x)\, \varphi_j(| x |)$. If $f \in C_c^\infty(\mathbb{R}^n)$, obviously

$$D_1(f) = \sum_{j=0}^\infty D(f^j).$$

We estimate each term of the sum separately. Fix $j$, and put $t = 2^{-j}$. Then

$$\begin{aligned} D(f^j) &= t^{-i} D((f^j)_t) \\ &= 2^{ij} D(f_t \cdot \varphi(| x |)). \end{aligned}$$

Now $\operatorname{supp}(f_t \cdot \varphi(|x|))$ is contained in the ball of radius 2 about 0. So

$$| D(f_t \cdot \varphi(|x|))| \leqslant C_1 \cdot \sup_{|\alpha| \leqslant k_2} \sup | D^\alpha(f_t \cdot \varphi)|.$$

If we use the rule for differentiating products, and the fact that the first $k_2$ derivatives of $\varphi$ are uniformly bounded, we find that this is bounded by

$$C_2 \cdot 2^{j(k_2+n)} \sup_{|\alpha| \leqslant k_2} \sup_{2^{j-1} < |x| < 2^{j+1}} | D^\alpha(f)|.$$

(The power of two comes from the fact that we are differentiating some dilation of $f$, and the range of $|x|$ from the support of $\varphi$.) So

$$| D(f^j)| \leqslant C_2 \cdot 2^{j(i+k_2+n)} \sup_{|\alpha| \leqslant k_2} \sup_{2^{j-1} < |x| < 2^{j+1}} | D^\alpha(f)|.$$

For the values of $x$ in question,

$$\begin{aligned} 2^{j(i+k_2+n)} &\leqslant (2 \cdot |x|)^{(i+k_2+n+1)} \cdot 2^{-j} \\ &\leqslant (2 \cdot |x|)^{k_1} \cdot 2^{-j}. \end{aligned}$$

Hence

$$| D(f^j)| \leqslant C_3 \cdot 2^{-j} \sup_{|\alpha| \leqslant k_2} \sup_x | x^{k_1} D^\alpha(f)|;$$

so

$$\left| \sum_{j=0}^{\infty} D(f^j) \right| \leqslant C_3 \cdot \sup_{|\alpha| \leqslant k_2} \sup_x | x^{k_1} D^\alpha(f)|.$$

This proves the estimate for $D_1$, and hence for $D$. Q.E.D.

If $\alpha$ is a multi-index, let $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$.

LEMMA 2.3. *Let $\theta$ be a distribution on a neighborhood $\Omega$ of 0 in $\mathbb{R}^n$, admitting an asymptotic expansion $\theta \sim \sum t^i D_i$. If $\alpha$ and $\beta$ are multi-indices, then $\theta \circ x^\alpha$ ($\theta$ composed with multiplication by $x^\alpha$) and $\theta \circ D^\beta$ admit asymptotic expansions. More precisely*:

$$\begin{aligned} \theta \circ x^\alpha &\sim \sum t^i (D_{i-|\alpha|} \circ x^\alpha) \\ \theta \circ D^\beta &\sim \sum t^i (D_{i+|\beta|} \circ D^\beta). \end{aligned}$$

*Proof.* Formally these are obvious; the necessary estimates of remainders are trivial. Q.E.D.

COROLLARY 2.4. *Let $D = \sum_{|\alpha| \leqslant N} g_\alpha D^\alpha$ be a differential operator on $\Omega$, with $g_\alpha \in C^\infty(\Omega)$. Let $g_\alpha \sim \sum_\beta g_{\alpha\beta} x^\beta$ be the Taylor expansion of $g_\alpha$ at 0 ($g_{\alpha\beta} \in$ Y). Let $\theta$ be a distribution on $\Omega$ admitting an asymptotic expansion $\theta \sim \sum_{i=r}^\infty t^i D_i$ at 0. Then $\theta \circ D$ admits the asymptotic expansion*

$$\theta \circ D \sim \sum_{i=r-N}^{\infty} t^i \sum_{k=r-i}^{N} \Big( \sum_{|\alpha|-|\beta|=k} g_{\alpha\beta} D_{i+k} \circ x^\beta D^\alpha \Big).$$

*Proof.* Notice first that the inner sum is finite. So formally the result follows from Lemma 2.3; the necessary estimates are a straightforward consequence of Taylor's theorem with remainder, and are left to the reader. Q.E.D.

DEFINITION 2.5. Let $\theta$ be a distribution on a neighborhood $\Omega$ of 0 in $\mathbb{R}^n$, admitting an asymptotic expansion $\theta \sim \sum t^i D_i$. The *asymptotic support* of $\theta$ at 0, $AS(\theta)$, is the closure of the union of the supports of the $\hat{D}_i$.

It is easy to see that $AS(\theta)$ is contained in the wave front set of $\theta$ at zero.

COROLLARY 2.5. *In the setting of Corollary 2.4,*

$$AS(\theta \circ D) \subseteq AS(\theta).$$

*If $D$ is multiplication by a function which does not vanish at zero, then equality holds.*

*Proof.* If $T$ is a tempered distribution on $\mathbb{R}^n$, then

$$(T \circ x^\beta D^\alpha)^\wedge = c_{\alpha\beta} \hat{T} \circ D^\beta x^\alpha,$$

where $c_{\alpha\beta}$ is an appropriate power of $i$. In particular,

$$\operatorname{supp}(T \circ x^\beta D^\alpha)^\wedge \subseteq \operatorname{supp} \hat{T}.$$

The first statement follows immediately. For the second, note simply that $D^{-1}$ is an operator of the same sort, at least in a neighborhood of zero. Q.E.D.

Of course the second statement can be generalized enormously, using some sort of ellipticity condition on $D$; this is standard for the wavefront set. Since we don't need such results, they are omitted.

EXAMPLE. Let $C \subseteq \mathbb{R}^n$ be a closed cone, $\Omega$ a neighborhood of 0, and $g$ a function on $C \cap \Omega$ which is smooth on a neighborhood of $C \cap \Omega$. For $f \in C_c^\infty(\Omega)$, put

$$\theta(f) = \int_{C \cap \Omega} f(x)\, g(x)\, dx,$$

$dx$ being the usual measure on $\mathbb{R}^n$. Then $\theta$ is a distribution on $\Omega$, and has an asymptotic expansion $\theta \sim \sum_{i=0}^{\infty} t^i D_i$. Here $D_i$ is defined as follows: let $g \sim \sum g_\alpha x^\alpha$ be the Taylor expansion of $g$ at 0. Then

$$D_i(f) = \int_C f(x) \Big( \sum_{|\alpha|=i} g_\alpha x^\alpha \Big) \, dx.$$

This is obvious if $g = 1$, and the general case follows from Corollary 2.4. This example will be used in the next section to obtain asymptotic expansions of characters.

## 3. Asymptotic Expansions of Invariant Eigendistributions

Let $G$ be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Consider an invariant eigendistribution $\Theta$ on $G$. Let $\mathfrak{z}$ be the center of the universal enveloping algebra $U(\mathfrak{g}_c)$. In this section we will lift the distribution $\Theta$ to a distribution on the Lie algebra and show that it has an asymptotic expansion according to the definition in section 2. In particular $\Theta$ could be the character of an irreducible representation. In this case we will use the asymptotic expansion to obtain information about the representation.

We start by collecting some of the facts about invariant eigendistributions that we will use later.

Theorem 3.1. *Let*

$$\Omega = \{X \in \mathfrak{g} \colon |\operatorname{Im} \lambda| < \pi \text{ for any eigenvalue } \lambda \text{ of } \operatorname{ad} X\}.$$

*Then $\Omega$ is a $G$-invariant neighborhood of* 0 *in* $\mathfrak{g}$.

(1) *The exponential map* $\exp\colon \mathfrak{g} \to G$ *is an analytic diffeomorphism when restricted to $\Omega$ onto the open set $\mathscr{W} = \exp \Omega$. Put*

$$j(X) = \det((\exp(\operatorname{ad} X/2) - \exp(-\operatorname{ad} X/2))/\operatorname{ad} X).$$

Define $\xi(X) = j(X)^{1/2}$, with the square root chosen so that $\xi(0) = 1$. *Then the Haar measure $dx$ and the Euclidean measure $dX$ are related by*

$$dx = \xi(X)^2 \, dX.$$

*Let $\varphi \in C^\infty(\Omega)$. Define $f_\varphi \in C^\infty(\mathscr{W})$ by*

$$f_\varphi(\exp X) = \xi(X)^{-1} \varphi(X).$$

*Then*

$$\int \varphi_1 \varphi_2 \, dX = \int f_{\varphi_1} f_{\varphi_2} \, dx.$$

*If $\Theta$ is a distribution on $\mathscr{W}$ we can define $\theta$ a distribution on $\Omega$ by the relation*

$$\theta(\varphi) = \Theta(f_\varphi).$$

*If $\Theta$ is G-invariant then so is $\theta$. From here on assume $\Theta$ is invariant.*

(2) *Let $I(\mathfrak{g}_c)$ be the ring of invariant polynomials on $\mathfrak{g}_c$. Then there is an algebra isomorphism*

$$z \mapsto p_z$$

*between $\mathfrak{z}$ and $I(\mathfrak{g}_c)$ such that if $\Theta$ lifts to $\theta$ then $z \cdot \Theta$ lifts to $\partial(p_z)\theta$. In particular, if*

$$z \cdot \Theta = \chi(z)\Theta \qquad \text{for all} \quad z \in \mathfrak{z}$$

*for some character $\chi: \mathfrak{z} \to$ Y, then*

$$\partial(p)\theta = \chi(p)\theta$$

*where $\chi(p_z) = \chi(z)$.*

*In other words, if $\Theta$ is an eigendistribution for $\mathfrak{z}$ then $\theta$ is an eigendistribution for $\partial(I(\mathfrak{g}_c))$.*

*Proof.* In this form these results are due to Harish-Chandra. A summary can be found in [2]. The proofs are in [3], [5], and [6].

We will denote by $\mathscr{N}$ the set of nilpotent elements in $\mathfrak{g}$. By definition

$$\mathscr{N} = \{X \in \mathfrak{g}\text{: ad } X \text{ is nilpotent}\}.$$

We will often identify $\mathfrak{g}$ with its dual $\mathfrak{g}^*$ by the Cartan-Killing form $B(\cdot, \cdot)$. We define the Fourier transform as

$$\mathscr{F}_{\mathfrak{g}} f(X) = \int_{\mathfrak{g}^*} e^{i\lambda(X)} f(\lambda)\, d\lambda$$

for any $f \in C_c^\infty(\mathfrak{g}^*)$ or $f \in \mathscr{S}(\mathfrak{g}^*)$ (the Schwartz space). Sometimes when there is no ambiguity in terms of the algebra used we will denote the Fourier transform by $\hat{f}$. Via the identification mentioned before we can also write

$$\mathscr{F}_{\mathfrak{g}} f(X) = \int_{\mathfrak{g}} e^{iB(X,Y)} f(Y)\, dY$$

where $f \in \mathscr{S}(\mathfrak{g})$.

THEOREM 3.2. *Let $\Theta$ be an invariant eigendistribution, $\theta$ the lifting to $C_c^\infty(\Omega)$ according to Theorem 3.1. Then $\theta$ has an asymptotic expansion. If $\theta \sim \sum t^i D_i$ then the $D_i$ are tempered, invariant and* supp $\hat{D}_i \subseteq \mathscr{N}$.

*Proof.* According to [5], $\theta$ is a function, analytic on $\mathfrak{g}' \cap \Omega$, locally $L^1$. Suppose $\mathfrak{h}_1, \ldots, \mathfrak{h}_r$ is a complete set of representatives of Cartan subalgebras.

Let $\Delta_i$ be the set of roots of $\mathfrak{g}_c$ with respect to $\mathfrak{h}_{i,c}$ and $P_i$ a positive system. We define

$$\pi_i(Z) = \prod_{\alpha \in P_i} \alpha(Z).$$

According to Harish-Chandra, for each $f \in C_c^\infty(\mathfrak{g})$ we define

$$\varphi_f^i(Z) = \overline{\pi_i(Z)} \int_{G/H_i} f(\mathrm{Ad}\, x \cdot Z)\, dx$$

where $H_i$ is the Cartan subgroup corresponding to $\mathfrak{h}_i$ and $Z \in \mathfrak{h}_i'$. The main property of $\varphi_f$ is that it defines a continuous map from $\mathscr{S}(g)$ to $\mathscr{S}(\mathfrak{h}')$ (see for example [12], vol. 2, section 8.4).

Let $f \in C_c^\infty(\Omega)$. If $\mathfrak{h}_i^j$ are the connected components of the sets $\mathfrak{h}_i'$, Weyl's integral formula implies

$$\theta(f) = \sum_{i,j} c_{ij} \int_{\mathfrak{h}_i^j \cap \Omega} \pi_i(Z)\, \theta_{ij}(Z)\, \varphi_f^i(Z)\, dZ$$

where $\theta_{ij} = \theta\,|_{\mathfrak{h}_i^j \cap \Omega}$. It is known that $\pi_i \theta_{ij}$ is an analytic function which extends to a neighborhood of the closure of $\mathfrak{h}_i^j \cap \Omega$; in fact it is a linear combination of terms consisting of polynomials multiplied by exponentials. By changing variables,

$$\theta(f_t) = \sum_{i,j} c_{ij} \int_{\mathfrak{h}_i^j \cap \Omega} \pi_i(tZ)\, \theta_{ij}(tZ)\, \varphi_f^i(Z)\, dZ.$$

By the example in section 1 this has an asymptotic expansion. By Lemmas 2.1 and 2.2 the $D_i$'s are homogeneous and therefore tempered. By the uniqueness of the asymptotic expansion the $D_i$'s are also invariant. Due to Theorem 3.1,

$$\partial(p)\theta = \chi(p)\theta \qquad \text{for} \quad p \in I(\mathfrak{g}_c).$$

Let $\deg p = s$, $p$ homogeneous of positive degree. Then

$$[\partial(p)\theta](f_t) = t^{-s}\theta((\partial(p)f)_t).$$

Thus, by the uniqueness of the expansion,

$$\partial(p)D_i = \chi(p)D_{i-s},$$

so for $r$ large enough,

$$\partial(p^r)D_i = 0.$$

This implies that (letting $I^+(\mathfrak{g}_c)$ denote the ideal of invariant polynomials with zero constant term)

$$\operatorname{supp} \hat{D}_i \subseteq \{X: p(X) = 0 \text{ for all } p \in I^+(\mathfrak{g}_c)\}.$$

On the other hand, it is well known that

$$\mathcal{N} = \{X: p(X) = 0 \text{ for all } p \in I^+(\mathfrak{g}_c)\}.$$

We now mention some facts about induced representations which will be used later.

Let $P = MAN$ be a parabolic subgroup and $\pi$ an irreducible representation of $M$ on a Hilbert space $\mathscr{H}$. Let $\nu \in \mathfrak{a}_c^*$ where $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ is the Lie algebra of $P$. Let $\pi_P$ be the representation of $P$ on $\mathscr{H}$ defined by

$$\pi_P(man) = e^{(\sqrt{-1}\nu - \rho)(\log a)}\, \pi(m)$$

where $\rho(X) = \frac{1}{2} \operatorname{tr}(\operatorname{ad} X|_{\mathfrak{n}})$.

Let $\pi_\nu = \operatorname{Ind}_P{}^G \pi_P$.

LEMMA 3.3. *Let* $\Theta = \operatorname{tr} \pi$ *and* $\Theta_\nu = \operatorname{tr} \pi_\nu$. *If* $f \in C_c{}^\infty(G)$ *and* $\varphi \in C_c{}^\infty(P)$, *then*

$$\operatorname{tr} \pi_P(\varphi) = \int_{MAN} e^{(\rho+\sqrt{-1}\nu)(\log a)}\, \Theta(m)\, \varphi(man)\, dm\, da\, dn$$

$$\Theta_\nu(f) = \int_{MA} e^{(\rho+\sqrt{-1}\nu)(\log a)}\, \Theta(m) \int_{KN} f(kmank^{-1})\, dk\, dn\, dm\, da$$

$$= \int_{MA} e^{\sqrt{-1}\nu(\log a)}\, \Theta(m)\, D(ma) \int_{G/MA} f(xmax^{-1})\, dx\, dm\, da$$

*where* $D(\exp(X_{\mathfrak{m}} + X_{\mathfrak{a}})) = |\det(e^{\operatorname{ad}(X_{\mathfrak{m}}+X_{\mathfrak{a}})/2} - e^{-\operatorname{ad}(X_{\mathfrak{m}}+X_{\mathfrak{a}})/2})|_{\mathfrak{n}}|$ *for* $X_{\mathfrak{m}} \in \mathfrak{m}$, $X_{\mathfrak{a}} \in \mathfrak{a}$. *In particular let* $f^x(Z) = f(\operatorname{Ad} x \cdot Z)$ *and* $f|_P$ *be the function* $f$ *restricted tp* $P$. *Then*

$$\Theta_\nu(f) = \int_{G/P} \operatorname{tr} \pi_P(f^x|_P)\, dx.$$

*Proof.* This is well known when $P$ is minimal. The proof carries over with simple modifications. See [13].

Let $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ be a Cartan decomposition. Let $K$ be the compact group corresponding to $\mathfrak{k}$.

LEMMA 3.4. *Let* $\varphi \in C_c{}^\infty(\mathfrak{s})$, $\varphi \geqslant 0$, $\int_{\mathfrak{s}} \varphi(X)\, dX = 1$ *and* $f \in C^\infty(K)$. *Define*

$$\varphi_\epsilon(X) = \epsilon^{-\dim \mathfrak{s}}\, \varphi(\epsilon^{-1}X)$$

$$f_\epsilon(\exp Xk) = \varphi_\epsilon(X) f(k)$$

*where* $X \in \mathfrak{s}$, $k \in K$. *Let* $\pi$ *be an irreducible representation with character* $\Theta$. $\pi|_K$ *also has a character which we denote by* $\Theta^K$. *Let* $\psi \in C_c^\infty(\Omega \cap \mathfrak{k})$ *be such that* $\psi(Y) = f(\exp Y)$. *Let* $\theta$ *and* $\theta^K$ *be the distributions corresponding to* $\Theta$ *and* $\Theta^K$. *Let* $\xi$ *and* $\xi_K$ *be the functions in Theorem* 3.1 *corresponding to G and K. Then*

$$\lim_{\epsilon\to 0} \pi(f_\epsilon) = \pi_K(f) \qquad \text{(weakly)}$$

$$\lim_{\epsilon\to 0} \Theta(f_\epsilon) = \Theta^K(f)$$

$$\lim_{\epsilon\to 0} \theta(\psi_\epsilon) = \theta^K(\xi/\xi_K\psi)$$

*where* $\psi_\epsilon(X+Y) = \varphi_\epsilon(X)\,\psi(Y)$ *and* $X \in \mathfrak{s}$, $Y \in \mathfrak{k}$.

*Proof.* We have the relations

$$\begin{aligned}\pi(f_\epsilon) &= \int_G f_\epsilon(x)\,\pi(x)\,dx \\ &= \int_{\mathfrak{s}} D(X)\,\varphi_\epsilon(X)\,\pi(\exp X)\,dX \int_K f(k)\,\pi(k)\,dk\end{aligned}$$

where $D(X)$ is the Jacobian of the map $(X, k) \to \exp X \cdot k$. Since $\pi_K(f) = \int_K f(k)\,\pi(k)\,dk$ and $\int \varphi(X)\,dX = 1$, $\pi(f_\epsilon) - \pi_K(f) = \pi_K(f) \int_{\mathfrak{s}} [D(X)\,\pi(\exp X) - I]\,\varphi_\epsilon(X)\,dX$. Then for any $v \in \mathscr{H}$,

$$\|(\pi(f_\epsilon) - \pi_K(f))v\| \leqslant \|\pi_K(f)\| \int_{\mathfrak{s}} \|D(x)\,\pi(\exp X)v - v\|\,\varphi_\epsilon(X)\,dX.$$

By changing variables,

$$\int_{\mathfrak{s}} \|D(X)\,\pi(\exp X)v - v\|\,\varphi_\epsilon(X)\,dX = \int_{\mathfrak{s}} \|D(\epsilon X)\,\pi(\exp \epsilon X)v - v\|\,\varphi(X)\,dX.$$

Since $\varphi$ is compactly supported,

$$\|D(\epsilon X)\,\pi(\exp \epsilon X)v - v\| \leqslant C$$

for all $X \in \operatorname{supp} \varphi$ and $\epsilon < 1$. Thus we can apply the bounded convergence theorem to obtain

$$\lim_{\epsilon\to 0} \pi(f_\epsilon)v = \pi_K(f)v.$$

The fact that $\pi_K$ is of trace class is well known. We need part of the proof of this fact to prove the statements in the lemma. Let

$$\Omega = 1 - (X_1^2 + \cdots + X_r^2) \in U(\mathfrak{k})$$

where $X_1, \ldots, X_r$ is an orthonormal basis of $\mathfrak{k}$. Then let $\mathscr{H}$ be the space of the representation $\pi$ and $\{v_i\}_{i\in N}$ an orthonormal basis of $K$-finite vectors for $\mathscr{H}$. Then

$$\Theta(f_\epsilon) = \sum_i (\pi(f_\epsilon)v_i, v_i)$$
$$\Theta^K(f) = \sum_i (\pi_K(f)v_i, v_i)$$
$$\theta(\psi_\epsilon) = \sum_i (\pi(f_{\psi_\epsilon})v_i, v_i)$$
$$\theta^K(\psi) = \sum_i (\pi_K(f_\psi)v_i, v_i).$$

Similarly to the previous statement,

$$\pi(f_{\psi_\epsilon}) = \int_G f_{\psi_\epsilon}(x)\, \pi(x)\, dx = \int_{\mathfrak{g}} \xi(X)\, \psi_\epsilon(X)\, \pi(\exp X)\, dX$$

converges weakly to $\pi_K(f_{\xi/\xi_K\psi})$.

Thus any finite sum occuring in $\Theta(f_\epsilon)$ or $\theta(\psi_\epsilon)$ converges to the corresponding sum in $\Theta^K(f)$ and $\theta^K(\xi/\xi_K\psi)$ respectively.

Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{k}$ and let $\Lambda \subseteq \mathfrak{t}^*$ parametrize $\hat{K}$. For $\delta \in \Lambda$, let $\mathscr{H}_\delta$ denote the $\delta$-primary subspace of $\mathscr{H}$; we may assume that each $v_i$ lies in some $\mathscr{H}_\delta$. Then for $h \in C_c^\infty(G)$,

$$\sum_i (\pi(h)v_i, v_i) = \sum_{\delta\in\Lambda} c(\delta)^{-s} \sum_{v_i\in\mathscr{H}_\delta} (\pi(\Omega^s h)v_i, v_i)$$

where $c(\delta)$ is the value of $\Omega$ on $\delta$. Let $m(\delta)$ be the multiplicity of $\delta$ in $\mathscr{H}$ and $d(\delta)$ the dimension of the representation $\delta$. Then if $\| \pi(x)\| < C$ for $x \in \operatorname{supp}(h)$,

$$\sum_{|\delta|>N} c(\delta)^{-s} \sum_{v_i\in\mathscr{H}_\delta} |(\pi(\Omega^s h)v_i, v_i)| \leqslant \| \Omega^s h \|_{1,G} \cdot C \cdot \sum_{|\delta|>N} m(\delta)\, c(\delta)^{-s}\, d(\delta).$$

It is known that

$$\sum_\delta m(\delta)\, c(\delta)^{-s}\, d(\delta)$$

converges if $s$ is large enough.

In order to complete the proof we have to show that $\| \Omega^s f_\epsilon \|_{1,G}$ and $\| \Omega^s f_{\psi_\epsilon} \|_{1,G}$ are bounded independently of $\epsilon$. We note

$$\Omega^s f_\epsilon = \varphi_\epsilon \cdot \Omega^s f,$$

so by a change of variables

$$\| \Omega^s f_\epsilon \|_{1,G} \leqslant \| \Omega^s f \|_{1,K}\, .$$

$$\Omega^s f_{\psi_\epsilon}(x) = \left(1 - \sum \frac{\partial^2}{\partial t_i^2}\right)^s \Big|_{t_i=0} (\xi^{-1}\psi_\epsilon)(\log(x \exp(-t_1X_1 - \cdots - t_rX_r)).$$

By using the Campbell-Baker-Hausdorff formula we can write

$$\log(\exp Y \exp X \exp(-t_1X_1 - \cdots - t_rX_r)) = M_{\mathfrak{k}}(Y, X, t_iX_i) + M_{\mathfrak{s}}(Y, X, t_iX_i)$$

where $M_{\mathfrak{k}} \in \mathfrak{k}$ and $M_{\mathfrak{s}} \in \mathfrak{s}$ are absolutely convergent infinite series in Lie brackets of $X$, $Y$, $t_iX_i$. We note that each term of $M_{\mathfrak{s}}$ contains at least one $Y$. Then

$$\| \Omega^s f_{\psi_\epsilon} \|_{1,G} = \int_{\mathfrak{k}+\mathfrak{s}} \left(1 - \sum \frac{\partial^2}{\partial t_i^2}\right)^s \Big|_{t_i=0} \xi(M_{\mathfrak{k}} + M_{\mathfrak{s}})\, \varphi_\epsilon(M_{\mathfrak{s}})\, \psi(M_{\mathfrak{k}})\, dY\, dX.$$

By changing variables to $Y = \epsilon Y'$ we get

$$\int_{\mathfrak{k}+\mathfrak{s}} \left(1 - \sum \frac{\partial^2}{\partial t_i^2}\right)^s \Big|_{t_i=0} \xi(M_{\mathfrak{s}}(\epsilon Y, X, t_iX_i) + M_{\mathfrak{k}}(\epsilon Y, X, t_iX_i)) \cdot \varphi(\epsilon^{-1}M_{\mathfrak{s}}(\epsilon Y, X, t_iX_i))\, \psi(\epsilon Y, X, t_iX_i)\, dY\, dX.$$

Then $\epsilon^{-1}M_{\mathfrak{s}}(\epsilon Y, X, t_iX_i)$ is analytic at $\epsilon = t_1 = \cdots = t_r = 0$. The rest of the proof is straightforward.

Let $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ be a parabolic subalgebra and $\mathfrak{p}^* = \mathfrak{m}^* + \mathfrak{a}^* + \mathfrak{n}^*$ be its dual.

THEOREM 3.5. *Let $P = MAN$ be a parabolic, $\nu \in \mathfrak{a}_c^*$ and $\pi \in \hat{M}$ an irreducible representation of $M$ on a Hilbert space. Let $\pi_\nu$ be the induced representation, $\Theta$ and $\Theta_\nu$ the corresponding characters. Let $\theta$ and $\theta_\nu$ be their lifts to the algebras $\mathfrak{g}$ and $\mathfrak{m}$. Let $\Theta_P$ be the character of $\pi_P$ defined earlier and $\theta_P$ its lifting to $\mathfrak{p}$.*

*For $\varphi \in C_c^\infty(\mathfrak{g})$ define $\varphi^x$ to be*

$$\varphi^x(Z) = \varphi(\mathrm{Ad}\, x \cdot Z)$$

*and $\varphi|_{\mathfrak{p}}$ the restriction of $\varphi$ to $\mathfrak{p}$. Then*

$$\theta_\nu(\varphi) = \int_K \theta_P(\xi_P/\xi \cdot \varphi^x|_{\mathfrak{p}})\, dx. \tag{1}$$

*Let $\theta_P \circ \xi_P/\xi \sim \sum t^i D_i$ and $\theta_P \sim \sum t^i E_i$. Then*

$$\theta_\nu \sim \sum t^{\dim \mathfrak{p} - \dim \mathfrak{g} + i} \int_K D_i(\varphi^x|_{\mathfrak{p}})\, dx \tag{2}$$

*and*

$$AS(\Theta_\nu) \subseteq \mathrm{Ad}\, K \cdot \{\lambda \in \mathfrak{g}^* : \lambda|_{\mathfrak{p}} \in AS(\Theta_P)\}. \tag{3}$$

*Assume that $E_i$ satisfy the following. Let $\varphi \geqslant 0$. Then if* supp $\varphi \cap$ supp $\hat{E}_j = \varnothing$ *for $j < r$ and $\hat{E}_r(\varphi) \neq 0$, then $\hat{E}_r(\varphi) > 0$. Then equality holds in* (3). (*This is e.g. true for representations which are unitarily induced from limits of discrete series*).

*Finally,*

$$AS(\Theta_P) = \{\lambda \in \mathfrak{p}^*: \lambda|_{\mathfrak{a}+\mathfrak{n}} = 0 \textit{ and } \lambda|_{\mathfrak{m}} \in AS(\Theta)\}. \tag{4}$$

*Proof.* We note that

$$\begin{aligned} \xi_P(X) &= |\det_{\mathfrak{p}}((e^{\operatorname{ad} X/2} - e^{-\operatorname{ad} X/2})/\operatorname{ad} X)|^{1/2} \\ &= \xi_M(X_{\mathfrak{m}})\, R(X_{\mathfrak{m}} + X_{\mathfrak{a}})^{1/2} \end{aligned}$$

and

$$\xi(X) = \xi_M(X_{\mathfrak{m}}) \cdot R(X_{\mathfrak{m}} + X_{\mathfrak{a}})$$

where $X = X_{\mathfrak{m}} + X_{\mathfrak{a}} + X_{\mathfrak{n}} \in \mathfrak{p}$ and

$$R(X_{\mathfrak{m}} + X_{\mathfrak{a}}) = |\det_{\mathfrak{n}}((e^{\operatorname{ad} X/2} - e^{-\operatorname{ad} X/2})/\operatorname{ad} X)|.$$

Then (1) and (2) follows from these formulas and Lemma 3.3. Relation (3) is a simple consequence of these formulas. We note that if the $E_i$'s satisfy the positivity condition then so do the $D_i$'s by the formula in Corollary 2.4.

Suppose $\lambda \notin AS(\Theta_\nu)$ but $\lambda \in \operatorname{Ad} K \cdot \{\lambda \in \mathfrak{g}^*: \lambda|_{\mathfrak{p}} \in AS(\Theta_P)\}$. Then for any $\varphi \in C_c^\infty(\mathfrak{g}^*)$, supp $\varphi$ contained in a small enough neighborhood of $\lambda$,

$$\int_K D_i(\varphi^x|_{\mathfrak{p}})\, dx = 0$$

for all $i$. Let $r$ be such that $\lambda \in \operatorname{supp} \hat{D}_r$ but $\operatorname{Ad} K \cdot \lambda \notin \operatorname{supp} \hat{D}_i$ with $i < r$. Then there is $\varphi \in C_c^\infty(\mathfrak{g}^*)$ such that $\varphi \geqslant 0$, $\varphi(\lambda) \neq 0$ and

$$\hat{D}_r(\varphi) \neq 0$$

while

$$\operatorname{supp}(\varphi^x|_{\mathfrak{p}}) \cap \operatorname{supp} \hat{D}_i = \varnothing, \qquad i < r \text{ and all } x \in K.$$

Then, it follows

$$\int_K \hat{D}_r(\varphi^x)\, dx \neq 0$$

since

$$\hat{D}_r(\varphi^x) \geqslant 0 \text{ for all } x.$$

We now relate the support of $\theta_P$ with the support of $\theta$. From Lemma 3.3 we deduce the relation

$$\theta_P(\varphi) = \theta((\mathscr{F}_{\mathfrak{a}}^{-1}\varphi)_{\mathfrak{n}}(\nu))$$

where $\mathscr{F}_a$ is the Fourier transform with respect to $\mathfrak{a}$ and

$$\varphi_{\mathfrak{n}}(X) = \int_{\mathfrak{n}} \varphi(X + X_{\mathfrak{n}})\, dX_{\mathfrak{n}}, \qquad X \in \mathfrak{m} + \mathfrak{a}.$$

Assume

$$\theta(\psi_t) \sim \sum t^j F_j(\psi)$$

for $\psi \in C_0^\infty(\mathfrak{m})$. Then we have the relation

$$\theta_P(\mathscr{F}_{\mathfrak{p}}\varphi)_t = \theta(\mathscr{F}_{\mathfrak{m}}(\varphi(\cdot, t\nu, 0))_t)$$

where $\varphi \in C_c^\infty(\mathfrak{p}^*)$ and the three coordinates refer to the decomposition $\mathfrak{p}^* = \mathfrak{m}^* + \mathfrak{a}^* + \mathfrak{n}^*$.

Let

$$\varphi(\cdot, t\nu, 0) \sim \sum (\partial(p_{s,\nu})|_0 \varphi) t^s$$

be the Taylor expansion of $\varphi$ in the variable $t$.

Thus

$$E_j(\mathscr{F}_{\mathfrak{p}}\varphi) = \sum_{s \leqslant j} \partial(p_{s,\nu})|_{\lambda_{\mathfrak{a}}=0} F_{j-s}(\mathscr{F}_{\mathfrak{m}}(\varphi\,|_{\mathfrak{n}})).$$

This implies relation (4) in the theorem.

Let $\delta \in \hat{K}$ be an irreducible representation of $K$. Then we can associate to each such $\delta$ an orbit of a regular element in $\mathfrak{k}^*$ by the coadjoint action. We identify $\hat{K}$ with this set of orbits.

THEOREM 3.6. *Let $\pi$ be an irreducible representation of G on a Hilbert space $\mathscr{H}$. Let $\Theta$ be its character and $\Theta^K$ be the character of $\pi_K = \pi|_K$. Then $\Theta^K$ has an asymptotic expansion at* 0. *Let*

$$K(\pi) = \{\delta \in \mathfrak{k}^* : \textit{there is a sequence } t_n\delta_n \to \delta \textit{ such that } t_n \to 0 \textit{ and } \delta \textit{ occurs with positive multiplicity in } \pi_K\}.$$

*Then*

$$AS(\theta^K) = K(\pi). \tag{1}$$

*Also*

$$AS(\theta^K) \subseteq \{\delta \in \mathfrak{k}^* : \textit{there is } \lambda \in AS(\theta) \textit{ such that } \lambda|_{\mathfrak{k}} = \delta\}. \tag{2}$$

*If $\pi$ is a discrete series representation then equality holds.* (Related results may be found in [8], Theorem 6.2. A weak version of (2) was recently proved by Howe and Wallach (unpublished).)

*Proof.* We first show that $\theta^K$ has an asymptotic expansion at zero when $\Theta$ is a discrete series character. Thus we assume rank $\mathfrak{g}$ = rank $\mathfrak{k}$. We identify $\mathfrak{g}$ and $\mathfrak{g}^*$ via the Cartan-Killing form. Then

$$\partial(p)\theta = p(\lambda)\theta \qquad \text{for} \quad p \in I(\mathfrak{g}_c)$$

and $\lambda$ a regular elliptic element. By the work of Harish-Chandra we may assume $\theta$ is a globally defined (not just on $C_c^\infty(\Omega)$) tempered eigendistribution. Its Fourier transform is a linear combination of measures supported on orbits of regular elliptic elements, in fact by [9] just one such orbit. Let $f \in C_c^\infty(\mathfrak{g}^*)$. Choose a Cartan subalgebra $\mathfrak{t} \subseteq \mathfrak{k}$ and let $T$ be its Cartan subgroup. Then ([9])

$$\theta(\mathscr{F}_{\mathfrak{p}} f) = \varphi_f^T(\lambda)$$

where $\varphi_f$ and its properties were explained during the proof of Theorem 3.2. Since

$$(\mathscr{F}_{\mathfrak{g}} f)_t = t^{-\dim\mathfrak{g}} \mathscr{F}_{\mathfrak{g}}(f_{t^{-1}}),$$

$$\theta((\mathscr{F}_{\mathfrak{g}} f)_t) = t^{-\dim\mathfrak{g}} \theta(\mathscr{F}_{\mathfrak{g}}(f_{t^{-1}})) = t^{-(\dim\mathfrak{g} - \operatorname{rank}\mathfrak{g})/2} \varphi(t_f{}^T\lambda).$$

As already mentioned, $\varphi_f{}^T \in \mathscr{S}(\mathfrak{t}^i)$ where $\mathfrak{t}^i$ are the connected components of $\mathfrak{t}^{*\prime}$. Thus

$$\varphi_f{}^T(t\lambda) = \sum_{i \leqslant N} t^i E_i(f) + t^{N+1} O(f)$$

where $|O(f)| \subseteq \nu_\xi(f)$ is a seminorm in the Schwartz topology.

In particular we apply this to $\psi_\epsilon$ defined in Lemma 3.4. We recall that

$$\lim_{\epsilon \to 0} \theta(\psi_\epsilon) = \theta^K(\xi/\xi_K \psi)$$

so that

$$\lim_{\epsilon \to 0} \varphi^T_{\mathscr{F}_{\mathfrak{g}}^{-1}\psi_\epsilon}(\lambda) = \theta^K(\xi/\xi_K\psi). \tag{*}$$

For any function $f \in C_c^\infty(\mathfrak{k}^*)$ we define $f_K \in C^\infty(\mathfrak{g}^*)$ by

$$f_K(\lambda) = f(\lambda|_{\mathfrak{k}}).$$

Let $f = \mathscr{F}_{\mathfrak{k}}^{-1}\psi$ where $\psi$ is as in Lemma 3.4. Then

$$\lim_{\epsilon \to 0} \mathscr{F}_{\mathfrak{g}}^{-1}\psi_\epsilon = f_K.$$

We show that $f_K$ coincides with a Schwartz function along the support of

$\varphi_{\cdot}^{T}(\lambda)$. Indeed let $\mathfrak{g}^* = \mathfrak{k}^* + \mathfrak{s}^*$ be the Cartan decomposition. Let $X \in \mathfrak{g}^*$ be such that

$$-\eta < -B(X, X) = |B(X_{\mathfrak{k}}, X_{\mathfrak{k}})| - B(X_{\mathfrak{s}}, X_{\mathfrak{s}}) < +\eta.$$

Then

$$B(X_{\mathfrak{s}}, X_{\mathfrak{s}}) < \eta + |B(X_{\mathfrak{k}}, X_{\mathfrak{k}})|.$$

Thus, on the set

$$\mathscr{U}_\eta = \{X: |B(X, X)| < \eta\}$$

for any polynomial $p \in S(\mathfrak{g}_c)$ there is $p_K \in S(\mathfrak{k}_c)$ such that

$$|p(X)| \leqslant |p_K(X_{\mathfrak{k}})| \qquad \text{for all} \quad X \in \mathscr{U}_\eta .$$

Let $h_1 \in C_c^\infty(\mathbb{R})$, $h_1 \geqslant 0$ such that supp $h_1 \subset (-1, +1)$. Let

$$h(X) = h_1(B(X, X)).$$

Then

$$hf_K \in \mathscr{S}(\mathfrak{g}^*)$$

and

$$hf_K = f_K \text{ on } \mathscr{U}_{1/2}$$

if we assume $h_1 \equiv 1$ on $(-\frac{1}{2}, +\frac{1}{2})$.

Thus $\varphi_{f_K}^T(t\lambda)$ makes sense for $t$ small enough and $\lambda$ regular. In addition, for any seminorm $\nu_\xi$ on $\mathscr{S}(\mathfrak{g}^*)$ there is a seminorm $\nu_{\xi'}$ on $\mathscr{S}(\mathfrak{k}^*)$ such that

$$\nu_\xi(hf_K) \leqslant \nu_{\xi'}(f).$$

The set $\mathscr{U}_{1/2}$ contains the $G$ orbit of $t\lambda$ for $t$ small enough as well as the set of nilpotents $\mathscr{N}$. This implies that

$$\lim_{\epsilon\to 0} \varphi^T_{\mathscr{F}_{\mathfrak{g}}^{-1}\psi_\epsilon}(t\lambda) = \varphi^T_{hf_K}(t\lambda)$$

$$\lim_{\epsilon\to 0} E_i(\mathscr{F}_{\mathfrak{g}}^{-1}\psi_\epsilon) = E_i(hf_K), \qquad \text{all } i.$$

Finally, since $\psi_\epsilon = \varphi_\epsilon \otimes \psi$ we get

$$\mathscr{F}_{\mathfrak{g}}^{-1}\psi_\epsilon = \mathscr{F}_{\mathfrak{s}}^{-1}\varphi_\epsilon \otimes \mathscr{F}_{\mathfrak{k}}^{-1}\psi.$$

We can find a bound on $O(\psi_\epsilon)$ uniform in $\epsilon < 1$. We conclude

$$\lim_{\epsilon\to 0} \varphi^T_{\mathscr{F}_{\mathfrak{g}}^{-1}\psi_\epsilon}(t\lambda) = \varphi^T_{f_K}(t\lambda)$$

and

$$\varphi^T_{f_K}(t\lambda) \sim \sum E_i(f_K)t^i.$$

Thus by (*) we get

$$\theta^K(\xi/\xi_K \psi_t) = t^{-(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})/2}\, \varphi^T_{f_K}(t\lambda)$$

so

$$\theta^K(\xi/\xi_K \psi_t) \sim \sum t^{i-(\dim \mathfrak{g} - \operatorname{rank} \mathfrak{g})/2}\, E_i(f_K)$$

where $\psi = \mathscr{F}_{\mathfrak{k}} f$.

Let $S_i = \operatorname{supp} E_i$. We have to identify the union of the supports of the distributions

$$f \mapsto E_i(f_K).$$

Let

$$K(\theta) = \{\delta \in \mathfrak{k}^*\colon \text{there is } \lambda \in AS(\theta) \text{ such that } \lambda\,|_{\mathfrak{k}} = \delta\}.$$

We want to show that $AS(\theta^K) = K(\theta)$. The inclusion $AS(\theta^K) \subseteq K(\theta)$ is clear. Assume $\delta \in K(\theta)$ but $\delta \notin AS(\theta^K)$. Then there is a neighborhood of $\delta$, $V_\delta \subseteq \mathfrak{k}^*$ such that for every $f \in C_c^\infty(\mathfrak{k}^*)$ with $\operatorname{supp} f \subseteq V_\delta$,

$$E_i(f_K) = 0 \qquad \text{for all } i.$$

On the other hand, there is $\lambda$ such that $\lambda\,|_{\mathfrak{k}} = \delta$ and $\lambda \in \bigcup_i \operatorname{supp} E_i$. Let $r$ be the smallest integer such that $\{\lambda \in \mathfrak{g}^*\colon \lambda\,|_{\mathfrak{k}} = \delta\} \cap \operatorname{supp} E_r \neq \varnothing$. Then we can shrink $V_\delta$ so that

$$\{\lambda : \lambda\,|_{\mathfrak{k}} \in V_\delta\} \cap \operatorname{supp} E_i = \varnothing \qquad \text{for} \quad i < r.$$

Choose $f, g \in C_c^\infty(V_\delta)$, $f \geqslant 0$, $g \leqslant 0$, and a neighborhood $U_\delta$ of $\delta$ so that $f \geqslant \epsilon$ and $g \leqslant -\epsilon$ on $U_\delta$. Let $\lambda_0 \in \operatorname{supp} E_r$ be such that $\lambda_0\,|\mathrm{i} \in U_\delta$. Let $U_{\lambda_0}$ be a neighborhood of $\lambda_0$ such that if $\lambda \in U_{\lambda_0}$ then $\lambda\,|_{\mathfrak{k}} \in U_\delta$. Let $h \in C_c^\infty(U_{\lambda_0})$ be such that $|\, h \,| \leqslant \epsilon$. Then $E_i(f_K - h)$ and $E_i(h - g_K) = 0$ for $i < r$. Since $\varphi_f{}^T$ is positive on positive functions,

$$E_r(f_K - h) \geqslant 0, \qquad E_r(h - g_K) \geqslant 0.$$

Thus $E_r(g_K) \leqslant E_r(h) \leqslant E_r(f_K)$ so $E_r(h) = 0$ for all $h \in C_c^\infty(U_{\lambda_0})$ a contradiction. Thus

$$K(\theta) = AS(\theta^K).$$

The case when $\Theta$ is the character of a unitarily induced representation from a representation in the limits of the discrete series is identical since in that case $\theta$

is the Fourier transform of a $\varphi_f^L$ with $L$ a non-compact Cartan subgroup. We now deal with the case when $\Theta$ is not necessarily unitarily induced. Let $\Theta_\nu = \mathrm{Ind}_P{}^G(\pi \otimes \nu \otimes 1)$. If $P = MAN$ then the same proof as in Lemma 3.3 and Theorem 3.4 shows that, for $\psi \in C^\infty(K)$,

$$\mathrm{tr}\, \pi_K(\psi) = \int_K \mathrm{tr}\, \pi_{M_K}(\psi^x \mid_{M_K})\, dx.$$

Here $M_K = M \cap K$ and $\mathfrak{m}_K$ is its Lie algebra. For $\psi \in C_c^\infty(\mathfrak{k} \cap \Omega)$ this formula reads

$$\theta_\nu{}^K(\psi) = \int_K \theta^{M_K}(\xi_K/\xi_{M_K}\, \psi^x \mid_{\mathfrak{m}_K})\, dx \tag{3}$$

where $\xi_K$ and $\xi_{M_K}$ are the functions defined in Theorem 3.1. Thus if $\theta^{M_K}$ has an asymptotic expansion at 0, $\theta_\nu{}^K$ also has an asymptotic expansion. The formulas mentioned in this theorem plus the ones in Theorem 3.5 imply the following. Suppose $\pi$ is in the limits of the discrete series. Let

$$\theta_\nu \sim \sum t^i E_i$$

and

$$\theta_\nu{}^K \circ \xi/\xi_K \sim \sum t^j F_j\,.$$

Then for $f \in C_c^\infty(\mathfrak{k}^*)$,

$$F_j(\mathscr{F}_{\mathfrak{k}} f) = \hat{E}_j(f_K). \tag{4}$$

(This holds for unitary $\nu$ by the arguments above, and in general by analytic continuation.)

Since any character is a linear combination of such $\Theta_\nu$ the inclusion (2) follows from formula (4). Using formula (3) and the statements about $AS(\Theta_\nu)$ in Theorem 3.5 and the equality in (2) for discrete series we can conclude equality in (2) for the character of just one induced representation.

Finally we show $K(\pi) = AS(\theta^K)$. Let $\delta \in \hat{K}$ and $\chi_\delta$ be its character. Then for $\varphi \in C^\infty(K)$,

$$\Theta^K(\varphi) = \sum_{\delta \in \hat{K}} m(\delta)\, \chi_\delta(\varphi).$$

On the other hand

$$\theta^K(\varphi_t) \sim \sum t^i D_i(\varphi) \qquad \text{for} \quad \varphi \in C_c^\infty(\mathfrak{k} \cap \Omega).$$

Since the $D_i$ are $K$-invariant we may assume $\varphi^x = \varphi$ for all $x \in K$. Substituting $\mathscr{F}_{\mathfrak{k}}\varphi$ for $\varphi$ we get

$$\theta^K(\mathscr{F}_{\mathfrak{k}}\varphi) = \sum_{\delta\in\hat{K}} m(\delta)\, d(\delta)\, \varphi(\delta)$$

$$\theta^K(\mathscr{F}_{\mathfrak{k}}\varphi_t) = \sum_{\delta\in\hat{K}} m(\delta)\, d(\delta)\, \varphi(t\delta)$$

where $d(\delta)$ is the dimansion of the representation $\delta$. Thus

$$\sum_{\delta\in\hat{K}} m(\delta)\, d(\delta)\, \varphi(t\delta) \sim \sum t^i \hat{D}_i(\varphi).$$

Suppose $\mu \in K(\pi)$ but $\mu \notin AS(\theta^K)$. Then there is a neighborhood of $V_\mu$ of $\mu$ such that supp $\varphi \subseteq V_\mu$ implies $\hat{D}_i(\varphi) = 0$ for all $i$. But then

$$\lim_{t\to 0} \sum_{\delta} m(\delta)\, d(\delta)\, \varphi(t\delta) = 0.$$

Let $\varphi \geqslant 0$. There is a sequence $t_n\delta_n \to \mu$. Assume $\varphi \equiv 1$ in a small neighborhood of $\mu$. Then

$$\sum_{\delta} m(\delta)\, d(\delta)\, \varphi(t_n\delta) \geqslant m(\delta_n)\, d(\delta_n)\, \varphi(t_n\delta_n) \geqslant 1$$

for $n$ large enough, a contradiction. Conversely, let $\mu \in AS(\theta^K)$ but $\mu \notin K(\pi)$. Then there is a set $B_\mu(\epsilon) = \{\delta \in \mathfrak{k}^*: \|\delta\| = 1, \|\delta - (\mu/\|\mu\|)\| < \epsilon\}$ such that, with at most finitely many exceptions, if $m(\delta) > 0$, then $\delta \notin V_\mu(\epsilon) = \{\gamma: \alpha\gamma \in B_\mu(\epsilon)$ for some $\alpha \in \mathbb{R}^+\}$. Let $\varphi$ be such that supp $\varphi \subseteq \mathrm{Ad}\, K \cdot V_\mu(\epsilon)$. Then

$$\sum m(\delta)\, d(\delta)\, \varphi(t\delta) \equiv 0 \qquad \text{for } t \text{ small}$$

so all $\hat{D}_i(\varphi) = 0$. Thus $\mu \notin AS(\theta^K)$, a contradiction. The proof is now complete.

We now determine the asymptotic $K$-types for the discrete series more precisely. Consider $G$ such that rank $G$ = rank $K$. Assume $\mathfrak{t} \subseteq \mathfrak{k}$ is a Cartan subalgebra. Let $\mathfrak{t}'$ be the set of regular elements and choose compatible orderings for the root systems $\Delta(\mathfrak{g}_c, \mathfrak{t}_c)$ and $\Delta(\mathfrak{k}_c, \mathfrak{t}_c)$. Let $\mathfrak{t}^+$ be the set of regular elements in a positive chamber for $\mathfrak{k}$ and $\mathfrak{t}^+ = \bigcup_{j=1}^{s} \mathfrak{t}^j$ where $\mathfrak{t}^j$ are positive chambers for $\mathfrak{g}$.

According to [4] the discrete series is parametrized by a lattice $L \subseteq \mathfrak{t}^*$. Using the identification between $\mathfrak{g}$ and $\mathfrak{g}^*$ given by $B(\ ,\ )$ we can identify any $\lambda \in L$ with a $Z_\lambda \in \mathfrak{t}^j$ such that, according to [9], the eigendistribution $\theta_\lambda$ defined before, equals the Fourier transform of $\varphi_f{}^j(Z_\lambda)$ (up to a constant). Then as pointed out earlier, it is known that $\varphi_f{}^j \in \mathscr{S}(\mathfrak{t}^j)$ so $\varphi_f{}^j(tZ_\lambda) \sim \sum t^i D_{i,\lambda}(f)$ at $t = 0$. Clearly, m supp $D_{i,\lambda} = AS(\Theta_\lambda)$.

PROPOSITION 3.7. *Let*

$$\eta = \{X \in \mathfrak{g}: X = \lim t_i \,\mathrm{Ad}\, x_i Z_\lambda \textit{ where } t_i \to 0\}.$$

*Then $\eta = AS(\Theta_\lambda)$ and $\eta$ does not depend on $\lambda$ but only on the chamber $\mathfrak{t}^j$ which contains $Z_\lambda$.*

*Proof.* Consider any $X \in \eta$. Clearly $X$ is nilpotent. We may assume $X$ is such that $X \notin \operatorname{cl} \operatorname{Ad} G \cdot (Y) - \operatorname{Ad} G \cdot (Y)$ for any other $Y \in \eta$. It is enough to show $O(X) = \operatorname{Ad} G \cdot X$ is contained in the support of some $D_{i,\lambda}$. We will show that the $G$-invariant measure on $O(X)$ is the limit of $t^\alpha \varphi_f{}^j(tZ)$ for some $\alpha$ and a certain class of $f \in C^\infty(\mathfrak{g})$ when $t \to 0^+$. Since $AS(\Theta_\lambda) \subseteq \eta$ trivially, this will complete the proof. We will use the following facts without proof.

(The transverse described in Lemma 3.8 was first introduced by Harish-Chandra. As stated here, Lemma 3.8 was first proved by R. Rao in connection with some results (unpublished) on the measures supported on nilpotent orbits.)

LEMMA 3.8. *Let $q = \dim O(X)$. Consider the Lie triple $\{X, H, Y\}$. Let $\mathfrak{z}_Y = \operatorname{Cent}_\mathfrak{g} Y$ and $\mathscr{U} = X + \mathfrak{z}_Y$. Then the map $\psi\colon G \times \mathscr{U} \to \mathfrak{g}$ given by $\psi(x, v) = \operatorname{Ad} x \cdot v$ is a submersion. In particular, for any $v \in \mathscr{U}$, $\mathscr{U}$ is transverse to $O(v)$ and $\mathscr{U}^G = \operatorname{Ad} G \cdot \mathscr{U}$ is open. $O(X) \cap \mathscr{U}$ is a finite set. Let $\beta \in C_c^\infty(G \times \mathscr{U})$. Then there is $f_\beta \in C_c^\infty(\mathscr{U}^G)$ such that $\beta \mapsto f_\beta$ is well defined and onto $C_c^\infty(\mathscr{U}^G)$. Then*

$$\int_{G/G_v} f_\beta(\operatorname{Ad} x \cdot v)\, dx = \int_G \int_{\mathscr{U} \cap O(v)} \beta(x, u)\, d\omega(u)\, dx.$$

*For $v$ semisimple, $O(v) \cap \mathscr{U}$ is closed and $d\hat{E}(u)$ is a measure defined by some $C^\infty$-form. For $v = X$ it is a linear combination of delta functions.*

*If $t > 0$ then*

$$f_\beta(tv) = t^{-(n-q)/2} f_{\beta_t}(v)$$

*where $\beta_t = \beta \circ \varphi$ and $\varphi\colon G \times \mathscr{U} \to G \times \mathscr{U}$ is given by*

$$\varphi(x, X + u) = (x\gamma_t^{-1}, X + u_t), \qquad \gamma_t = \exp(-\tfrac{1}{2} \log tH)$$

*and $u_t = t\gamma_t u$.*

We get back to the proof of Proposition 3.7.

Assume $Z_0 \in \mathfrak{t}^j$ such that $O(tZ_0)$ contains $X$ in its closure. Then we may assume $O(Z) \cap \mathscr{U} \neq \varnothing$. Assume

$$O(Z) \cap \mathscr{U} = \{X + u\colon u \text{ in some closed submanifold of } \mathfrak{z}_Y\}.$$

Then

$$O(tZ) \cap \mathscr{U} = \{X + t \operatorname{Ad} \gamma_t \cdot u\colon X + u \in O(Z) \cap \mathscr{U}\}.$$

We claim that $\| u \|$ has to be bounded. Indeed, the set $O(tZ) \cap \mathscr{U}$ can have only elements in its closure such that $\dim O(v) = q$. Otherwise there would be

$Y \in \eta$ such that cl $O(Y) - O(Y)$ contains $X$. By dimension considerations and the fact that such a set is algebraic, the boundary of $O(tZ) \cap \mathscr{U}$ is finite. Assume that $\| u \|$ is unbounded. Let $\mathfrak{z}_Y = \sum_{i \geqslant 0} V_i$ where $V_i$ are eigenspaces for ad $H$ with eigenvalue $-i$. Then if $u = \sum u_i$,

$$t \operatorname{Ad} \gamma_t u = \sum t^{1+(i/2)} u_i .$$

If $\| u \|$ is unbounded, we can find sequences $u_s$ and $t_s \to 0$ such that $\| t_s u_s \| \to \infty$. This gives a contradiction. It follows that, since $O(Z) \cap \mathscr{U}$ is also closed, that $O(Z) \cap \mathscr{U}$ is conpact. Thus

$$\begin{aligned} t^{(n-q)/2} \int_{G/G_Z} f_\beta(t \operatorname{Ad} xZ)\, dx &= \int_{G/G_Z} f_{\beta_t}(\operatorname{Ad} xZ)\, dx \\ &= \int_G \int_{\mathscr{U} \cap O(Z)} \beta_t(x, u)\, d\omega(u)\, dx \\ &= \int_G \int_{\mathscr{U} \cap O(Z)} \beta(x, X + t\gamma_t u)\, d\omega(u)\, dx. \end{aligned}$$

Since $\mathscr{U} \cap O(Z)$ is compact and $X + t\gamma_t u \to X$ as $t \to 0$, it follows at once that $O(Z) \cap \mathscr{U} = \{X\}$ and

$$\lim_{t \to 0^+} t^{(n-q)/2} \int_{G/G_{Z_0}} f_\beta(t \operatorname{Ad} xZ_0)\, dx = c_{Z_0} \int_{G/G_X} f_\beta(\operatorname{Ad} x \cdot X)\, dx$$

and $c_{Z_0} > 0$. In terms of the function $\varphi_f$ we can write

$$\lim_{t \to 0^+} t^{-\alpha} \varphi_f{}^j(tZ_0) = c_{Z_0} \int_{G/G_X} f(\operatorname{Ad} x \cdot X)\, dx$$

if supp $f \subseteq \mathscr{U}^G$. Or we can write

$$\varphi_f{}^j(0; Z_0{}^\alpha) = c_{Z_0} \int_{G/G_X} f(\operatorname{Ad} x \cdot X)\, dx, \qquad \operatorname{supp} f \subseteq \mathscr{U}^G.$$

But now since $\varphi_f{}^j$ is a Schwartz function, it follows that

$$\lim_{t \to 0^+} \varphi_f{}^j(tZ; Z_0{}^\alpha) = c_{Z_0} \int_{G/G_X} f(\operatorname{Ad} x \cdot X)\, dx, \qquad \operatorname{supp} f \subseteq \mathscr{U}^G.$$

This is enough to conclude that $X \in$ cl $O(tZ)$ for any other $Z \in \mathfrak{t}^j$ as well. These arguments complete the proof.

COROLLARY 3.9. *Let $\theta$ be an invariant distribution supported on a set of nilpotent elements $\eta$. Suppose $\theta$ is homogeneous of degree $n - \alpha$. Let $n - r$ be the largest dimension of an orbit contained in $\eta$. Then $\alpha \geqslant (n - r)/2$. In particular, if $\alpha = (n - r)/2$ then $\theta$ is a linear combination of measures.*

*Proof.* Let $X \in \eta$. It is enough to consider what happens when we consider $\theta\,|_{\mathscr{U}G}$. Let $dx$ be the Haar measure on $G$, $dZ$ and $du$ the Euclidean measures on $\mathfrak{g}$ and $\mathfrak{z}_Y$ respectively. We can lift $\theta$ to a distribution $dx \otimes \tau_\theta$ where $\tau_\theta$ is a distribution on $\mathscr{U}$ by

$$(dx \otimes \tau_\theta)(\beta) = \theta(f_\beta).$$

Then, if $\theta$ is supported on $O(X)$, $\tau_\theta$ is supported on

$$O(X) \cap \mathscr{U} = \{X\}$$

by the previous argument. Thus $\tau_\theta = D^s\delta_X$ where $s$ is a multiindex and $D^s$ a differential operator on $C^\infty(\mathfrak{z}_Y)$. Let $\beta(x, X + u) = \beta_1(x)\,\beta_2(X + u)$. We have the relations

$$\theta(R_t f_\beta) = t^\alpha \theta(f_\beta) = t^\alpha \int_G \beta_1(x)\,dx \cdot \tau_\theta(\beta_2),$$

$$\theta(R_t f_\beta) = \theta(t^{-(n-r)/2} f_{\beta_t}) = t^{-(n-r)/2} \int_G \beta_1(x\gamma_t)\,dx \cdot \tau_\theta(\beta_{2,t})$$

where

$$\beta_{2,t}(X + u) = \beta_2\left(X + \sum_{j\geqslant 0} t^{1+(j/2)} u_j\right).$$

Thus

$$\tau_\theta(\beta_{2,t}) = t^{\alpha+(n-r)/2}\,\tau_\theta(\beta_2)$$

for all $\beta_2 \in C_c^\infty(\mathscr{U})$. By using the fact that $\tau_\theta$ is a derivative of the delta function and the relation

$$D_j^{s_j}(\beta_{2,t})|_{u=0} = (t^{1+(j/2)})^{|s_j|}\, D_j{}^s\beta\,|_{u=0}$$

for any multiindex $s_j$ and $D_j$ a differential operator on $C^\infty(V_j)$, we get the relation,

$$\sum |s_j| \left(1 + \frac{j}{2}\right) = \alpha + (n-r)/2.$$

Thus, $\alpha + (n - r)/2 > 0$ if one of the $|s_j| > 0$ and $\alpha + (n - r)/2 = 0$ if and only if $|s_j| = 0$ for all $j$. This completes the proof.

The set described in Proposition 3.7 is being computed on a case-by-case basis by D. Peterson; his results are complete at least for $SU(p, q)$.

## 4. Relations with Primitive Ideals and Coherent Continuation

Let $\pi$ be an irreducible admissible representation of $G$, $\Theta_\pi$ its character, and $\theta_\pi$ the corresponding distribution near 0 on $\mathfrak{g}$. In this section we will relate the asymptotic expansion of $\theta_\pi$ to the primitive ideal $I_\pi$ associated to $\pi$. Suppose (as we may without changing $\Theta_\pi$ or $I_\pi$) that $\pi$ is realized on a Hilbert space $\mathscr{H}$, and that $\pi|_K$ is unitary. Let $X \subseteq \mathscr{H}$ denote the subspace of $K$-finite vectors. Then $X$ is a module for the (complexified) enveloping algebra $U(\mathfrak{g}_c)$ of $\mathfrak{g}$; we denote this action by $\pi$ also. The ideal $I_\pi \subseteq U(\mathfrak{g}_c)$ is defined to be the annihilator of $X$. We have a homogeneous ideal $\mathrm{gr}(I_\pi) \subseteq S(\mathfrak{g}_c)$, the symmetric algebra of $\mathfrak{g}_c$. The *associated cone* $\mathscr{V}(I_\pi)$ is defined to be the zero variety of $\mathrm{gr}(I_\pi)$ inside $\mathfrak{g}_c^*$ (which is in a natural way the maximal spectrum of $S(\mathfrak{g}_c)$. The *Gelfand-Kirillov dimension* $d(I_\pi)$ may be defined to be the (complex) dimension of $\mathscr{V}(I_\pi)$. One knows that $\mathscr{V}(I_\pi)$ is a finite union of nilpotent orbits of the complex adjoint group $G_{\mathbb{C}}$ in $\mathfrak{g}_c$. (For all this see [1].) In interpreting the following theorem, one should bear in mind the (easily verified) fact that if $X \in \mathfrak{g}^*$, then

$$\dim_{\mathbb{C}}(G_{\mathbb{C}} \cdot X) = \dim_{\mathbb{C}}(G \cdot X).$$

THEOREM 4.1. *With notation as above,*

$$AS(\Theta_\pi) \subseteq \mathscr{V}(I_\pi) \cap \mathfrak{g}^*.$$

*Furthermore,*

$$d(I_\pi) = \dim_{\mathbb{R}} AS(\Theta_\pi) = \dim_{\mathbb{C}} \mathscr{V}(I_\pi) = \dim_{\mathbb{R}} \mathscr{V}(I_\pi) \cap \mathfrak{g}^*.$$

*The first term in the asymptotic expansion of the lift $\theta_\pi$ of $\Theta_\pi$ to $\mathfrak{g}$ is a linear combination of Fourier transforms of invariant measures on nilpotent orbits of dimension $d(I_\pi)$, and hence has homogeneity degree $-\frac{1}{2}d(I_\pi)$.*

*Proof.* Write $\theta_\pi \sim \sum_{i=-r}^{\infty} t^i D_i$. To prove the first assertion, it suffices to show that if $p \in S(\mathfrak{g})$ is a polynomial function on $\mathfrak{g}_c^*$ vanishing on $\mathscr{V}(I_\pi)$, and $i \geqslant -r$, then there is some integer $N = N(p, i) \geqslant 1$ such that

$$\hat{D}_i \circ p^N = 0.$$

Since such a polynomial lies in $\sqrt{\mathrm{gr}(I_\pi)}$ by the Nullstellensatz, we may as well assume $p \in \mathrm{gr}(I_\pi)$. By a standard argument, if this holds for $p_1$ and $p_2$, then it holds for $p_1 + p_2$; so we may assume $p$ is homogeneous, say of degree $s$. Let $\partial(p)$ be the constant coefficient linear differential operator on $\mathfrak{g}$ associated to $p$; then what we must show is

$$D_i \circ \partial(p)^N = 0. \tag{4.2}$$

Let $\{U_j\}_{j=0}^{\infty}$ be the canonical filtration of $U(\mathfrak{g}_c)$, so that $U_j/U_{j-1} \simeq S^j(\mathfrak{g}_c)$. Choose an element $u \in I_\pi \cap U_s$ such that $p = u + U_{s-1} \in U_s/U_{s-1} \simeq S^s(\mathfrak{g}_c)$. Consider $u$ as a left invariant differential operator on $G$. If $\mathscr{H}^\infty$ is the space of smooth vectors in $\mathscr{H}$, then $\pi(u)$ is a well defined operator on $\mathscr{H}^\infty$, and if $f \in C_c^\infty(G)$, then

$$\pi(u \cdot f) = \pi(f)\,\pi(u).$$

Choosing an orthonormal basis $\{v_i\}$ of $\mathscr{H}$ consisting of $K$-finite vectors, we find

$$\begin{aligned}\Theta_\pi(u \cdot f) &= \sum \langle \pi(u \cdot f)v_i, v_i\rangle \\ &= \sum \langle \pi(f)\,\pi(u)v_i, v_i\rangle = 0\end{aligned}$$

since $u \in I_\pi$; so $\Theta_\pi \circ u = 0$. Let $\tilde{u}$ be the lift of the distribution $u$ to $\mathfrak{g}$ defined as in section 3; then

$$\theta_\pi \circ \tilde{u} = 0. \tag{4.3}$$

Clearly $\tilde{u}$ is a differential operator of order $s$. Write

$$\tilde{u} = \sum_{i=-s}^{\infty} \tilde{u}_i,$$

with

$$\tilde{u}_i = \sum_{\substack{|\beta|-|\alpha|=i \\ |\alpha|\leqslant s}} c_{\alpha\beta} X^\beta D^\alpha$$

in terms of some basis $\{X_i\}$ of $\mathfrak{g}$. By the Campbell-Baker-Hausdorff formula, one sees easily that

$$\tilde{u}_{-s} = \partial(p).$$

We now establish (4.2) by induction on $i$. By (4.3) and Corollary 2.4,

$$0 = \theta_\pi \circ \tilde{u} \sim \sum_{j=-r-s} t^j E_j,$$

where

$$E_j = \sum_{m+i=j} D_m \circ \tilde{u}_i.$$

Since $E_j = 0$, we have for each $i$ an equation of the form

$$D_i \circ \partial(p) = D_i \circ \tilde{u}_{-s} = -\sum_{j=-r}^{i-1} D_j \circ \tilde{u}_{i-j-s}. \tag{4.4}$$

Suppose then that (4.2) is known for $j < i$. Choose $N = N(p, i)$ so large that for $-r \leqslant j \leqslant i-1$,

$$\tilde{u}_{i-j-s} \circ \partial(p)^{N-1} = \partial(p)^{N(p,j)} Q_j ,$$

for some differential operator $Q_j$; this is possible since $\tilde{u}_{i-j-s}$ has polynomial coefficients. Applying $\partial(p)^{N-1}$ to both sides of (4.4), we get

$$\begin{aligned} D_i \circ \partial(p)^N &= -\sum_{j=-r}^{i-1} D_j \circ \tilde{u}_{i-j-s}\, \partial(p)^{N-1} \\ &= -\sum_{j=-r}^{i-1} D_j \circ \partial(p)^{N(p,j)} Q_j \\ &= 0 \end{aligned}$$

by induction. This proves the first assertion of the theorem. By the remarks preceding the theorem, it follows that

$$d(I_\pi) = \dim_{\mathbb{C}} \mathscr{V}(I_\pi) \geqslant \dim_{\mathbb{R}} \mathscr{V}(I_\pi) \cap \mathfrak{g}^* \geqslant \dim_{\mathbb{R}} AS(\Theta_\pi). \tag{4.5}$$

Write $2d' = \dim_{\mathbb{R}} AS(\Theta_\pi)$. By Corollary 3.9, $AS(\Theta_\pi)$ cannot support a homogeneous invariant distribution of degree less than $-d'$, so the order $-r$ of $\theta_\pi$ at 0 (recall that $\theta_\pi \sim \sum_{i=-r}^{\infty} t^i D_i$; to say that $\theta_\pi$ has order $-r$ means $D_{-r} \neq 0$) satisfies $r \leqslant d'$. Suppose we can show that $2r \geqslant d(I_\pi)$. This will force equalities in (4.5), proving the second assertion; and the last will follow from Corollary 3.9. So it is enough to show $2r \geqslant d(I_\pi)$. Let $\Theta_\pi{}^K$ be the $K$-character of $\pi$, and $\theta_\pi{}^K$ its lift to $\mathfrak{k}$. By Theorem 3.6,

$$\theta_\pi{}^K \sim \sum_{i=-s}^{\infty} t^i E_i;$$

we assume $E_{-s} \neq 0$. The proof of Theorem 3.6 shows how to compute the $E_i$ from the $D_i$; in particular we have $s \leqslant r$, so it suffices to show that $2s \geqslant d(I_\pi)$. Choose $f \in C_c{}^\infty(\mathfrak{k})$, invariant under $\mathrm{Ad}(K)$, so that $\hat{f} \geqslant 0$, and $\hat{f}(\mu) \geqslant 1$ for $|\mu| \leqslant 1$ (in some norm on $\mathfrak{k}^*$). Then $(f_t)\hat{} \geqslant 1$ for $|\mu| \leqslant t^{-1}$. With notation as in the proof of Theorem 3.6, this gives

$$\begin{aligned} \theta_\pi{}^K(f_t) &= \sum_{\mu \in \hat{K}} m_\mu (f_\alpha)\hat{}(\mu) \cdot \dim(\mu) \\ &\geqslant \sum_{\substack{\mu \in \hat{K} \\ |\mu| \leqslant t^{-1}}} m_\mu \cdot \dim(\mu). \end{aligned}$$

By Theorem 1.2 of [11], this last term satisfies

$$\sum_{\substack{\mu \in \hat{K} \\ |\mu| \leqslant t^{-1}}} m_\mu \cdot \dim(\mu) \geqslant A(t^{-1})^{\frac{1}{2}d(I_\pi)}.$$

Thus $-s \leqslant -\frac{1}{2}d(I_\pi)$ as desired. Q.E.D.

We turn now to the behavior of asymptotic expansions under coherent continuation. For more details of the definitions, see [7], [10], or [11]. Fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, and some translate $\Lambda \subseteq \mathfrak{h}_c^*$ of the lattice of weights of finite dimensional representations of $G$. Let $\{\Theta(\lambda) \mid \lambda \in \Lambda\}$ be a coherent family of virtual characters of $G$. This means that each $\Theta(\lambda)$ is an integral combination of irreducible characters of infinitesimal character $\lambda$, and if $F$ is a finite dimensional representation of $G$ with weights $\Delta(F) \subseteq \mathfrak{h}_c^*$ (counted with multiplicity) then

$$\Theta(\lambda) \cdot \Theta(F) = \sum_{\mu \in \Delta(F)} \Theta(\lambda + \mu). \tag{4.6}$$

Fix a system $\Delta^+ \subseteq \Delta(\mathfrak{g}_c, \mathfrak{h}_c)$ of positive roots. We will assume in addition that whenever $\lambda$ is dominant (i.e. $2\langle\alpha, \lambda\rangle/\langle\alpha, \alpha\rangle$ is not a negative integer for $\alpha \in \Delta^+$) then $\Theta(\lambda)$ is either 0 or the character of an irreducible representation $\pi(\lambda)$, and that the latter is the case whenever $\lambda$ is nonsingular. Any irreducible representation $\pi$ occurs as some $\pi(\lambda)$ in such a coherent family, which is unique up to obvious equivalences. In this case the non-zero $\pi(\lambda)$ have annihilators $I_{\pi(\lambda)}$ with a common Gelfand-Kirillov dimension $2d$; and every constituent of each virtual character $\Theta(\lambda)$ has Gelfand-Kirillov dimension at most $2d$ (cf. [11]). Accordingly we can write

$$\theta_\lambda \sim \sum_{i=-d}^{\infty} t^i D_i(\lambda).$$

Now the $D_i(\lambda)$ were constructed in section 3 from the formulas for $\theta_\lambda$. Since these formulas depend nicely on $\lambda$ (cf. [7], [10]), we deduce

PROPOSITION 4.7. *Let $\{\Theta_\lambda \mid \lambda \in \Lambda\}$ be a coherent family of virtual characters as above, with $\Theta_\lambda$ the character of an irreducible representation $\pi(\lambda)$ such that $d(I_{\pi(\lambda)}) = 2d$ whenever $\lambda$ is dominant and regular. For fixed $i$, the distributions*

$$\{D_i(\lambda) \mid \lambda \in \Lambda\}$$

*span a finite dimensional space. Let $\{E_{ij}\}$ be a basis of this space. Then*

$$D_i(\lambda) = \sum_j p_{ij}(\lambda) E_{ij},$$

*with $p_{ij}$ a polynomial function on $\mathfrak{h}^*$, homogeneous of degree $i + \frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h})$. If $i = -d$, then the $p_{ij}$ are harmonic.*

*Proof.* Everything but the last statement follows from the remarks preceding the proposition and straightforward computation. For the last statement, notice that if $F$ is a finite dimensional representation of $G$, lifting (4.6) to $\mathfrak{g}$ gives

$$\theta(\lambda) \cdot T(F) = \sum_{\mu \in \Delta(F)} \theta(\lambda + \mu);$$

here $T(F)$ is a smooth function near 0 of $\mathfrak{g}$, and

$$T(F)(0) = \dim(F).$$

Applying Corollary 2.4, we get

$$D_{-s}(\lambda) \cdot \dim(F) = \sum_{\mu \in \Delta(F)} D_{-s}(\lambda + \mu),$$

$$(\dim F) \cdot \sum p_{-s,j}(\lambda) E_{-s,j} = \sum_{\mu \in \Delta(F)} \sum_j p_{-s,j}(\lambda + \mu) E_{-s,j}\,.$$

Since the $E_{-s,j}$ are linearly independent, this gives

$$\dim(F) \cdot p_{-s,j}(\lambda) = \sum_{\mu \in \Delta(F)} p_{-s,j}(\lambda + \mu).$$

By Lemma 4.3 of [11], it follows that $p_{-s,j}$ is harmonic. (Since $p_{-s,j}$ is already known to be a polynomial function, one could give a simple direct proof.) Q.E.D.

COROLLARY 4.8. *With notation as in Proposition 4.7, there is a harmonic polynomial $c$, homogeneous of degree $\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h}) - d$, such that for $\lambda$ dominant,*

$$\lim_{t \to \infty} t^{-d} \sum_{|\mu| \leqslant t} \dim \mu \cdot m_{\cdot}(\mu) = c(\mu).$$

*Here $m_\lambda(\mu)$ is the multiplicity of $\mu$ in $\pi(\lambda)$.*

*Proof.* Write $\theta^K(\lambda) \sim \sum_{i=-d}^{\infty} t^i E_i$. By (4) in the proof of Theorem 3.6, Theorem 4.1, and Proposition 4.7, $\hat{E}_{-d}$ is a linear combination of homogeneous measures on $\mathfrak{k}^*$, with coefficients homogeneous harmonic polynomials in $\lambda$ of degree $\frac{1}{2}\dim(\mathfrak{g}/\mathfrak{h}) - d$. In particular, if $f \in C_c^\infty(\mathfrak{k})$ is $\mathrm{Ad}(K)$ invariant, there is a harmonic polynomial $c_f$ of the specified degree, such that

$$\lim_{t \to \infty} t^{-d} \sum_{|\mu| \leqslant t} \dim \mu \cdot m_{\cdot}(\mu) \hat{f}(t^{-1}\mu) = c_f(\lambda).$$

The corollary follows by an obvious approximation argument. Q.E.D.

This corollary establishes Conjecture 5.1 of [11].

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