On the minimal diameter of hyperbolic surfaces

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Goal: study the minimal possible diameter of hyperbolic surfaces with high genus $g$: asymptotic to $1 \times \log g$.

Small diameter $\approx$ highly connected objects.

Random (say, 3-regular) graphs are also very connected: probabilistic method, very common in combinatorics.

Probabilistic method in hyperbolic geometry.
I The diameter of 3-regular random graphs
II Notions of hyperbolic geometry
III The diameter of hyperbolic surfaces
IV Ideas of the proof
V Perspectives
The diameter of 3-regular random graphs

- $G_n$ obtained from $n$ vertices with 3 half-edges each, by matching the half-edges uniformly at random (connected with proba $1 - O(1/n)$).
- Diameter: maximal graph distance between two vertices.

**Theorem (Bollobas–Fernandez de la Vega, 1982)**

$$\frac{\text{diam}(G_n)}{\log_2 n} \xrightarrow{(P)} 1.$$  

- Lower bound: a ball of radius $r$ has size at most $3 \times 2^r$, so the diameter is $\geq \log_2 n$. 

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Minimal diameter of hyperbolic surfaces
Diameter of random graphs: proof

- Upper bound: it is enough to prove that for any two fixed vertices \( v_1, v_2 \):

\[
P \left( (1 + \varepsilon) \log_2 n \leq d_{G_n}(v_1, v_2) < +\infty \right) = o \left( \frac{1}{n^2} \right).
\]

- Explore balls of radius \( r = \frac{1+\varepsilon}{2} \log_2 n \) around \( v_1 \) and \( v_2 \), and try to connect them.

- If no "bad step", we would have \( |\partial B_r(v_1)| = 3 \times 2^r = 3n^{\frac{1+\varepsilon}{2}} \).
But $\mathbb{P}(\text{bad step at time } i) \leq \frac{i+2}{3n}$, with independence over $i$.

Consequence: with probability $1 - o\left(\frac{1}{n^2}\right)$:
- $O(1)$ bad steps in the ball of radius $\frac{1-\varepsilon}{2} \log_2 n$ bad steps,
- $o\left(n^{\frac{1+\varepsilon}{2}}\right)$ bad steps between distances $\frac{1-\varepsilon}{2} \log_2 n$ and $\frac{1+\varepsilon}{2} \log_2 n$,
- so $|\partial B_r(v_1)| \geq \delta n^{\frac{1+\varepsilon}{2}}$ w.h.p., and the same is true around $v_2$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{Diagram illustrating the relationship between $B_r(v_1)$ and $B_r(v_2)$.}
\end{figure}
If \( \mathcal{B}_r(v_1) \cap \mathcal{B}_r(v_2) \neq \emptyset \), we are done.

If not, each loose half-edge on \( \partial \mathcal{B}_r(v_1) \) has probability \( \frac{|\partial \mathcal{B}_r(v_2)|}{n} \geq \delta n^{-\frac{1-\varepsilon}{2}} \) to be connected to \( \mathcal{B}_r(v_2) \). So

\[
\mathbb{P}( \mathcal{B}_r(v_1) \text{ and } \mathcal{B}_r(v_2) \text{ not directly linked} ) \leq \left( 1 - \delta n^{-\frac{1-\varepsilon}{2}} \right)^{\frac{1+\varepsilon}{2}} \\
\leq \exp \left( -\delta n^\varepsilon \right) \\
= o \left( \frac{1}{n^2} \right),
\]

and \( d_{\mathcal{G}_n}(v_1, v_2) \leq 2r + 1 \leq (1 + \varepsilon) \log_2 n \) with very high probability.
Hyperbolic geometry

- The *hyperbolic plane* $\mathbb{H}$ can be seen as the unit disk, equipped with the metric

$$ds^2 = \frac{4dx^2}{1 - |x|^2}.$$

- **Curvature:** $|B_\varepsilon(x)| = \pi\varepsilon^2 - \frac{\pi}{12}\varepsilon^4 K(x) + o(\varepsilon^4)$.

- Riemann uniformization theorem: $\mathbb{H}$ is the unique simply connected surface with constant curvature equal to $-1$.
A compact hyperbolic surface $S$ is a 2d manifold equipped with a Riemannian metric with constant curvature $-1$. We consider closed surfaces, i.e. no boundary.

Gauss–Bonnet formula: $\int_S K(x)\,dx = 2\pi(2 - 2g)$, where $g$ is the genus of the surface, i.e. the number of holes. So $g \geq 2$.

Equivalent definitions:
- $S$ is locally isometric to $\mathbb{H}$,
- $S$ is a quotient of $\mathbb{H}$ (by a nice enough group action),
- $S$ is a surface equipped with a conformal structure.
Existence, but no uniqueness: for $g \geq 2$, hyperbolic metrics on a genus $g$ surface form a $(6g - 6)$-dimensional space $M_g$ called the *moduli space*.

One way to build a lot of them is to use *pants*.

For any $\ell_1, \ell_2, \ell_3 \geq 0$, there is a unique surface isomorphic to the sphere minus 3 disjoint disks, such that:

- the boundaries of the three disks are closed geodesics with lengths $\ell_1, \ell_2, \ell_3$;
- the curvature is $-1$ outside of the boundary.
By gluing $2g - 2$ pairs of pants such that the lengths of the boundaries match two by two, we can build many hyperbolic surfaces.

6g − 6 degrees of freedom: $3g - 3$ for the lengths of the cycles, and $3g - 3$ for the twists.

Conversely, every hyperbolic surface of genus $g$ can be cut by $3g - 3$ closed geodesics into $2g - 2$ pairs of pants.
Given a hyperbolic surface $S$, several natural quantities to look at, and try to optimize over the moduli space:
- diameter,
- spectral gap (eigenvalues of the Laplacian),
- Cheeger constant (isoperimetric inequalities),
- systole (length of the smallest closed geodesics).

All of these measure the "connectivity" of the surface.

In the context of hyperbolic surfaces, non-optimal bounds (constant factors) often obtained via arithmetic constructions [Brooks, Buser, Kim, Sarnak...].

A typical graph is very connected, so random graphs (like uniform 3-regular graphs) are close to optimal for these quantities.
Diameter of hyperbolic surfaces

- Diameter: maximal distance between two points of $S$.
- Easy: $\sup_{S \in M_g} \text{diam}(S) = +\infty$.

Lower bound for the minimal diameter: volume growth argument

- By Gauss-Bonnet, $\text{Area}(S) = 2\pi(2g - 2)$.
- In $\mathbb{H}$, the area of balls is $\text{Area}(B_r(x)) = 2\pi(\cosh(r) - 1)$.
- $S$ is a quotient of $\mathbb{H}$, so $\text{Area}(B_r(x)) \leq 2\pi(\cosh(r) - 1)$ in $S$.
- So if $B_r(x)$ covers $S$, then $\cosh(r) - 1 \geq 2g - 2$, so

$$\inf_{S \in M_g} \text{diam}(S) \geq \cosh^{-1}(2g - 1) = \log g + O(1).$$

- Best lower bound [Bavard 1996]: also $\log g + O(1)$. 
Main theorem

Theorem (B.–Curien–Petri, 2019)

We have

\[
\min_{S \in M_g} \text{diam}(S) = (1 + o(1)) \log g.
\]

- Construction: random gluing of pants!
- Start from \(2g - 2\) pants with perimeters \((a, a, a)\), and glue the \(6g - 6\) holes uniformly at random to obtain \(S_{g,a}\).
- Twist 0: the "centers" of two neighbour pants have the same projections on the boundary.

We show \(\text{diam}(S_{g,a}) \sim \frac{1}{\beta_a} \log g\) w.h.p., where \(\beta_a < 1\), but \(\beta_a \to 1\) as \(a \to \infty\).
For a fixed, the diameter of a pair of pants is a constant. Enough to bound distances between the centers of the pants.

Quick bound: $\text{diam}(S_{a,g}) \leq 2d_a \text{diam}(G_{2g-2}) \sim 2d_a \log_2 n$. After computing $d_a$ for $a \to +\infty$, we get $\approx 1.38 \log g$.

Not optimal: sometimes, there is a much shorter path.
Adapting the explorations

- Instead of using random graphs as a black box, adapt the proof!
- Adapt the exploration to the *hyperbolic* metric, instead of the graph distance.
- Ideal situation: the neighbourhood of one center looks like an infinite tree of pants $T_a$. We need to understand its growth!
Growth of the infinite tree of pants

- $B_r$: ball of radius $r$ around a center of the pants tree $T_a$. Let $|B_r|$ be the number of pants whose center is in $B_r$.

**Lemma**

We have $|B_r(T_a)| \sim C_a e^{\beta_a r}$, where $\beta_a \to 1$ as $a \to +\infty$.

- Sketch of proof: pants can be decomposed in two right-angled hexagons.

- Gluings with *twist* 0, so the red "weldings" match on neighbour pants.
Growth of the infinite pair of pants

- Hence, the tree of pants is the gluing of two copies of an infinite tree of right-angled hyperbolic hexagons:

- Above: infinite tree of hexagons for increasing values of $a$.
- The growth of hexagon trees corresponds to orbital counting for a subgroup of $PSL_2(\mathbb{R})$ generated by reflexions. This is well understood by geometers [Patterson–Sullivan, McMullen...]
As for graphs, we want to show, for any centers $c_1, c_2$:

$$\mathbb{P}\left(d_{hyp}(c_1, c_2) \geq \left(\frac{1+\varepsilon}{\beta_a}\right) \log g\right) = o\left(\frac{1}{g^2}\right).$$

We explore the balls of radius $r = \frac{1+\varepsilon}{2\beta_a} \log g$ around $c_1$ and $c_2$ for the hyperbolic metric on the infinite tree of pants.

As for graphs, we can bound the number of "bad" steps: the volume and boundary of $B_r(c_1)$ are at least a constant times what they would be in the tree of pants.

So $|\partial B_r(c_1)| \geq \delta \exp \left(\beta_a \frac{1+\varepsilon}{2\beta_a} \log g\right) = \delta g^{\frac{1+\varepsilon}{2}}$, and the same is true for $c_2$.

As for graphs, this implies that with very high probability, there is an edge between $B_r(c_1)$ and $B_r(c_2)$, so

$$d_{hyp}(c_1, c_2) \leq 2r + O(1) = \frac{1+\varepsilon}{\beta_a} \log g + O(1).$$
Error term? For random graphs $O(\log \log n)$. It should also be true here (for $a = \log \log n$).

Other natural models of random surfaces:
- Brooks–Makover surfaces (built by uniformizing random triangulations with unconstrained genus): diameter $\sim 2 \log g$ [BCP 2019+],
- Weil–Petersson random surfaces ($\approx "Lebesgue measure"$ on the space of hyperbolic surfaces): diameter $\leq 40 \log g$ [Mirzakhani 2013].

Study other quantities? Maximal spectral gap for hyperbolic surfaces? Cheeger constant?

Hypercubic manifolds in higher dimensions?
THANK YOU!