Midterm 1 Solutions Duration: 45 minutes

This test has ***5 questions*** *on* ***7 pages****, for a total of 50 points.*

- Read all the questions carefully before starting to work.
- Put your final answer in the boxes provided for each question where there is an answer-box provided.
- All questions are long-answer. You should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ____

Question:	1	2	3	4	5	Total
Points:	6	21	7	8	8	50
Score:						

1. **You must show all your work in order to receive full marks for these questions.**

3 marks (a) Compute

$$\arcsin\left(\sin\left(\frac{-10\pi}{3}\right)\right).$$

Solution :
Notice that

$$\sin\left(\frac{-10\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$$

and so, since $\pi/3 \in [-\pi/2, \pi/2]$,

$$\arcsin\left(\sin\left(\frac{-10\pi}{3}\right)\right) = \frac{\pi}{3}.$$

3 marks (b) Find where the following function is continuous

$$f(x) = \sqrt{\arcsin(x) - \pi/4}.$$

Solution :
The function is a composition of continuous functions, provided we restrict ourselves to values of x in the domain of $\arcsin(x)$ for which $\arcsin(x) \geq \pi/4$. This is precisely those x with

$$x \in \left[-\frac{1}{\sqrt{2}}, 1\right].$$

2. **You must show all your work in order to receive full marks for these questions.**

6 marks (a) If $f(x) = x^4 \cdot g(x)$ and $g(1) = -2$, while $g'(1) = 4$, then compute $f'(1)$.

Solution :

We have

$$f'(x) = 4x^3 \cdot g(x) + x^4 \cdot g'(x),$$

by the Product Rule. It follows that

$$f'(1) = 4 \cdot (-2) + 1 \cdot 4 = -4.$$

7 marks (b) Compute

$$\lim_{x \to 2} \frac{x^4 - 16}{x^3 - 8}$$

Solution :

We have

$$\frac{x^4 - 16}{x^3 - 8} = \frac{(x-2)(x+2)(x^2+4)}{(x-2)(x^2+2x+4)}$$

and so

$$\lim_{x \to 2} \frac{x^4 - 16}{x^3 - 8} = \lim_{x \to 2} \frac{(x-2)(x+2)(x^2+4)}{(x-2)(x^2+2x+4)} = \lim_{x \to 2} \frac{(x+2)(x^2+4)}{x^2+2x+4} = \frac{4 \cdot 8}{12} = \frac{8}{3}.$$

8 marks (c) Without using L'Hôpital's rule, compute the following limit:

$$\lim_{x\to-\infty}\left(\sqrt{3x^2-4x}-\sqrt{3x^2+x}\right)$$

Solution :

We have

$$\lim_{x\to-\infty}\left(\sqrt{3x^2-4x}-\sqrt{3x^2+x}\right)=\lim_{x\to-\infty}\left(\sqrt{3x^2-4x}-\sqrt{3x^2+x}\right)\cdot\frac{\left(\sqrt{3x^2-4x}+\sqrt{3x^2+x}\right)}{\left(\sqrt{3x^2-4x}+\sqrt{3x^2+x}\right)}$$

and so

$$\lim_{x\to-\infty}\left(\sqrt{3x^2-4x}-\sqrt{3x^2+x}\right)=\lim_{x\to-\infty}\frac{3x^2-4x-(3x^2+x)}{\left(\sqrt{3x^2-4x}+\sqrt{3x^2+x}\right)}$$

whence

$$\lim_{x\to-\infty}\left(\sqrt{3x^2-4x}-\sqrt{3x^2+x}\right)=\lim_{x\to-\infty}\frac{-5x}{\left(\sqrt{3x^2-4x}+\sqrt{3x^2+x}\right)}.$$

Remembering that $\sqrt{x^2}=|x|$ and so $\sqrt{x^2}=-x$ for x negative, we divide top and bottom by $\sqrt{x^2}$ to get

$$\lim_{x\to-\infty}\left(\sqrt{3x^2-4x}-\sqrt{3x^2+x}\right)=\lim_{x\to-\infty}\frac{5}{\left(\sqrt{3-4/x}+\sqrt{3+1/x}\right)}=\frac{5}{2\sqrt{3}}.$$

7 marks 3. Find the derivative of the function $f(x) = \sqrt{4x-3}$ from the definition of the derivative.

Solution :

We have

$$f'(x) = \lim_{h \to 0} \frac{\sqrt{4(x+h)-3} - \sqrt{4x-3}}{h}$$

and so

$$f'(x) = \lim_{h \to 0} \frac{\left(\sqrt{4(x+h)-3} - \sqrt{4x-3}\right)\left(\sqrt{4(x+h)-3} + \sqrt{4x-3}\right)}{h\left(\sqrt{4(x+h)-3} + \sqrt{4x-3}\right)}.$$

It follows that

$$f'(x) = \lim_{h \to 0} \frac{4(x+h)-3-(4x-3)}{h\left(\sqrt{4(x+h)-3} + \sqrt{4x-3}\right)} = \lim_{h \to 0} \frac{4h}{h\left(\sqrt{4(x+h)-3} + \sqrt{4x-3}\right)}$$

and so

$$f'(x) = \lim_{h \to 0} \frac{4}{\sqrt{4(x+h)-3} + \sqrt{4x-3}} = \frac{2}{\sqrt{4x-3}}.$$

8 marks 4. Find the slope of the tangent line at the curve given by the equation

$$x^y = y^{2x}$$

at the point $(2, 16)$.

Solution :

Taking logarithms, we have that

$$y\log(x) = 2x\log(y).$$

Differentiating this with respect to x,

$$y'\log(x) + \frac{y}{x} = 2x \cdot \frac{y'}{y} + 2\log(y).$$

Substituting $x = 2$ and $y = 16$, it follows that

$$y'\log(2) + 8 = \frac{y'}{4} + 2\log(16),$$

and so

$$y' = \frac{2\log(16) - 8}{\log(2) - 1/4} = \frac{8\log(2) - 8}{\log(2) - 1/4}.$$

$\boxed{\text{8 marks}}$ 5. Determine with proof the values of a and b for which the function

$$f(x) = \begin{cases} x^3 + ax + b & , \text{ if } x \leq 0 \\ x^3 \cos\left(\frac{1}{x}\right) & , \text{ if } x > 0 \end{cases}$$

is differentiable at $x = 0$.

Solution :

Notice that this function is differentiable away from $x = 0$. To be differentiable at $x = 0$, it is necessary that it be continuous there. We have

$$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} x^3 + ax + b = b = f(0).$$

Now

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^3 \cos\left(\frac{1}{x}\right).$$

Since

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1,$$

for all $x > 0$, it follows that, for such x,

$$-x^3 \leq x^3 \cos\left(\frac{1}{x}\right) \leq x^3.$$

Since

$$\lim_{x \to 0^+} (-x^3) = \lim_{x \to 0^+} x^3 = -0,$$

we may appeal to the Squeeze Theorem to conclude that

$$\lim_{x \to 0^+} x^3 \cos\left(\frac{1}{x}\right) = 0.$$

It follows that

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} x^3 \cos\left(\frac{1}{x}\right) = 0$$

and hence, in order for $f(x)$ to be continuous at $x = 0$, we require that $b = 0$.

To check for differentiability at $x = 0$, we use the definition of derivative. We have that

$$f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h},$$

if this limit exists. Now, using that $b = f(0) = 0$,

$$\lim_{h \to 0^-} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^-} \frac{h^3 + ah}{h} = \lim_{h \to 0^-} h^2 + a = a.$$

On the other hand,

$$\lim_{h \to 0^+} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0^+} \frac{h^3 \cos\left(\frac{1}{h}\right)}{h} = \lim_{h \to 0^+} h^2 \cos\left(\frac{1}{h}\right).$$

Once again using the Squeeze Theorem, we can show that

$$\lim_{h \to 0^+} h^2 \cos\left(\frac{1}{h}\right) = 0$$

and so, for $f(x)$ to be differentiable at $x = 0$, it follows that $a = 0$.