

**Midterm 1 Solutions      Duration: 45 minutes***This test has **5 questions** on **7 pages**, for a total of 50 points.*

- Read all the questions carefully before starting to work.
- Put your final answer in the boxes provided for each question where there is an answer-box provided.
- All questions are long-answer. You should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: \_\_\_\_\_ Last Name: \_\_\_\_\_

Student-No: \_\_\_\_\_ Section: \_\_\_\_\_

Signature: \_\_\_\_\_

Question:	1	2	3	4	5	Total
Points:	6	21	7	8	8	50
Score:						

**Student Conduct during Examinations**

- Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCCard for identification.
- Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
- Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
- Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
  - speaking or communicating with other examination candidates, unless otherwise authorized;
  - purposely exposing written papers to the view of other examination candidates or imaging devices;
  - purposely viewing the written papers of other examination candidates;
  - using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
  - using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
- Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
- Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
- Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

1. You must show all your work in order to receive full marks for these questions.

3 marks

(a) Compute

$$\arcsin\left(\sin\left(\frac{-10\pi}{3}\right)\right).$$

**Solution :**

Notice that

$$\sin\left(\frac{-10\pi}{3}\right) = \sin\left(\frac{2\pi}{3}\right) = \sin\left(\frac{\pi}{3}\right)$$

and so, since  $\pi/3 \in [-\pi/2, \pi/2]$ ,

$$\arcsin\left(\sin\left(\frac{-10\pi}{3}\right)\right) = \frac{\pi}{3}.$$

3 marks

(b) Find where the following function is continuous

$$f(x) = \sqrt{\arcsin(x) - \pi/4}.$$

**Solution :**

The function is a composition of continuous functions, provided we restrict ourselves to values of  $x$  in the domain of  $\arcsin(x)$  for which  $\arcsin(x) \geq \pi/4$ . This is precisely those  $x$  with

$$x \in \left[-\frac{1}{\sqrt{2}}, 1\right].$$

2. You must show all your work in order to receive full marks for these questions.

6 marks

(a) If  $f(x) = x^4 \cdot g(x)$  and  $g(1) = -2$ , while  $g'(1) = 4$ , then compute  $f'(1)$ .

**Solution :**

We have

$$f'(x) = 4x^3 \cdot g(x) + x^4 \cdot g'(x),$$

by the Product Rule. It follows that

$$f'(1) = 4 \cdot (-2) + 1 \cdot 4 = -4.$$

7 marks

(b) Compute

$$\lim_{x \rightarrow 2} \frac{x^4 - 16}{x^3 - 8}$$

**Solution :**

We have

$$\frac{x^4 - 16}{x^3 - 8} = \frac{(x - 2)(x + 2)(x^2 + 4)}{(x - 2)(x^2 + 2x + 4)}$$

and so

$$\lim_{x \rightarrow 2} \frac{x^4 - 16}{x^3 - 8} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)(x^2 + 4)}{(x - 2)(x^2 + 2x + 4)} = \lim_{x \rightarrow 2} \frac{(x + 2)(x^2 + 4)}{x^2 + 2x + 4} = \frac{4 \cdot 8}{12} = \frac{8}{3}.$$

8 marks
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(c) Without using L'Hôpital's rule, compute the following limit:

$$\lim_{x \rightarrow -\infty} \left( \sqrt{3x^2 - 4x} - \sqrt{3x^2 + x} \right)$$

**Solution :**

We have

$$\lim_{x \rightarrow -\infty} \left( \sqrt{3x^2 - 4x} - \sqrt{3x^2 + x} \right) = \lim_{x \rightarrow -\infty} \left( \sqrt{3x^2 - 4x} - \sqrt{3x^2 + x} \right) \cdot \frac{(\sqrt{3x^2 - 4x} + \sqrt{3x^2 + x})}{(\sqrt{3x^2 - 4x} + \sqrt{3x^2 + x})}$$

and so

$$\lim_{x \rightarrow -\infty} \left( \sqrt{3x^2 - 4x} - \sqrt{3x^2 + x} \right) = \lim_{x \rightarrow -\infty} \frac{3x^2 - 4x - (3x^2 + x)}{(\sqrt{3x^2 - 4x} + \sqrt{3x^2 + x})}$$

whence

$$\lim_{x \rightarrow -\infty} \left( \sqrt{3x^2 - 4x} - \sqrt{3x^2 + x} \right) = \lim_{x \rightarrow -\infty} \frac{-5x}{(\sqrt{3x^2 - 4x} + \sqrt{3x^2 + x})}.$$

Remembering that  $\sqrt{x^2} = |x|$  and so  $\sqrt{x^2} = -x$  for  $x$  negative, we divide top and bottom by  $\sqrt{x^2}$  to get

$$\lim_{x \rightarrow -\infty} \left( \sqrt{3x^2 - 4x} - \sqrt{3x^2 + x} \right) = \lim_{x \rightarrow -\infty} \frac{5}{\left( \sqrt{3 - 4/x} + \sqrt{3 + 1/x} \right)} = \frac{5}{2\sqrt{3}}.$$

7 marks
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3. Find the derivative of the function  $f(x) = \sqrt{4x - 3}$  from the definition of the derivative.

**Solution :**

We have

$$f'(x) = \lim_{h \rightarrow 0} \frac{\sqrt{4(x+h) - 3} - \sqrt{4x - 3}}{h}$$

and so

$$f'(x) = \lim_{h \rightarrow 0} \frac{\left(\sqrt{4(x+h) - 3} - \sqrt{4x - 3}\right) \left(\sqrt{4(x+h) - 3} + \sqrt{4x - 3}\right)}{h \left(\sqrt{4(x+h) - 3} + \sqrt{4x - 3}\right)}.$$

It follows that

$$f'(x) = \lim_{h \rightarrow 0} \frac{4(x+h) - 3 - (4x - 3)}{h \left(\sqrt{4(x+h) - 3} + \sqrt{4x - 3}\right)} = \lim_{h \rightarrow 0} \frac{4h}{h \left(\sqrt{4(x+h) - 3} + \sqrt{4x - 3}\right)}$$

and so

$$f'(x) = \lim_{h \rightarrow 0} \frac{4}{\sqrt{4(x+h) - 3} + \sqrt{4x - 3}} = \frac{2}{\sqrt{4x - 3}}.$$

8 marks
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4. Find the slope of the tangent line at the curve given by the equation

$$x^y = y^{2x}$$

at the point  $(2, 16)$ .

**Solution :**

Taking logarithms, we have that

$$y \log(x) = 2x \log(y).$$

Differentiating this with respect to  $x$ ,

$$y' \log(x) + \frac{y}{x} = 2x \cdot \frac{y'}{y} + 2 \log(y).$$

Substituting  $x = 2$  and  $y = 16$ , it follows that

$$y' \log(2) + 8 = \frac{y'}{4} + 2 \log(16),$$

and so

$$y' = \frac{2 \log(16) - 8}{\log(2) - 1/4} = \frac{8 \log(2) - 8}{\log(2) - 1/4}.$$

8 marks

5. Determine with proof the values of  $a$  and  $b$  for which the function

$$f(x) = \begin{cases} x^3 + ax + b & , \text{ if } x \leq 0 \\ x^3 \cos\left(\frac{1}{x}\right) & , \text{ if } x > 0 \end{cases}$$

is differentiable at  $x = 0$ .

**Solution :**

Notice that this function is differentiable away from  $x = 0$ . To be differentiable at  $x = 0$ , it is necessary that it be continuous there. We have

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^3 + ax + b = b = f(0).$$

Now

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^3 \cos\left(\frac{1}{x}\right).$$

Since

$$-1 \leq \cos\left(\frac{1}{x}\right) \leq 1,$$

for all  $x > 0$ , it follows that, for such  $x$ ,

$$-x^3 \leq x^3 \cos\left(\frac{1}{x}\right) \leq x^3.$$

Since

$$\lim_{x \rightarrow 0^+} (-x^3) = \lim_{x \rightarrow 0^+} x^3 = 0,$$

we may appeal to the Squeeze Theorem to conclude that

$$\lim_{x \rightarrow 0^+} x^3 \cos\left(\frac{1}{x}\right) = 0.$$

It follows that

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^3 \cos\left(\frac{1}{x}\right) = 0$$

and hence, in order for  $f(x)$  to be continuous at  $x = 0$ , we require that  $b = 0$ .

To check for differentiability at  $x = 0$ , we use the definition of derivative. We have that

$$f'(0) = \lim_{h \rightarrow 0} \frac{f(0+h) - f(0)}{h},$$

if this limit exists. Now, using that  $b = f(0) = 0$ ,

$$\lim_{h \rightarrow 0^-} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^-} \frac{h^3 + ah}{h} = \lim_{h \rightarrow 0^-} h^2 + a = a.$$

On the other hand,

$$\lim_{h \rightarrow 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \rightarrow 0^+} \frac{h^3 \cos\left(\frac{1}{h}\right)}{h} = \lim_{h \rightarrow 0^+} h^2 \cos\left(\frac{1}{h}\right).$$

Once again using the Squeeze Theorem, we can show that

$$\lim_{h \rightarrow 0^+} h^2 \cos\left(\frac{1}{h}\right) = 0$$

and so, for  $f(x)$  to be differentiable at  $x = 0$ , it follows that  $a = 0$ .