1 Talk six: Odds and ends

Speakers: Amin Soofiani

We wish to prove the following:

**Theorem:** Let $G$ be an affine algebraic group; then there exists $n > 0$ such that $G$ is isomorphic to a closed subgroup of $\text{GL}(n)$.

We begin with:

**Proposition:** Let $G$ be an affine algebraic group, acting on an affine variety $X$ via $\phi : G \times X \to X$, and let $F$ be a finite-dimensional subspace of $k[X]$.

1. There is a finite-dimensional subspace $E$ of $k[X]$ which contains $F$, and which is stable under all translations $\tau_x, x \in G$.

2. $F$ itself is stable under all $\tau_x, x \in G$, if and only if $\phi^* F \subseteq k[G] \otimes F$.

We saw the proof last time.

*Proof. (of theorem)* Let us write $k[G] = \langle f_1, \ldots, f_n \rangle$. By the proposition, there exists a finite-dimensional subspace $E \subseteq k[X]$ including $f_1, \ldots, f_n$, which is stable under all translations $\tau_x$. Applying the proposition again (with $F = E$), we may assume that $E = \langle f_1, \ldots, f_r \rangle$.

By the second part of the proposition, we can write

$$\phi^* f_i = \sum_j m_{ij} \otimes f_j$$
for some \( m_{ij} \in k[G] \). Recall the action of the translations \( \rho_x \) and \( \lambda_x \) on \( k[G] \):

\[
(\rho_x f)(y) = f(yx), \quad (\lambda_x f)(y) = f(x^{-1}y),
\]

and we get

\[
(\rho_x f_i)(y) = f_i(yx) = \sum_j m_{ij}(x) f_j(y),
\]

and so the matrix of \( \rho_x \mid_E \) relative to the basis \( f_1, \ldots, f_n \) is \( M = (m_{ij}) \).

Now, for \( x \in G \), consider the map \( \psi : G \to GL(n, k) \) defined by \( \psi(x) = (m_{ij}(x)) \); we claim that \( \psi \) is injective. Indeed, \( g \in \ker \psi \) if and only if, for all \( i \), one has

\[
\rho_g f_i = f_i \iff \rho_g f = f \quad \forall f \in k[G],
\]

and so \( g = e \). We now claim that there is an isomorphism \( \psi' : G \to \psi(G) \).

We calculate

\[
f_i(yx) = \sum_j m_{ij}(x) f_j(y),
\]

so for \( y = e \) we have

\[
f_i(x) = \sum_j m_{ij}(x) f_j(e),
\]

and so the \( m_{ij} \) generate \( k[G] \). Hence \( \psi^* : k[GL(n, k)] \to k[G], T_{ij} \mapsto m_{ij} \) is surjective, hence \( k[G] \cong k[G'] \), so \( G \cong G' \), and we are done. \( \Box \)

Speaker: Atharva Korde

Recall from before: let \( G \) be an affine algebraic group and \( H \) a closed normal subgroup. Then the quotient of \( G \) by \( H \) is an affine algebraic group.

**Proof.** Let \( G_1, G_2 \) be linear algebraic groups with respective closed subgroups \( H_1, H_2 \); then \( H_1 \times H_2 \) is a closed subgroup of \( G_1 \times G_2 \) and that there is an isomorphism of homogeneous \( G_1 \times G_2 \)-spaces:

\[
(G_1 \times G_2)/(H_1 \times H_2) \cong G_1/H_1 \times G_2/H_2.
\]

We can similarly make \( G/H \) a \( G \times G \)-homogeneous space: for \( (x, y) \in G \times G \), \( gH \in G/H \), we put \( (x, y).gH = xgy^{-1}H \).

By the universal property of the quotient, there is a map \( \mu : (G \times G)/(H \times H) \cong (G/H) \times (G/H) \to G/H \); we can use \( \mu \) to define multiplication on \( G/H \).

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Recall from before that we had a finite-dimensional rational representation $V$ of $G$, and nonzero vector $v \in V$, such that

$$H = \{x \in G : x.v \in kv\}.$$

Let us define

$$\tilde{V} = \sum_{\chi \in \text{Hom}(H, G_m)} V_{\chi} = \{w \in W : h.w = \chi(h)w\};$$

then

$$V = \bigoplus_{\chi} V_{\chi},$$

and $v \in \tilde{V}$; moreover, $\tilde{V}$ is also a rational representation of $G$. If $g \in G$ and $w \in V_{\chi}$, then $gw \in V_{\chi'}$, where

$$\chi'(h) = \chi(g^{-1}hg).$$

Indeed,

$$h(gw) = g(g^{-1}hg)w = g(\chi(g^{-1}hg)w) = \chi(g^{-1}hg)(gw) = \chi'(h)(gw).$$

Now: if $W = \{\psi \in \text{End}(\tilde{V}) : \psi(V_{\chi}) \subseteq V_{\chi}\}$, then $G$ acts on $W$ via

$$(g.\psi)(w) = g(\psi(g^{-1}w));$$

so if $w \in V_{\chi}$, then $g^{-1}w \in V_{\chi'}$, so $\psi(g^{-1}w) \in V_{\chi'}$, hence $g(\psi(g^{-1}w)) \in V_{\chi}$.

If $g \in G$ acts as the identity on $W$, then $(g\psi)(w) = \psi(w)$ for all $\psi \in W$, and so $\psi \circ g^{-1} = g^{-1} \circ \psi$ for all $\psi \in W, w \in \tilde{V}$.

Let us take $\psi \in \text{End}(V_{\chi})$ for some $\chi$, and 0 elsewhere; for instance, the character defined by $h.v = \phi(h)v, v \in \tilde{V}$. Then $g^{-1}$ acts by a scalar on $V_{\chi}$, hence on $V_{\phi}$, which implies that $g^{-1} \in H$ and hence $g \in H$. Thus $H = \ker [G \to \text{GL}(W)]$, inducing a morphism of algebraic groups

$$\lambda : G/H \to \text{GL}(W);$$

the proof ends by showing that $\lambda$ is an isomorphism; we will omit this step. \hfill \Box

Speaker: Simone Coccia

Let us recall the notion of a root system: let $E$ be a real, Euclidean vector space with scalar product $(\cdot, \cdot)$. For $\alpha \in E$, let $L_\alpha$ denote the hyperplane orthogonal to $\alpha$. For $\alpha, \beta \in E$, put

$$r_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha,$$
\[ \langle \beta, \alpha \rangle = \frac{2(\beta, \alpha)}{(\alpha, \alpha)}. \]

**Definition:** A root system is a subset \( \Delta \subseteq E \) such that:

1. \( \Delta \) is finite and 0 \( \notin \Delta \).
2. For all \( \alpha \in \Delta \) one has \( r_\alpha(\Delta) \subseteq \Delta \).
3. For all \( \alpha, \beta \in \Delta \), one has \( \langle \alpha, \beta \rangle \in \mathbb{Z} \).

The **rank** of a root system is defined to be the integer \( \dim \text{span}_R(\Delta) \).

For \( \alpha \in \Delta, c \in \mathbb{R} \), let us suppose \( c\alpha \in \Delta \). We must have

\[ \langle \alpha, c\alpha \rangle \in \mathbb{Z}, \]

so \( \frac{2}{c} \in \mathbb{Z} \); also

\[ \langle c\alpha, \alpha \rangle \in \mathbb{Z}, \]

so \( 2c \in \mathbb{Z} \) also. It follows that \( c = 1 \) or \( 2 \); we will call \( \Delta \) **reduced** if \( \alpha, c\alpha \in \Delta \) implies \( c = \pm 1 \). An **isomorphism** of root systems is a bijection \( \phi : \Delta \to \Delta' \) such that

\[ \langle \phi(\alpha), \phi(\beta) \rangle = \langle \alpha, \beta \rangle \]

for all \( \alpha, \beta \in \Delta \).

We will pick up here next week.