1 Talk four: Connectedness and related topics

Speaker: Amin Soofiani

We begin with some basic results.

**Proposition:** Let $G$ be an algebraic group.

1. There is a unique irreducible component $G^0$ of $G$ that contains the identity; moreover, $G^0$ is a closed normal subgroup of finite index.

2. $G^0$ is the unique connected component of $G$ containing 1.

3. Any closed subgroup of $G$ of finite index contains $G^0$.

**Proof.** Let $X_1, \ldots, X_n$ be the irreducible components of $G$ which contain 1. As the product of irreducible components is itself irreducible, we have that the product

$$X_1 \cdots X_n$$

is an irreducible component containing 1, and so without loss of generality equals $X_1$; this proves the first claim.

Now, let $x \in G^0$; then because $(G^0)^{-1}$ is an irreducible component of $G$ which contains 1, it follows from our previous work that $(G^0)^{-1} = G^0$ and thus that $x^{-1} \in G^0$. Furthermore, $G^0x = G^0$, and so $G^0$ must be a subgroup of $G$. Conjugation by $x \in G$ is a homeomorphism, so the image of $G^0$ under this automorphism is an irreducible component containing 1, and so equals $G^0$; thus $G^0$ is normal. Finally, the number of cosets of $G^0$ in $G$ equals the number of connected components of $G$, which...
must be finite (because \( G \) is an affine variety). The second point is a consequence of
the fact that \( G \) is irreducible if and only if it is connected.
For the last point, suppose \( H \leq G \) is some closed subgroup of finite index, and put
\( H^0 = H \cap G^0 \). Because \( H^0 \) is both open and closed in \( G^0 \), and \( G^0 \) is connected, it
follows that \( H^0 = G^0 \), and so \( G^0 \subseteq H \). \( \square \)

**Corollary:** All connected components of \( G \) have the same dimension.

**Lemma:** Let \( U, V \subseteq G \) be dense, open subsets; then \( UV = G \).

**Proof.** Let \( x \in G \); a map \( V \to xV^{-1} \) is obtained by composing inversion with left-
multiplication by \( x \). As the intersection of nonempty open, dense subsets in an affine
variety is again nonempty, we have for every \( y \in U \cap xV^{-1} \) that
\[
y = xv^{-1} \text{ for some } v \in V \iff x = yv \in UV.\]
\( \square \)

**Lemma:** Let \( H \) be a subgroup of \( G \). Then:

1. \( \bar{H} \) is a subgroup of \( G \).
2. If \( H \) contains a nonempty open subset of \( \bar{H} \), then \( H \) is closed.

**Proof.** Let \( x \in H \); then \( H = xH \subseteq x\bar{H} \), hence \( \bar{H} \subseteq x\bar{H} \), so \( H\bar{H} = \bar{H} \). If \( x \in \bar{H} \), then \( Hx \subseteq \bar{H} \), so
\[
\bar{H}x = \overline{Hx} \subseteq \bar{H},
\]
and clearly
\[
(\bar{H})^{-1} = \overline{H^{-1}} = \bar{H}.
\]
To prove the second point, let \( U \) be a nonempty open subset of \( H \); we can write
\[
H = \bigcup_{x \in \bar{H}} Ux.
\]
This means that \( H \) is open in \( \bar{H} \), so
\[
H = HH = \bar{H},
\]
and so \( H \) is closed. \( \square \)

**Proposition:** Let \( \phi : G \to G' \) be a homomorphism of algebraic groups. Then:
1. \( \ker \phi \) is a closed normal subgroup of \( G \);
2. \( \phi(G) \) is a closed subgroup of \( G' \); and
3. \( \phi(G^0) = (\phi(G))^0 \).

**Proof.** The first point is immediate from the observation \( \ker \phi = \phi^{-1}(\{0\}) \). To show the second point, we observe that \( \phi(G) \) is locally closed (by Chevalley’s theorem?), and so is closed in \( G' \).

Finally, because we have just shown that \( \phi(G^0) \) is a closed subgroup of \( G' \), which is connected (because \( \phi \) is continuous), which has finite index in \( \phi(G) \). By our proposition, this means that \( (\phi(G))^0 \subseteq \phi(G^0) \), and so \( (\phi(G))^0 = \phi(G^0) \) as claimed.

**Proposition:** Let \((X_i, \phi_i)_{i \in I}\) be a family of irreducible varieties together with morphisms \( \phi_i : X_i \to G \) for \( i \in I \). Denote by \( H \) the smallest closed subgroup of \( G \) containing the images of all \( Y_i = \phi(X_i) \), and assume that all \( Y_i \) contain 1. Then:

1. \( H \) is connected.
2. There exists a positive integer \( n \), together with \( a = (a(1), \ldots, a(n)) \in I^n \) and \( \epsilon(k) = \pm 1 \) for \( 1 \leq k \leq n \), such that
   \[ H = Y_{a(1)}^{\epsilon(1)} \cdots Y_{a(n)}^{\epsilon(n)}. \]

**Corollary:** Assume that \((G_i)_{i \in I}\) is a family of closed connected subgroups of \( G \). Then the subgroup \( H \) generated by them is closed and connected, and furthermore there is an integer \( n \) and tuple \( a = (a(1), \ldots, a(n)) \in I^n \) such that
   \[ H = G_{a(1)} \cdots G_{a(n)}. \]

**Corollary:** If \( H \) and \( K \) are two closed subgroups of \( G \), one of which is connected, then the commutator group \([H, K]\) is connected.

**Examples:**

1. The groups \( G_a, G_m, \) and \( D_n := \prod_{i=1}^n G_m \) are connected, as is \( \text{SL}(n) \); we see this by observing that \( \text{SL}(n) \) is defined as the zero locus of \( \det g - 1 \), which is an irreducible polynomial and so defines an irreducible (hence connected) variety.
2. An example of a disconnected group is given by \( \text{SO}(n) = O_n \cap \text{SL}(n) \); this is a subgroup of \( O_n \) of index 2, whenever the characteristic of the base field is not 2. By a previous proposition, disconnectedness of \( O(n) \) follows from the fact that \( (O(n))^0 \subseteq \text{SO}(n) \).

Onto our next topic – \( G \)-spaces.

**Definition:** A \( G \)-variety or \( G \)-space is a variety \( X \) on which \( G \) acts as a permutation group, with the action being given by a morphism of varieties. A homogeneous space is a \( G \)-space on which the \( G \)-action is transitive.

**Definition:** Let \( X \) and \( Y \) be \( G \)-spaces. A morphism \( \phi : X \rightarrow Y \) is called a \( G \)-morphism or \( G \)-equivariant map if \( \phi(g.x) = g.\phi(x) \) for all \( x \in X, g \in G \). The isotropy group of \( x \in X \) is its stabilizer under the group action, viz.

\[
G_x = \{ g \in G : g.x = x \}.
\]

The isotropy group is a closed subgroup of \( G \).

**Examples:**

1. If \( X = G \), we have an action of \( G \) on \( X \) by conjugation, \( g.x = g x g^{-1} \). The orbits are conjugacy classes, and the isotropy groups are centralizers.

2. If \( X = G \), we also have an action of \( G \) on \( X \) by left-multiplication, \( g.x = gx \). The orbit is all of \( G \), and the stabilizer is the identity subgroup.

3. If \( V \) is a finite-dimensional vector space and \( G \rightarrow \text{GL}(V) \) is any rational representation, then we can take \( X = V \).

**Lemma:** Let \( X \) be any \( G \)-space. Then:

1. Every orbit is locally closed, and

2. There exist closed orbits.

**Example:** let us consider the action of the groups \( \text{GL}(n), \text{SL}(n), \) and \( D_n \) on \( A \) from the previous example. There are two \( \text{SL}(n) \) orbits, namely, \( \{0\} \) and \( A \setminus \{0\} \); it follows immediately that the same statements are true for \( \text{GL}(n) \). The number of orbits of \( D_n \) is \( 2^n \) (exercise).

The talk will continue next week, with the same speaker.