1 Talk three: Derivations, differentials, and Lie algebras

Speaker: Zach Goldthorpe

We begin by discussing general tangent spaces on algebraic varieties. Throughout, $K$ is an algebraically closed field (of arbitrary characteristic).

Motivation: Let $X \subset A^n_K$ be a closed subvariety of affine $n$-space, so that we can write $K[X] = K[T_1,\ldots,T_n]/I$ for some finitely-generated ideal $I = (f_1,\ldots,f_s)$. Given $v \in A^n_K$, we can consider the line $L = \{x + tv : t \in K\}$ for any $x \in X$. In this situation, $v$ is said to be tangent at $x$ if

$$\sum_{j=1}^n v_j \frac{\partial f}{\partial T_j}(x) = 0.$$

Recall that a derivation $D : A \to M$ of a $K$-algebra $A$ into a (left) $A$-module is any $K$-linear map satisfying the Leibniz property


Let us denote by $\text{Der}_K(A,M)$ the left $A$-module of derivations $A \to M$.

If $v$ is tangent to $X$ at $x$, we can define

$$D_v := \sum_{j=1}^n v_j \frac{\partial}{\partial T_j},$$

which is certainly a derivation; if $\mathfrak{m}_x \subset K[T_1,\ldots,T_n]$ is the (maximal) ideal generated by those functions which vanish on $X$, then $D_v I \subseteq \mathfrak{m}_x$. The map $f \mapsto D_v f(x)$
vanishes on $I$, and so it gives us a map $K[X] \to K$; more precisely, it is a derivation $K[X] \to K_x$, where $K_x$ is $K$ itself viewed as a $K[X]$-module under the action $f \mapsto f(x)$.

This gives us a bijective correspondence between the tangent vectors to $X$ at $x$, and the space $\text{Der}_K(K[X], K_x)$.

**Definition:** If $X$ is affine and $x \in X$, define the tangent space at $x$ to be $T_x X := \text{Der}_K(K[X], K_x)$. Recall that if $\phi : X \to Y$ is a map of affine varieties, then we get a map $d\phi_x : T_x X \to T_{\phi x} Y$ via $D \mapsto D \circ \phi^*$, called the differential of $\phi$ at $x$. This satisfies the chain rule $d(\phi \circ \psi)_x = d\phi_x \circ d\psi_x$.

**Lemma:** One has $T_x X \cong (m_x/m_x^2)^*$. 

**Proof.** If $D \in T_x X$, then $D$ vanishes on $m_x^2$, and so induces a map $m_x/m_x^2 \to K$. Conversely, if $\ell : m_x/m_x^2 \to K$ is any map, we obtain a derivation via $f \mapsto \ell(f - f(x) + m_x^2)$. 

**Lemma:** Let $O_x = O_{X,x}$ be the ring of regular functions on $X$ at $x$; then the map $K[X] \to O_x$ induces an isomorphism $\alpha : \text{Der}_K(O_x, K) \to \text{Der}_K(K[X], K_x)$.

**Proof.** (sketch) Let $D : K[X] \to K$ be an arbitrary derivation. If $f \in O_x$, then $f = g/h$ where $g, h \in k[X]$ and $h(x) \neq 0$, and so by the quotient rule

$$Df = \frac{h(x)Dg - g(x)Dh}{h(x)^2}.$$ 

**Corollary:** If $\phi : X \to Y$ is an isomorphism of $X$ into an affine open subvariety of $Y$, then we have an isomorphism $d\phi : T_x X \to T_{\phi x} X$.

**Definition:** If $X$ is an arbitrary variety and $x \in X$, then define the tangent space of $X$ at $x$ to be 

$$T_x X := \lim_{\to} T_x U,$$

the direct limit taken over all neighbourhoods $U$ of $x$.

Now: we move onto Lie algebras.

Let $G$ be an algebraic group, and let $\lambda, \rho : G \to \text{GL}(K[G])$ be the translation maps 

$$(\lambda(x)f)(y) = f(x^{-1}y) \text{ and } (\rho(x)f)(y) = f(yx);$$
this gives a (left) action of $G$ on $K[G]$ (in both cases). It follows from categorical considerations that $K[G] \otimes_K K[G] \cong K[G \times G]$. If we take

$$m : K[G] \otimes_K K[G] \to K[G],$$

then $m$ acts on $F \in K[G \times G]$ via

$$(mF)(x) = F(x, x);$$

we then put $I = \ker m$. For $x \in G$, both $\lambda(x) \otimes \lambda(x)$ and $\rho(x) \otimes \rho(x)$ stabilize $I$ and $I^2$ both, and so we obtain operators $\lambda(x), \rho(x)$ on $\Omega_G := I/I^2$.

We have a derivation $d : K[G] \to \Omega_G$ via $df = f \otimes 1 - 1 \otimes f$; an easy calculation shows that $d$ commutes with all $\lambda(x)$ and $\rho(x)$.

Given $x \in G$, the map $y \mapsto xyz^{-1}$ induces a linear automorphism $\text{Ad}_x : T_eG \to T_eG$.

**Definition:** Set $D_G = \text{Der}_K(K[G], K[G])$; then $D_G$ is a Lie algebra, with Lie bracket $[D, D'] = D \circ D' - D' \circ D$.

**Lemma:** There is an isomorphism of $K[G]$-modules $\Psi : D_G \to K[G] \otimes K T_eG$ satisfying

1. $\Psi \circ \lambda(x) \circ \Psi^{-1} = \lambda(x) \otimes \text{id}_{T_eG}$ and $\Psi \circ \rho(x) \circ \Psi^{-1} = \rho(x) \otimes \text{id}_{T_eG}$.

2. For any $f \in K[G]$, write $\Delta f = \sum_i f_i \otimes g_i$, where

$$\Delta : K[G] \to K[G] \otimes_K K[G]$$

is defined $$(\Delta f)(x, y) = f(xy)$. Then for $X \in T_eG$, we have

$$\Psi^{-1}(1 \otimes X)f = -\sum_i f_i(Xg_i).$$

**Definition:** If $G$ is an algebraic group, we denote by $\text{Lie}(G)$ its Lie algebra, namely the subalgebra of $D_G$ fixed under the action of $\lambda(x)$.

**Corollary:** [Springer, proposition 4.4.5] Define $\alpha : D_G \to T_eG$ via

$$(\alpha D)(f) = Df(e).$$

Then $\alpha$ induces an isomorphism from $\text{Lie}(G)$ to $T_eG$.

**Corollary:** One has $\dim \text{Lie}(G) = \dim G$.

**Examples:**
1. \( G = \mathbb{G}_a \); in this case, \( K[G] = K[T] \), and the derivations commuting with all maps \( T \mapsto T + a \) for \( a \in K \) are generated by \( X := \frac{d}{dT} \). So then \( \mathfrak{g} = \text{span}_K X \cong K \) with trivial Lie bracket.

2. \( G = \mathbb{G}_m \); in this case, \( K[G] = K[T^{\pm 1}] \), and the derivations commuting with all maps \( T \mapsto aT \) for \( a \in K^\times \) are generated by \( X := T \frac{d}{dT} \). So then \( \mathfrak{g} = \text{span}_K X \) with trivial Lie bracket, again.

3. \( G = \text{GL}_n \); in this case, \( K[G] = K[T_{ij}, \det(T_{ij})^{-1}] \). For \( X = [X_{ij}] \), define

\[
D_x T_{ij} = -\sum_{\ell=1}^n T_{i\ell} X_{\ell j},
\]

which is a determinant that commutes with \( \lambda \). Here, \( \text{Lie}(G) = \mathfrak{gl}_n \).

**Lemma:** Let \( G \) be a linear algebraic group with multiplication \( \mu : G \times G \to G \) and inversion \( \iota : G \to G \). Then

\[
d\mu_{(e,e)} : \mathfrak{g} \oplus \mathfrak{g} \to \mathfrak{g}
\]

is the map \( (X, Y) \mapsto X + Y \), and

\[
d\iota_e : \mathfrak{g} \to \mathfrak{g}
\]

is the map \( X \to -X \). Moreover, if \( \phi : G \to \text{GL}(V) \) is a finite-dimensional rational representation of \( G \), its differential \( d\phi : \mathfrak{g} \to \mathfrak{gl}(V) \) is a representation of \( \mathfrak{g} = \text{Lie}(G) \) in \( V \).

**Fact:** If \( \text{Ad} : G \to \text{GL}(\mathfrak{g}) \) is the usual adjoint representation (with \( \mathfrak{g} = \text{Lie}(G) \)), then \( d\text{Ad} = \text{ad} \), where \( \text{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}) \) is defined \( \text{ad}(X)(Y) = [X, Y] \).

Finally, if we have two rational representations \( \phi_i : G_i \to \text{GL}(V_i) \) (for \( i = 1, 2 \)), then \( \phi_1 \oplus \phi_2 \) and \( \phi_1 \otimes \phi_2 \) are both rational representations of \( G_1 \times G_2 \), respectively on the spaces \( V_1 \oplus V_2 \) and \( V_1 \otimes V_2 \). Moreover, one has

\[
d(\phi_1 \oplus \phi_2) = d\phi_1 \oplus d\phi_2
\]

and

\[
d(\phi_1 \otimes \phi_2)(X_1, X_2)(v_1, v_2) = (d\phi_1(X_1)v_1) \otimes v_2 + v_1 \otimes d\phi_2(X_2)v_2.
\]