Algebraic Groups with Diagram of form $A_1$

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Today we shift setting to the case where $F$ is a field, of characteristic different from 2 and not algebraically closed. It might be that, under these hypothesis, $F$ is not perfect either, hence we will work with a fixed separable closure $F^s$ instead of an algebraic closure.

**Definition 1.** Let $(X, R, X^\vee, R^\vee)$ be a root datum and $D$ a basis of $R$ with corresponding system of positive roots $R^+$. Then $(R^+)^\vee$ is still a system of positive roots, corresponding to the basis $D^\vee$ of $R^\vee$.

**Definition 2.** We call an indexed root datum a sextuple $(X, D, X^\vee, D^\vee, D_0, \tau)$, where $(X, D, X^\vee, D^\vee)$ is a based root datum and $(D, D_0, \tau)$ is the index of $G$ relative to $F$, meaning that $\tau$ is a continuous action of $\Gamma_s$ on $X$ that stabilizes $D_0 \subseteq D \subseteq X$.

This is the kind of root datum we need to describe fully the structure of an algebraic group $G$ over our field $F$ as the usual root datum would describe only one form of $G$ namely, the one where $D_0 = \emptyset$ and $\tau$ is the trivial action, which is the case of $G$ being $F$-split (i.e. $G$ contains a maximal $F$-torus that is $F$-split).

Given the indexed root datum $\Psi = (X, D, X^\vee, D^\vee, D_0, \tau)$ of a semi-simple group $G$ over $F$ we can always consider the associated simply connected $F$-split group $H$ with root datum $\Phi = (X, D, X^\vee, D^\vee, 0, \emptyset)$ and we know there will be a $z \in \mathbb{Z}^1(\Gamma_F, \text{Aut}_{F^s}(M_2))$ such that $G = H_z$ We say that $G$ of inner type if $z \in \text{Inn}(H)$.

Now we proceed to talk about the rational forms of algebraic group with a particular Dynkin diagram.

Let $G$ be a semi-simple algebraic group over a field $F$ of which we fix a separable closure, $F^s$. We assume that $G$ is of inner type and it’s Dynkin $\mathcal{D}$ diagram is of type $A_1$, for example the root datum of $G$ is $\Psi_G = (\mathbb{Z}_2^2, R, (\mathbb{Z}_2^2)^\vee, R^\vee)$ where $R$ is spanned by $[-1, 1]$ in $\mathbb{Z}_2^2$, and let $H$ be the $F$-split with same root datum as $G$, so that $G = H_z$ the twist of $H$ by a cocycle in $\mathbb{Z}_1(F, \text{Inn}(H))$.

We may assume that $H = \text{GL}_2$ and $\text{Inn}(H) = \text{GL}_2/Z(\text{GL}_2) = \text{PGL}_2$ (see Springer 7.4.7 (2)), hence studying $G$ as a $F$ form of $H$ means studying the forms of $\text{PGL}_2$.

Let us take a cocycle $z \in H^1(\Gamma_F, \text{Aut}_{F^s}(M_2)) = H^1(\Gamma_F, \text{PGL}_2(F^s))$ (By Skolem-Noether) that we know being in bijection with the set $\Phi(F^s/F, M_2)$ of all rational forms of the matrix algebra of $M_2$, we proceed to construct one of these forms.
We start by considering the ring of algebraic functions $F^*[M_2]$ and, following the construction we construct a ($z$-twisted) group action of $\Gamma_{F^*}$ $*_z$ by setting, for $f \in F^*[M_2]$ and $\gamma \in \Gamma_F$.

$$\gamma *_z f = z(\gamma)(\gamma.f).$$

This is indeed a group action as, using the cocycle condition we get:

$$(\gamma \delta)_z f = z(\gamma)(\gamma.z(\delta))((\gamma \delta).f)$$

$$= z(\gamma)(\gamma.z(\delta))(\gamma.(\delta.f))$$

$$= z(\gamma)(\gamma.(z(\delta)(\delta.f)) = \gamma *_z (\delta *_z f)).$$

So that now we have the ring of rational functions of our form, given by $F^*[M_2]^{*(\Gamma_F)} = F[M_2]_z = \{ f \in F^*[M_2] | \forall \gamma \in \Gamma_F, \gamma *_z f = f \}$. Meaning that $H^1(F, \text{PGL}_2)$ classifies the isomorphism classes of all associative $F$–algebras that are isomorphic to $M_2$ over the separable closure $F_s$, i.e. all the central simple algebras over $F$. Conversely, Assume we have a quaternion algebra $A$ defined over $F$, then it must be $A \otimes_F F^s \cong M_2(F^s)$ via some isomorphism map $a$ defined over $F^s$, starting with $a$ we define a cocycle $c_a \in H^1(\Gamma_F, \text{Aut}(M_2))$ by taking $c_a(\tau) = a_\tau. a^{-1}$, indeed one has:

$$c_a(\sigma \tau) = a(\sigma \tau). a^{-1} = a_\sigma.(\tau.a^{-1}) = a_\sigma.(1) \sigma.(\tau.a^{-1}) = a_\sigma.(a^{-1}a_\sigma.\tau.a^{-1})$$

$$= a_\sigma.a^{-1}.a_\sigma.(\tau.a^{-1}) = c_a(\sigma).a_\sigma.(\tau.a^{-1}) = c_a(\sigma).a_\sigma.a_\tau.(\tau.a^{-1}) = c_a(\sigma).c_a(\tau).$$

Finding out that we have a 1-to-1 correspondence between rational forms of $M_2(F)$ and quaternion algebras over $F$.

This same kind of reasoning works for algebraic groups with Dynkin diagram of type $A_{n-1}$ for any $n$, as long as the group is of inner type, In the case of $G$ not being of inner type, than we would have to study the automorphism of $\text{GL}_n$ given by $x \mapsto t(x^{-1})$ which is a little more complicated, and revolves around studying the behaviour of involutions of quaternion algebras $K$ over a quadratic extension of $F$, and we find would find in the end that in this case we can describe our forms as the unitary subgroups $U(h) = \{ g \in M | g = h.g.t(kh) \}$ of $M = M_n(K)$ where $\kappa$ is an involution of $M$ that acts non trivially on its center.

**Example 1.** (What we did for $D = A_1$, but getting our hands dirty)

We recall that the underlying space of $M_2$ over $F^s$ is just $K^2_F$, and can view $M_2(K) = \text{Hom}_{\text{Alg}_F}(F[x_{11}, x_{12}, x_{21}, x_{22}], F^s)$ which gives a (partial) Hopf-algebra structure to the ring $F[x_{11}, x_{12}, x_{21}, x_{22}] = F[M_2]$. We are here interested in its co-multiplication map $\mu$, which arises from the standard matrix multiplication rule:

$$([a_{ij}][b_{hk}]) = \sum_{j=h} a_{ij} b_{hk}.$$
Meaning we have a map \( m : M_2 \times M_2 \to M_2 \) that can be rewritten as:

\[
\begin{align*}
\text{Hom}_{\text{Alg}/F}(F[x_{11}, x_{12}, x_{21}, x_{22}], -) \times \text{Hom}_{\text{Alg}/F}(F[y_{11}, y_{12}, y_{21}, y_{22}], -) & \xrightarrow{m} \text{Hom}_{\text{Alg}/F}(F[z_{11}, z_{12}, z_{21}, z_{22}], -) \\
\text{Hom}_{\text{Alg}/F}(F[x_{11}, x_{12}, x_{21}, x_{22}] \otimes F[y_{11}, y_{12}, y_{21}, y_{22}], -) & \xrightarrow{m'} \text{Hom}_{\text{Alg}/F}(F[z_{11}, z_{12}, z_{21}, z_{22}], -) \\
\text{Hom}(+, -)(\text{Alg} \otimes \text{Alg} \otimes \text{Alg} \otimes \text{Alg}) & \xrightarrow{\pi} \text{Hom}_{\text{Alg}/F}(F[z_{11}, z_{12}, z_{21}, z_{22}], -)
\end{align*}
\]

So that \( \mu = \tilde{m} \circ \text{Id} \) and it’s given by:

\[
\mu(z_{ij}) = \sum_{h,k} x_{ij} y_{hk}.
\]

With the same procedure we can find the co-action of an element of PGL\(_2(F_2)\) induced by the cocycle \( z \). In order to look at a cocycle \( z \) with values in PGL\(_2\) we look for a cocycle \( u \) with values in SL\(_2\) and we will take \( z = \pi u \). Note the conjugacy action is already trivial on \( Z(SL_2) \) so nothing bad happens. Writing \( SL_2 = V(x_{11}x_{22} - x_{12}x_{21} - 1) \) the invariance condition by the \( z \)-twisted action of \( \gamma \) can be seen at the level of hom-sets as the following diagram:

\[
\begin{align*}
\text{Hom}_{\text{Alg}/F}(F[x_{11}, x_{12}, x_{21}, x_{22}], F^\gamma) & \xrightarrow{\text{Id}} \text{Hom}_{\text{Alg}/F}(F[x_{11}, x_{12}, x_{21}, x_{22}], F^\gamma) \\
\text{Hom}_{\text{Alg}/F}(F[x_{11}, x_{12}, x_{21}, x_{22}], F^\gamma) & \xrightarrow{\pi_x} \text{Hom}_{\text{Alg}/F}(F[x_{11}, x_{12}, x_{21}, x_{22}], F^\gamma) \\
\text{Hom}_{\text{Alg}/F}(F[x_{11}, x_{12}, x_{21}, x_{22}], F^\gamma) & \xrightarrow{\pi_x} \text{Hom}_{\text{Alg}/F}(F[x_{11}, x_{12}, x_{21}, x_{22}], F^\gamma) \\
\end{align*}
\]

Let us call \( F[x_{11}, x_{12}, x_{21}, x_{22}] \) as we will be interested in the element of \( z(\gamma) \in \text{PGL} \) for \( \gamma \in \Gamma_F \). We start at the level of Hom functors, and using the same kind of reasoning we used for \( \mu \) we look for the conjugacy co-action by the element \( z(\gamma) \in \text{PGL} \):

\[
(-)^{z(\gamma)} : F[x_{11}^{z(\gamma)}, x_{12}^{z(\gamma)}, x_{21}^{z(\gamma)}, x_{22}^{z(\gamma)}] \to F[\gamma^{z(\gamma)}_{11}, \gamma^{z(\gamma)}_{12}, \gamma^{z(\gamma)}_{21}, \gamma^{z(\gamma)}_{22}] \otimes F[x_{11}, x_{12}, x_{21}, x_{22}]
\]

which is the map defined by:

\[
\begin{align*}
x_{11}^{(\gamma)} & \mapsto \gamma_{11}x_{22} \otimes x_{11} + \gamma_{12}x_{21} \otimes x_{12} - \gamma_{11}x_{21} \otimes x_{12} - \gamma_{12}x_{22} \otimes x_{11}; \\
x_{12}^{(\gamma)} & \mapsto \gamma_{11} \otimes x_{12} + \gamma_{12} \otimes x_{22} - \gamma_{11}x_{22} \otimes x_{12} - \gamma_{12}x_{21} \otimes x_{11}; \\
x_{21}^{(\gamma)} & \mapsto \gamma_{21}x_{21} \otimes x_{12} + \gamma_{22} \otimes x_{21} - \gamma_{21}x_{21} \otimes x_{22} - \gamma_{22}x_{21} \otimes x_{12}; \\
x_{22}^{(\gamma)} & \mapsto \gamma_{11} \otimes x_{22} + \gamma_{12} \otimes x_{21} - \gamma_{11}x_{21} \otimes x_{12} - \gamma_{12}x_{22} \otimes x_{11};
\end{align*}
\]

Recalling that the action of \( \gamma \) (as a Galois element) is trivial at this level and will only act on the coefficients of our polynomial, in the end we obtain that the invariance condition for an element \( p(x) \) of the regular functions of \( M_2[F^\gamma] \) can be rewritten as follows:
\[1 \otimes p(\mathbf{x}) = p^\gamma(\mathbf{x}^{\pi(\gamma)}).\]

**Example 2.** (This time an actual example)

Let \( F \) be a field where \(-1 \not\in (F^*)^2\) aim to build an inner \( F(i) \)-form \( K \) of \( M_2(F) \). To do so we start by building a cocycle \( z : \text{Gal}(F(i)/F) \rightarrow \text{Aut}(M_2(F(i))) \)
calling everything with simpler names, we try to construct an element \( z \) of \( Z^1(\mathbb{Z}/2\mathbb{Z}, \text{PGL}_2(F(i))) \), and \( \text{Gal}(F(i)/F) = \{\text{Id}, \text{Id}_{F(i)}\} \). To find such element we just need to find \( z(\mathbf{x}) \). To have a good candidate we look at the cocycle condition:

\[
\text{Id}_{\text{PGL}_2(F(i))} = z(\text{Id}_{F(i)}) = z(\mathbf{x}) = z(\mathbf{x}^{\pi(\gamma)}) = z(\mathbf{x}^{\pi(\gamma)}) = z(\mathbf{x}^{\pi(\gamma)}).\]

Therefore \( z(\mathbf{x}) \) must be a matrix \( M \) in \( \text{PGL}_2 \) such that \( M\overline{M} = 1 \). It’s immediate to check that the matrix \( A = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} \) satisfies this requirement, so we can now look at the actions above and find what is our \( x^A \) (i.e. the form of the conjugacy co-action by \( z(\mathbf{x}) \) ) which will be given by \( x^A_{ij} \rightarrow (-1)^{i+j}x_{ij} \), and our invariance condition will be given by

\[p(x_{1,1}, x_{1,2}, x_{2,1}, x_{2,2}) = p(x_{1,1}, -x_{1,2}, -x_{2,1}, x_{2,2})\]

It is now quite clear that the invariant subspace of \( F(i)[M_2]_z \) is given by \( \mathbb{F}[x_{11}, ix_{12}, ix_{21}, x_{22}] \) which means the \( F \)-subalgebra of matrices in \( M_2(F(i)) \) of the form \( \begin{bmatrix} a & ib \\ ic & d \end{bmatrix} \) which one can identify with the quaternion algebra \( (-1, -1) \)

by setting \( i = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix} , j = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} , k = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.\)