DONALDSON-THOMAS THEORY of the QUANTUM FERMAT QUINTIC

Spec $\overline{\mathbb{Q}}$

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Joint work with YU-HSIANG LIU

arXiv: 1911.07949

arXiv: 2004.10346

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arXiv: 1409.4101

Slides:

personal.math.ubc.ca/~behrend/talks/specqbar.pdf

Work over Spec $\mathbb{C}$ (Apologies: not Spec $\overline{\mathbb{Q}}$)

$\mathbb{P}^4_q$: quantum projective 4-space is the non-comm. graded $\mathbb{C}$-algebra

$A = \mathbb{C}[t_0,\dots,t_4]_q = \mathbb{C}\langle t_0,\dots,t_4\rangle / t_i t_j = q^{n_{ij}} t_j t_i \qquad q = \sqrt[5]{1}$

$$N = (n_{ij}) = \begin{pmatrix} 0 & 1 & -1 & 1 & -1 \\ -1 & 0 & 1 & -1 & 1 \\ 1 & -1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & -1 & 0 \end{pmatrix} \in M_{5\times 5}(\mathbb{F}_5) \quad \text{skew-symmetric matrix}$$

Q : Quantum Fermat Quintic

$Q = \mathbb{C}[t_0,\dots,t_4]_q / t_0^5 + \dots + t_4^5 \qquad (t_0^5+\dots+t_4^5 \text{ central in } A)$

$A \twoheadrightarrow Q$: " $Q \hookrightarrow \mathbb{P}^4_q$ " quintic hypersurface

Goal: Count the number of coherent sheaves (e.g. fat points) on Q

Geometry associated to Q:

- abelian category of coh sheaves $= \mathrm{qgr}(Q)$
  $= (\text{f.g. graded left } Q\text{-modules}) / (\text{f.d. graded left } Q\text{-mods})$
- structure sheaf $= Q \longrightarrow$ define subschemes of Q

Theorem (Kanazawa): (1) $\mathrm{qgr}(Q)$ has global dimension 3

i.e. $\dim \mathrm{Ext}^i(E,F) < \infty \ \forall i$, $\mathrm{Ext}^i(E,F) = 0 \ \forall i > 3$.

" $\mathrm{qgr}(Q)$ is smooth of dimension 3 "

(2) if $\vec{1}$ is an eigenvector of N then

$\mathrm{qgr}(Q)$ is Calabi-Yau-3 i.e. $\mathrm{Ext}^i(E,F)^\vee = \mathrm{Ext}^{3-i}(F,E)$.

☹ We did not get very far with techiques from non-comm. geometry

☺ A has a huge centre :

$$\begin{array}{ccc} \mathbb{C}[t_0,\dots,t_4]_q/t_0^5+\dots+t_4^5 & & \\ \mathbb{C}[x_0,\dots,x_4]/x_0+\dots+x_4 & & \end{array} \qquad \begin{array}{ccc} Q & & \mathcal{A} = \mathbb{P}^4_q \\ | & & | \\ \mathbb{P}^3 \cong X & \xrightarrow[\Sigma x_i = 0]{} & \mathbb{P}^4 \end{array} \qquad \begin{array}{c} A^{(5)} = \mathbb{C}[t_0,\dots,t_4]_q^{(5)} \\ \uparrow \\ B = \mathbb{C}[x_0,\dots,x_4] \end{array} \qquad \begin{array}{c} t_i^5 \\ \uparrow \\ x_i \end{array}$$

$\mathcal{A}$: sheaf of non-comm. $\mathcal{O}_{\mathbb{P}^4}$-algebras, locally free of rank 625.

$$\mathcal{A} \cong \mathcal{O}_{\mathbb{P}^4} \oplus \mathcal{O}_{\mathbb{P}^4}(-1)^{121} \oplus \mathcal{O}_{\mathbb{P}^4}(-2)^{381} \oplus \mathcal{O}_{\mathbb{P}^4}(-3)^{121} \oplus \mathcal{O}_{\mathbb{P}^4}(-4)$$

$\mathcal{A} \otimes_{\mathcal{O}_{\mathbb{P}^4}} \mathcal{A} \xrightarrow{\text{mult.}} \mathcal{A} \xrightarrow{\text{proj}} \mathcal{O}_{\mathbb{P}^4}(-4)$ is a perfect pairing.

<u>symmetric</u> because $\vec{1}$ is an eigenvector of N.

$Q = \mathcal{A}|X$, $X \hookrightarrow \mathbb{P}^4$ hyperplane $\Sigma x_i = 0$

$Q \otimes_{\mathcal{O}_X} Q \longrightarrow \mathcal{O}_X(-4) = \omega_X$ symmetric perfect pairing.

<u>Defn.</u> A <u>Calabi-Yau-3 pair</u> is (X, Q)

X : smooth projective $\mathbb{C}$-scheme of dimension 3

Q : locally free, finite rank sheaf of $\mathcal{O}_X$-algebras with a symmetric perfect pairing $Q \otimes_{\mathcal{O}_X} Q \longrightarrow \omega_X$.

e.g. every comm. CY3 is a pair $(X, \mathcal{O}_X)$, $\mathcal{O}_X \otimes_{\mathcal{O}_X} \mathcal{O}_X \to \mathcal{O}_X = \omega_X$

- $\operatorname{qgr} Q$ = (coh sheaves of $\mathcal{O}_X$-modules with a left Q-module structure)
- (coh Q-modules) is a CY3 category:

$$\operatorname{Ext}^i_Q(E, F)^\vee = \operatorname{Ext}^{3-i}_Q(\omega_Q \otimes_Q F, E)$$

$$\omega_Q = \mathcal{H}om_{\mathcal{O}_X}(Q, \omega_X) \cong Q \text{ as } Q\text{-bimodule.}$$

$\uparrow$ the pairing

Donaldson-Thomas Theory

Moduli spaces associated to pairs (X, Q), (they are comm!):

① $n \in \mathbb{N}$: $\mathrm{Hilb}^n Q$ = "Hilbert scheme of n points on (X, Q)"
= closed subscheme of $\mathrm{Quot}\, Q$, consisting of quotients $Q \twoheadrightarrow \mathcal{F}$ with left Q-module structure and length $\mathcal{F} = n$.

② $h \in \mathbb{Q}[u]$: $\mathrm{Coh}^h Q$ = moduli space of coherent $\mathcal{O}_X$-modules of Hilbert polynomial h, endowed with a left Q-module structure, <u>stable</u> as Q-mod

$\mathrm{Hilb}^n Q \longrightarrow \mathrm{Coh}^h Q$ — isomorphism onto a union of connected
$[Q \twoheadrightarrow \mathcal{F}] \longmapsto \ker(Q \to \mathcal{F})$ — components (in good cases)

DT theory is about $\mathrm{Coh}\, Q$, but can be abused to study $\mathrm{Hilb}^n Q$.

DT theory:
- make sense of $\#^{\mathrm{vir}} \mathrm{Coh}\, Q \in \mathbb{Z}$
- compute

Key idea: $\mathrm{Coh}\, Q$ behaves like a critical locus.

<u>Critical loci:</u> $f : M \to \mathbb{C}$ regular function, M: smooth $\mathbb{C}$scheme.

$$\begin{tikzcd} \mathrm{Crit}\, f \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & M \arrow[d, "df"] \\ M \arrow[r, "0"'] & \Omega_M \end{tikzcd}$$

Assume $\mathrm{Crit}\, f$ proper: Intersection theory
$\#^{\mathrm{vir}} \mathrm{Crit}\, f = I(M \cap_{\Omega_M} M)$.

e.g. $f = 0$, $\mathrm{Crit}\, f = M$ self-intersection: Gauß-Bonnet:

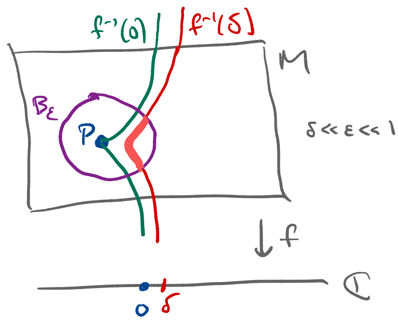
$$\#^{\mathrm{vir}} \mathrm{Crit}\, f = \int_{[M]} c_{\mathrm{top}}\, \Omega_M = (-1)^{\dim M} \int_{[M]} c_{\mathrm{top}}(T_M) = (-1)^{\dim M} \chi^{\mathrm{top}}(M)$$

General Case (Crit f singular): Singular Gauß-Bonnet
(micro-local index theorem, Kashiwara-MacPherson, early 1970s)

$$\#^{vir}\,\mathrm{Crit}\,f = \chi^{top}(\mathrm{Crit}\,f, \mu) \qquad \mu: \mathrm{Crit}\,f \to \mathbb{Z} \text{ weight fctn.}$$

$$P \in \mathrm{Crit}\,f: \quad \mu(P) = (-1)^{\dim M}\left(1 - \chi^{top}(\text{Milnor fibre of } f \text{ at } P)\right)$$

Milnor fibre:
intersection of a small ball in M around P
with a nearby fibre of f
The Milnor fibre carries the vanishing cycles



Coh Q behaves like a critical locus:

$$T_{\mathrm{Coh}\,Q}|_{\mathcal{F}} = \{ \mathcal{F}' \in \mathrm{Coh}\,Q(\mathbb{C}[\varepsilon]) \mid \mathcal{F}'/\varepsilon\mathcal{F}' = \mathcal{F} \}$$
$$= \{ \mathcal{F}' : \text{coh } Q[\varepsilon]\text{-module, flat}/\mathbb{C}[\varepsilon] \mid \mathcal{F}'/\varepsilon\mathcal{F}' = \mathcal{F} \}$$
$$= \mathrm{Ext}^1_Q(\mathcal{F}, \mathcal{F}) \qquad 0 \to \varepsilon\mathcal{F}' \to \mathcal{F}' \to \mathcal{F}'/\varepsilon\mathcal{F}' = \mathcal{F} \to 0$$
$$\cdot\varepsilon \quad \cong$$

Deformation theory: analytically near $\mathcal{F}$, Coh$Q$ is
the 0-locus of an analytic map

$$\mathrm{Ext}^1_Q(\mathcal{F},\mathcal{F}) \longrightarrow \mathrm{Ext}^2_Q(\mathcal{F},\mathcal{F})$$

whose lowest order term is the Yoneda square

$$[\mathcal{F} \to \mathcal{F}' \to \mathcal{F}] \longmapsto [\mathcal{F} \to \mathcal{F}' \to \mathcal{F}' \to \mathcal{F}]$$
$$e \longmapsto e \circ e$$

The CY3 condition:

$$\mathrm{Ext}^2_Q(\mathcal{F},\mathcal{F}) = \mathrm{Ext}^1_Q(\mathcal{F},\mathcal{F})^\vee$$

so $e \mapsto e \circ e$ is a differential, in fact the differential of

$e \mapsto \frac{1}{3} \mathrm{tr}(e \circ e \circ e)$ $\qquad$ $\mathrm{tr}: \mathrm{Ext}^3_Q(\mathcal{F},\mathcal{F}) = \mathrm{Hom}_Q(\mathcal{F},\mathcal{F})^\vee \longrightarrow \mathbb{C}$

In general, $\mathrm{Coh}\, Q$ is not a global critical locus, but carries a "symmetric obstruction theory" (classical shadow of a (-1)-shifted derived symplectic structure)

- $\mathrm{Coh}\, Q$ proper $\longrightarrow$ $\#^{\mathrm{vir}} \mathrm{Coh}^h Q \in \mathbb{Z}$ deformation invariant ($Q = \mathcal{O}_X$: R. Thomas ~2000, general case Lin)
- $\mathrm{Coh}\, Q$ has a constructible function $\nu: \mathrm{Coh}\, Q \longrightarrow \mathbb{Z}$ s.t. $\#^{\mathrm{vir}} \mathrm{Coh}^h Q = \chi^{\mathrm{top}}(\mathrm{Coh}^h Q, \nu)$ (B. ~2005)

<u>Define</u>: $\quad DT(\mathrm{Coh}^h Q) = \#^{\mathrm{vir}}(\mathrm{Coh}^h Q) = \chi^{\mathrm{top}}(\mathrm{Coh}^h Q, \nu)$.

Example: $Q = \mathcal{O}_X$

$$\mathcal{E}_X(z) = 1 + \sum_{n=1}^{\infty} \#^{\mathrm{vir}}(\mathrm{Hilb}^n X)\, z^n . \qquad M(z) = \prod_{i=1}^{\infty} \frac{1}{(1-z^i)^i} \quad (\text{McMahon})$$

$$\boxed{\mathcal{E}_X(z) = M(-z)^{\chi(X)}}$$

(X compact or not: use additivity of χ)

eg. comm. quintic $\qquad \mathcal{E}_X(z) = \prod_{i=1}^{\infty} (1-(-z)^i)^{200 i} \qquad \chi(X) = -200.$

$$= (1+z)^{200}(1-z^2)^{400}(1+z^3)^{600} \cdots$$

eg. $\#^{\mathrm{vir}} X = -\chi(X) = 200$

Localize:

$$\begin{array}{ccccccc} u_i^5 & \mathbb{C}[u_1,\dots,u_4]_q/\Sigma u_i^5=-1 & Q_0 & & Q & & A & \mathbb{C}[t_0,\dots,t_4]_q^{(5)} & t_i^5 \\ \uparrow & \uparrow & | & & | & & | & \uparrow & \uparrow \\ x_i & \mathbb{C}[x_1,\dots,x_4]/\Sigma x_i=-1 & \mathbb{A}^3\cong X_0 & \xrightarrow[x_0=1]{} & \mathbb{P}^3\cong X & \xrightarrow[\Sigma x_i=0]{} & \mathbb{P}^4 & \mathbb{C}[x_0,\dots,x_4] & x_i \end{array}$$

Here $u_i = \dfrac{t_0^4 t_i}{t_0^5} = \dfrac{t_0^4 t_i}{x_0}$ and $u_i u_j = q^{\bar{n}_{ij}} u_j u_i$ $\quad \bar{n}_{ij} = n_{ij} - n_{i0} - n_{0j}$

$\bar{N} \in M_{4\times 4}(\mathbb{F}_5)$ $\qquad \bar{N} = \begin{pmatrix} 0 & -2 & -1 & -2 \\ 2 & 0 & -1 & -1 \\ 1 & 1 & 0 & -2 \\ 2 & 1 & 2 & 0 \end{pmatrix}$.

Finite dim'l left Q_0-modules = $\mathbb{C}$-algebra morphisms $Q_0 \to M_{n\times n}\mathbb{C}$.

__n=1.__ at most one of $u_1,\dots,u_4 \neq 0$. Say $u_1 \neq 0$, $u_2=u_3=u_4=0$.

$u_1^5=-1 \Rightarrow u_1 = -q^i \quad i=0,\dots,4$

5 1-dim'l reps $S_0,\dots,S_4$ all supported over $\langle 1,-1,0,0,0\rangle \in X$

$\binom{5}{2}=10$ such points in $X \cong \mathbb{P}^3$

$\Rightarrow \# \mathrm{Hilb}^1 Q = 50$

__n=2.__ $\boxed{A_i}$ $\quad u_1 = \begin{pmatrix} -q^{i-2} & 0 \\ 0 & -q^i \end{pmatrix}$ $\quad u_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad u_3=u_4=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$0 \to S_{i-2} \to A_i \to S_i \to 0$ extension b/c $u_1u_2 = q^{-2}u_2u_1$

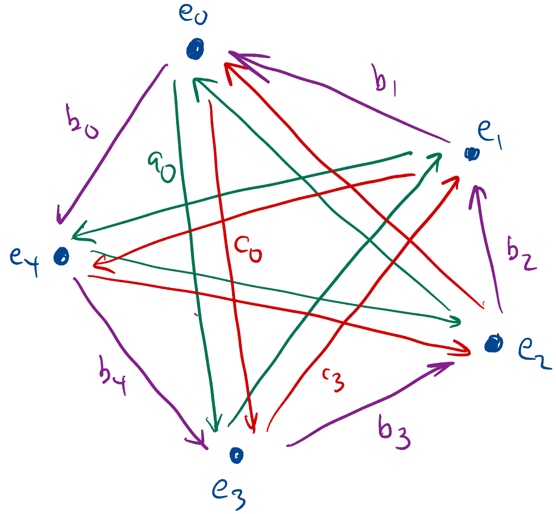
$\boxed{B_i}$ $\quad u_1 = \begin{pmatrix} -q^{i-1} & 0 \\ 0 & -q^i \end{pmatrix}$ $\quad u_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad u_2=u_4=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$0 \to S_{i-1} \to B_i \to S_i \to 0$ extension b/c $u_1u_3 = q^{-1}u_3u_1$

$\boxed{C_i}$ $\quad u_1 = \begin{pmatrix} -q^{i-2} & 0 \\ 0 & -q^i \end{pmatrix}$ $\quad u_4 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad u_2=u_3=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$0 \to S_{i-2} \to C_i \to S_i \to 0$ extension b/c $u_1u_4 = q^{-2}u_4u_1$

This information is encoded in the diagram: a <u>quiver</u>



The quiver encodes its <u>path algebra</u> $\mathcal{P}$

$\mathbb{C}$- vector space basis : paths of length ≥ 0
multiplication : concatenation

The above suggests (???)

$Q_0 \longrightarrow \mathcal{P}$ $\quad u_2 \mapsto \Sigma a_i \;;\; u_3 \mapsto \Sigma b_i \;;\; u_4 \mapsto \Sigma c_i$

need relations in $\mathcal{P}$:

$$\left.\begin{array}{lll} u_2u_3 = q^{-1}u_3u_2 & \rightsquigarrow & (\Sigma a_i)(\Sigma b_i) = q^{-1}(\Sigma b_i)(\Sigma a_i) \\ u_2u_4 = q^{-1}u_4u_2 & \rightsquigarrow & (\Sigma a_i)(\Sigma c_i) = q^{-1}(\Sigma c_i)(\Sigma a_i) \\ u_3u_4 = q^{-2}u_4u_3 & \rightsquigarrow & (\Sigma b_i)(\Sigma c_i) = q^{-2}(\Sigma c_i)(\Sigma b_i) \end{array}\right\} \begin{array}{l} \text{15} \\ \text{relations} \\ \mathcal{R} \end{array}$$

<u>Surprise</u>: $Q_0 \xrightarrow{\sim} \mathcal{P}/\mathcal{R}$; $\quad u_2 \mapsto \Sigma a_i \,,\; u_3 \mapsto \Sigma b_i \,,\; u_4 \mapsto \Sigma c_i$
$u_1 \longmapsto \Sigma q^i e_i / \sqrt[5]{x_1}$, is an isomorphism

$\rightarrow$ over an analytic open nbhd of $\langle 1,-1,0,0,0\rangle \in X \cong \mathbb{P}^3$
Q_0 and $\mathcal{P}/\mathcal{R}$ are isomorphic

$\rightsquigarrow$ closed subscheme of $\mathrm{Hilb}^n Q$, corresponding to fat points set-theoretically supported over $\langle 1,-1,0,0,0\rangle$, is governed by this quiver with relations $\mathcal{R}$.

The element of $\mathcal{P}$, a linear combination of cycles:

$$f = \left(\Sigma\, q^{i-1} b_i\right)\left(\Sigma a_i\right)\left(\Sigma c_i\right) - q^{-1}\left(\Sigma\, q^{i-1} b_i\right)\left(\Sigma c_i\right)\left(\Sigma a_i\right)$$

$$\mathcal{R} = \left(\partial_{a_i} f,\ \partial_{b_i} f,\ \partial_{c_i} f\right).$$

f is a potential for the quiver with relations.

Moduli spaces of representations of a quiver with potential are critical loci:

$$\mathrm{Rep}^n(\mathcal{P}, \mathcal{R}) = \mathrm{Crit}\Big(\mathrm{Rep}^n \mathcal{P},\quad \begin{aligned} \mathrm{Rep}^n \mathcal{P} &\longrightarrow \mathbb{C} \\ (V_i, \varphi_i) &\longmapsto \mathrm{tr}\, f(\varphi) \end{aligned}\Big)$$

Conclusion:

$$\zeta_Q(z) = 1 + \sum_{n \geq 1} \#^{vir}\left(\mathrm{Hilb}^n Q\right) z^n = \left(1 + \sum_{n\geq 1} \#^{vir}\, \mathrm{Rep}^n(\mathcal{P}, \mathcal{R})\, z^n\right)^{10} M(-t^5)^{-50}$$

~ END ~

Thank you!