Coordinates and Change of Basis

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Abstract

We discuss linear changes of coordinates.

Contents

Introduction

These notes are in somewhat preliminary form. The content is not yet stable. Watch for updates.

1 Vectors

In Figure 1, two coordinate systems in the plane are displayed. We can use either of the two coordinate systems to decribe vectors in the plane.

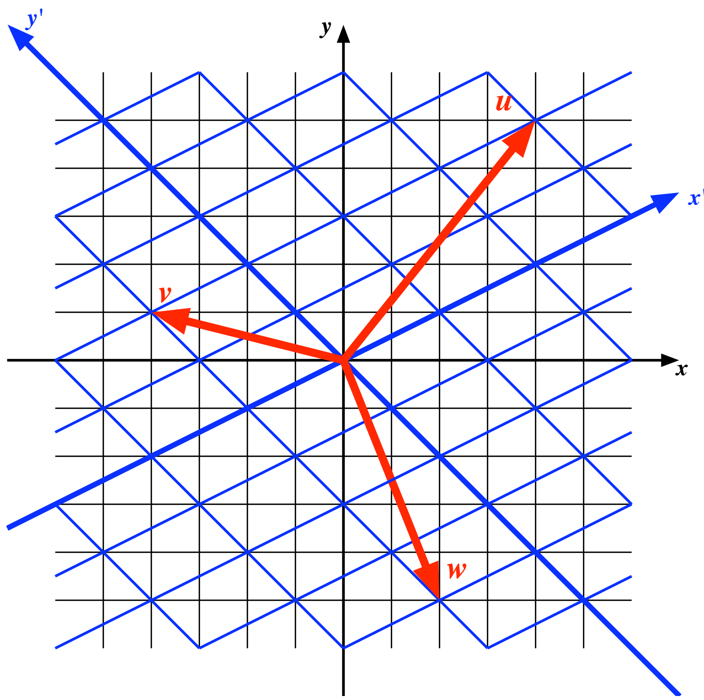

Figure 1: Three vectors

The vector $\vec{u} = \binom{4}{5}$ has standard coordinates $x = 4$ and $y = 5$. If we use the blue coordinate system, whose coordinate axes are labelled x' and y', the blue coordinates of $\vec{u}$ are $x' = 3$ and $y' = 2$. The notation is as follows:

$$[\vec{u}]_{\mathcal{S}} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \qquad [\vec{u}]_{\mathcal{B}} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \tag{1}$$

Similarely, we have

$$[\vec{v}]_{\mathcal{S}} = \begin{pmatrix} -4 \\ 1 \end{pmatrix} \qquad [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$[\vec{w}]_{\mathcal{S}} = \begin{pmatrix} 2 \\ -5 \end{pmatrix} \qquad [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

Every coordinate system is defined by a *basis*. The standard coordinate system is defined by the standard basis

$$\mathcal{S} = (\vec{e}_1, \vec{e}_2) = (\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix})$$

The new (blue) coordinate system is defined by the basis

$$\mathcal{B} = (\vec{u}_1, \vec{u}_2) = (\begin{pmatrix} 2 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \end{pmatrix})$$

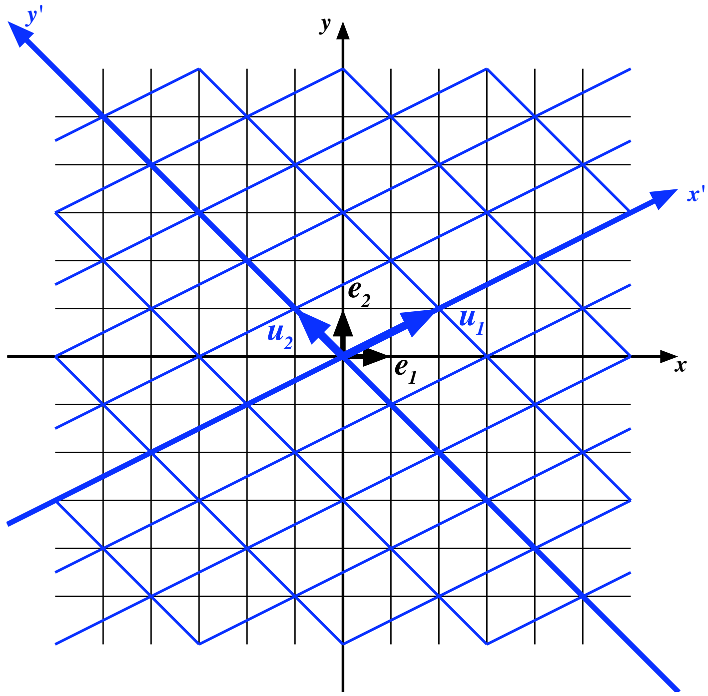

Figure 2: The blue coordinate system is defined by the basis $\mathcal{B} = (\vec{u}_1, \vec{u}_2)$, the standard coordinate system is defined by the basis $\mathcal{S} = (\vec{e}_1, \vec{e}_2)$

Referring back to Figure 1, we see that

$$\vec{u} = 3\vec{u}_1 + 2\vec{u}_2$$

and this is why the coordinates of $\vec{u}$ in the coordinate system defined by $\mathcal{B} = (\vec{u}_1, \vec{u}_2)$ are 3 an 2. Similarely,

$$\vec{v} = -\vec{u}_1 + 2\vec{u}_2 \quad \text{therefore} \quad [\vec{v}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

and

$$\vec{w} = -\vec{u}_1 - 4\vec{u}_2 \quad \text{therefore} \quad [\vec{w}]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -4 \end{pmatrix}$$

In general,

To find the coordinate vecctor $[\vec{a}]_{\mathcal{B}}$ of the vector $\vec{a}$ with respect to the coordinate system defined by the basis $\mathcal{B}$, express $\vec{a}$ as a linear combination $\vec{a} = x'\vec{u}_1 + y'\vec{u}_2$, and then read off: $[\vec{a}]_{\mathcal{B}} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

Of course every vector is equal to its coordinate vector in the standard basis:

$$\vec{u} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} \quad \text{so} \quad \vec{u} = 4\vec{e}_1 + 5\vec{e}_2 \quad \text{so} \quad [\vec{u}]_{\mathcal{S}} = \begin{pmatrix} 4 \\ 5 \end{pmatrix} = \vec{u}$$

1.1 How to convert between $[\vec{a}]_{\mathcal{S}}$ and $[\vec{a}]_{\mathcal{B}}$

Recall the above equation $\vec{a} = x'\vec{u}_1 + y'\vec{u}_2$. We can rewrite this as a matrix equation (recall that we can write linear combinations as matrix-vector products!)

$$\vec{a} = \begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

Remembering that $\vec{a} = [\vec{a}]_{\mathcal{S}}$ and $\left(\begin{smallmatrix} x' \\ y' \end{smallmatrix}\right) = [\vec{a}]_{\mathcal{B}}$, we can rewrite this as

$$[\vec{a}]_{\mathcal{S}} = \begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix} [\vec{a}]_{\mathcal{B}} \tag{2}$$

The matrix which appears in this equation:

$$P = \begin{pmatrix} | & | \\ \vec{u}_1 & \vec{u}_2 \\ | & | \end{pmatrix}$$

is called the *change of basis matrix* or *transition matrix* from the standard basis $\mathcal{S}$ to the new basis $\mathcal{B}$. This matrix P has the two basis vectors $\vec{u}_1$ and $\vec{u}_2$ as columns. With this notation, we can rewrite (2) as

$$\boxed{[\vec{a}]_{\mathcal{S}} = P\,[\vec{a}]_{\mathcal{B}}}$$

If we multiply by P^{-1} on the left, we obtain:

$$P^{-1}[\vec{a}]_{\mathcal{S}} = P^{-1}P[\vec{a}]_{\mathcal{B}} = I_2[\vec{a}]_{\mathcal{B}} = [\vec{a}]_{\mathcal{B}}$$

(Here I_2 is the 2×2 identity matrix.) So we get the other conversion formula

$$\boxed{[\vec{a}]_{\mathcal{B}} = P^{-1}\,[\vec{a}]_{\mathcal{S}}} \tag{3}$$

In our example, the matrix P is

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

and

$$P^{-1} = \tfrac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}$$

So, for example,

$$[\begin{pmatrix}4\\5\end{pmatrix}]_{\mathcal{B}} = \tfrac{1}{3}\begin{pmatrix}1 & 1\\-1 & 2\end{pmatrix}[\begin{pmatrix}4\\5\end{pmatrix}]_{\mathcal{S}} = \tfrac{1}{3}\begin{pmatrix}1 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix} = \begin{pmatrix}3\\2\end{pmatrix}$$

which is the same result (1), which we read off from the sketch. In general

$$\begin{pmatrix}x'\\y'\end{pmatrix} = [\vec{a}]_{\mathcal{B}} = \tfrac{1}{3}\begin{pmatrix}1 & 1\\-1 & 2\end{pmatrix}[\vec{a}]_{\mathcal{S}} = \tfrac{1}{3}\begin{pmatrix}1 & 1\\-1 & 2\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix} = \tfrac{1}{3}\begin{pmatrix}x+y\\-x+2y\end{pmatrix}$$

Or

$$\begin{aligned}x' &= \tfrac{1}{3}x + \tfrac{1}{3}y\\ y' &= -\tfrac{1}{3}x + \tfrac{2}{3}y\end{aligned}$$

1.2 In $\mathbb{R}^n$

If $\mathcal{B} = (\vec{u}_1, \ldots, \vec{u}_n)$ is a basis of $\mathbb{R}^n$, then for every vector $\vec{v} \in \mathbb{R}^n$, there exist *unique* numbers $x_1, \ldots, x_n \in \mathbb{R}$, such that

$$\vec{v} = x_1\vec{u}_1 + \ldots + x_n\vec{u}_n$$

These numbers are called the *coordinates* of $\vec{v}$ with respect to the basis $\mathcal{B}$, the column vector $\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ is called the *coordinate vector* of $\vec{v}$ with respect to the basis $\mathcal{B}$, notation

$$[\vec{v}]_{\mathcal{B}} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Given that $\mathcal{B} = (\vec{u}_1, \ldots, \vec{u}_n)$ is a basis of $\mathbb{R}^n$, then the matrix

$$P = \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{pmatrix}$$

with the basis vectors as columns is called the *transition matrix* or *change of basis matrix* from the standard basis $\mathcal{S}$ to $\mathcal{B}$.

Let us repeat our above calculation:

$$\begin{aligned}
[\vec{v}]_{\mathcal{S}} &= \vec{v} && \text{standard coordinate vector is equal to vector} \\
&= x_1\vec{u}_1 + \ldots + x_n\vec{u}_n && \text{this is the defining equation in the box} \\
&= \begin{pmatrix} | & & | \\ \vec{u}_1 & \cdots & \vec{u}_n \\ | & & | \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} && \text{linear combination is a matrix vector product} \\
&= P \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} && \text{definition of the change of basis matrix } P \\
&= P\,[\vec{v}]_{\mathcal{B}} && \text{definition of coordinate vector from above box}
\end{aligned}$$

We conclude

$$\boxed{[\vec{v}]_{\mathcal{S}} = P\,[\vec{v}]_{\mathcal{B}}}$$

$$\boxed{[\vec{v}]_{\mathcal{B}} = P^{-1}\,[\vec{v}]_{\mathcal{S}}}$$

1.3 An Application

Consider Figure 3.

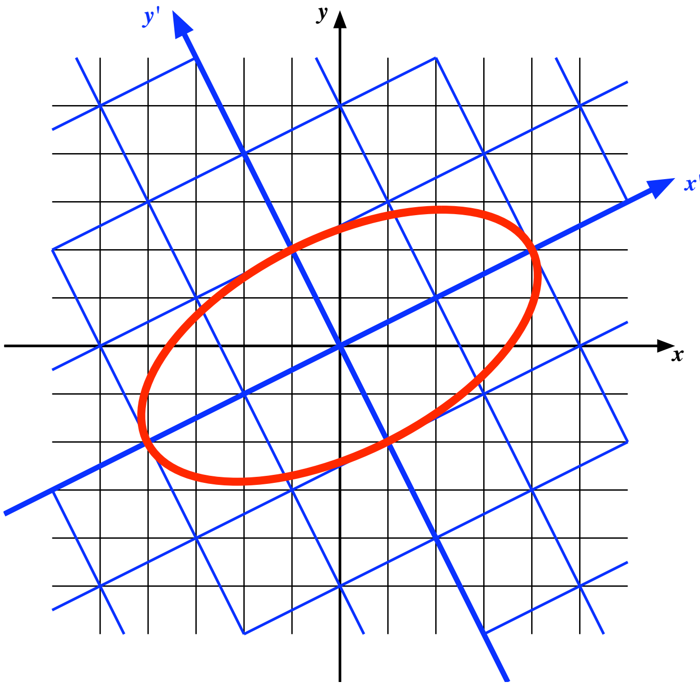

Figure 3: An ellipse

The (x', y')-coordinate system is well-suited for studying the ellipse. In fact, the equation for the ellipse is

$$\left(\frac{x'}{2}\right)^2 + (y')^2 = 1 \tag{4}$$

The transition matrix is

$$P = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

with inverse

$$P^{-1} = \tfrac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

And so

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \tfrac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$\begin{aligned} x' &= \tfrac{2}{5}x - \tfrac{1}{5}y \\ y' &= \tfrac{1}{5}x + \tfrac{2}{5}y \end{aligned}$$

Plugging this into the Equation (4), we get the standard equation for this ellipse

$$(\tfrac{1}{5}x - \tfrac{1}{10}y)^2 + (\tfrac{1}{5}x + \tfrac{2}{5}y)^2 = 1$$

which is equivalent to

$$8x^2 - 12xy + 17y^2 = 100 \tag{5}$$

Thus, choosing a good coordinate system and then changing basis, can simplify the study (in this case, finding the equation) of certain curves.

A central problem solved by linear algebra is the following: suppose an equation such as (5) of a conic section in standard coordinates if given. How can we find the (x', y')-coordinate system, whose axes are the major and minor axis of the ellipse? The method used is *diagonalization of symmetric matrices*.

1.4 Exercises

Exercise 1.1 In Figure 4, two vectors $\vec{v}_1$ and $\vec{v}_2$ are sketched, which form a basis $\mathcal{C} = (\vec{v}_1, \vec{v}_2)$ of $\mathbb{R}^2$.

(i) Find $[\vec{a}]_\mathcal{C}$, $[\vec{b}]_\mathcal{C}$ and $[\vec{c}]_\mathcal{C}$, geometrically, without doing any calculations.

(ii) Skech (in Figure 4) the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, given that

$$[\vec{u}]_\mathcal{C} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \qquad [\vec{v}]_\mathcal{C} = \begin{pmatrix} 2 \\ -3 \end{pmatrix} \qquad [\vec{w}]_\mathcal{C} = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

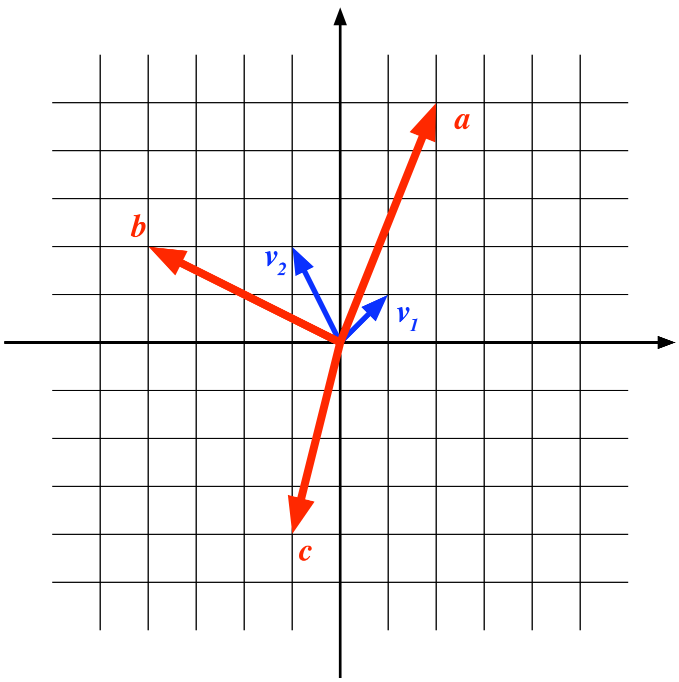

Figure 4: The figure for Exercise 1.1

Exercise 1.2 In Figure 5, two vectors $\vec{v}_1$ and $\vec{v}_2$ are sketched, which form a basis $\mathcal{C} = (\vec{v}_1, \vec{v}_2)$ of $\mathbb{R}^2$.

(i) Find $[\vec{a}]_\mathcal{C}$, $[\vec{b}]_\mathcal{C}$ and $[\vec{c}]_\mathcal{C}$, geometrically, without doing any calculations.

(ii) Skech (in Figure 5) the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, given that

$$[\vec{u}]_\mathcal{C} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \qquad [\vec{v}]_\mathcal{C} = \begin{pmatrix} 3 \\ 1 \end{pmatrix} \qquad [\vec{w}]_\mathcal{C} = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$$

Exercise 1.3 Given that $\mathcal{B} = \left(\begin{pmatrix} 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$, find

$$\left[\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right]_\mathcal{B} \qquad \left[\begin{pmatrix} -2 \\ -4 \end{pmatrix}\right]_\mathcal{B} \qquad \left[\begin{pmatrix} 6 \\ -1 \end{pmatrix}\right]_\mathcal{B} \qquad \left[\begin{pmatrix} a \\ b \end{pmatrix}\right]_\mathcal{B}$$

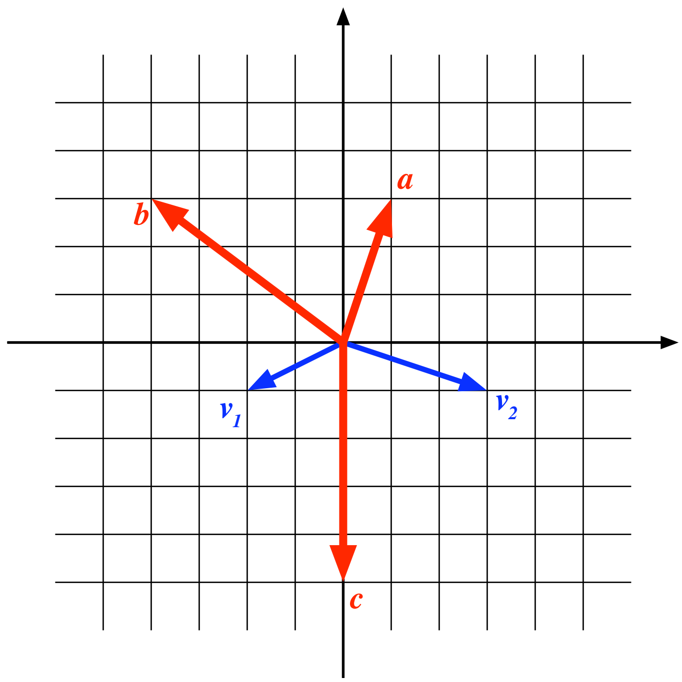

Figure 5: The figure for Exercise 1.2

Exercise 1.4 Given that $\mathcal{B} = (\left(\begin{smallmatrix}-2\\3\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}-1\\3\\1\end{smallmatrix}\right), \left(\begin{smallmatrix}1\\2\\1\end{smallmatrix}\right))$, find

$$[\left(\begin{smallmatrix}2\\4\\3\end{smallmatrix}\right)]_{\mathcal{B}} \qquad [\left(\begin{smallmatrix}-2\\-6\\1\end{smallmatrix}\right)]_{\mathcal{B}} \qquad [\left(\begin{smallmatrix}4\\-1\\3\end{smallmatrix}\right)]_{\mathcal{B}} \qquad [\left(\begin{smallmatrix}a\\b\\c\end{smallmatrix}\right)]_{\mathcal{B}}$$

Exercise 1.5 Consider the basis $\mathcal{B} = (\left(\begin{smallmatrix}-2\\3\end{smallmatrix}\right), \left(\begin{smallmatrix}2\\1\end{smallmatrix}\right))$ of $\mathbb{R}^2$. Suppose that

$$[\vec{a}]_{\mathcal{B}} = \left(\begin{smallmatrix}2\\-3\end{smallmatrix}\right) \qquad [\vec{b}]_{\mathcal{B}} = \left(\begin{smallmatrix}3\\-1\end{smallmatrix}\right) \qquad [\vec{c}]_{\mathcal{B}} = \left(\begin{smallmatrix}4\\5\end{smallmatrix}\right) \qquad [\vec{x}]_{\mathcal{B}} = \left(\begin{smallmatrix}s\\t\end{smallmatrix}\right)$$

Find $[\vec{a}]_{\mathcal{S}}$, $[\vec{b}]_{\mathcal{S}}$, $[\vec{c}]_{\mathcal{S}}$ and $[\vec{x}]_{\mathcal{S}}$.

Exercise 1.6 Suppose the standard (x, y)-coordinate system of $\mathbb{R}^2$ is rotated counterclockwise by an angle of $60°$, to yield the new (x', y')-coordinate system. Find the new coordinates of the points $\left(\begin{smallmatrix}2\\3\end{smallmatrix}\right)$, $\left(\begin{smallmatrix}-1\\1\end{smallmatrix}\right)$ and $\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)$.

Exercise 1.7 Suppose $[\left(\begin{smallmatrix}1\\2\end{smallmatrix}\right)]_{\mathcal{B}} = \left(\begin{smallmatrix}2\\-1\end{smallmatrix}\right)$ and $[\left(\begin{smallmatrix}2\\6\end{smallmatrix}\right)]_{\mathcal{B}} = \left(\begin{smallmatrix}-2\\3\end{smallmatrix}\right)$. Find $\mathcal{B}$.

2 Linear Operators

Consider the linear operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ wich is reflection across the line $L : x - 2y = 0$, see Figure 6. The (x', y')-coordinate system is defined by the basis

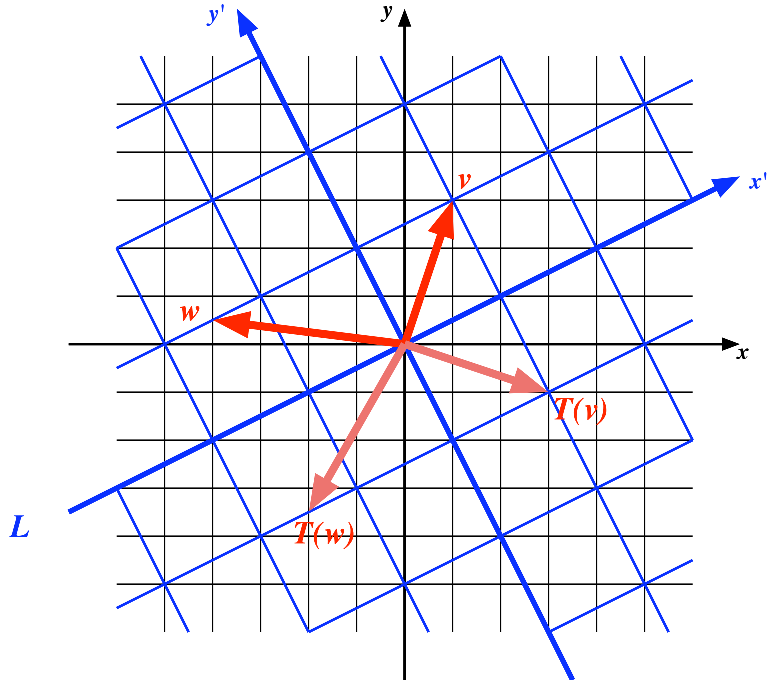

Figure 6: The reflection across the line $x - 2y = 0$. The original vectors are red, their images under the reflection are light red.

$\mathcal{B} = (\binom{2}{1}, \binom{-1}{2})$. The first basis vector $\binom{2}{1}$ is along the reflection mirror L, the second basis vector $\binom{-1}{2}$ is perpendicular to L.

The (x', y')-coordinate system is well-adapted to this particular transformation T. We can see that T preserves the x'-coordinate, but changes the sign on the y'-coordinate. For example, the vector $\vec{v}$ has coordinate vector $[\vec{v}]_{\mathcal{B}} = \binom{1}{1}$ and the reflected vector $T(\vec{v})$ has coordinate vector $[T(\vec{v})]_{\mathcal{B}} = \binom{1}{-1}$. Similarely, $[\vec{w}]_{\mathcal{B}} = \binom{-3/2}{1}$ and $[T(\vec{w})]_{\mathcal{B}} = \binom{-3/2}{-1}$. More generally, if a vector $\vec{a}$ has $[\vec{a}]_{\mathcal{B}} = \binom{x'}{y'}$, then $T(\vec{a})$ has $[T(\vec{a})]_{\mathcal{B}} = \binom{x'}{-y'}$.

The transformation $\binom{x'}{y'} \mapsto \binom{x'}{-y'}$ is accomplished by the matrix $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. This is the *matrix of T with respect to the basis $\mathcal{B}$*. Notation $[T]_{\mathcal{B}}$.

$$[T]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

The matrix $[T]_{\mathcal{B}}$ has the property that

$$\boxed{[T]_{\mathcal{B}}\,[\vec{a}]_{\mathcal{B}} = [T(\vec{a})]_{\mathcal{B}}} \qquad (6)$$

for every vector $\vec{a} \in \mathbb{R}^2$.

For our two example vectors this equation reads

$$\text{for } \vec{v}: \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\text{for } \vec{w}: \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} -3/2 \\ 1 \end{pmatrix} = \begin{pmatrix} -3/2 \\ -1 \end{pmatrix}$$

and in general

$$\text{for } \begin{pmatrix} x' \\ y' \end{pmatrix}: \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x' \\ -y' \end{pmatrix}$$

In words, we can reformulate Equation (6) as follows:

> The matrix of T with respect to $\mathcal{B}$ transforms the $\mathcal{B}$-coordinate vector of a vector into the $\mathcal{B}$-coordinate vector of the image vector.

or, put another way

> The matrix of T with respect to $\mathcal{B}$ describes the effect of T on $\mathcal{B}$-coordinate vectors.

2.1 How to find $[T]_\mathcal{B}$

Say the basis is $\mathcal{B} = (\vec{u}_1, \vec{u}_2)$. Since $\vec{u}_1 = 1\vec{u}_1 + 0\vec{u}_2$, we have $[\vec{u}_1]_\mathcal{B} = \binom{1}{0}$. Similarely, $[\vec{u}_2]_\mathcal{B} = \binom{0}{1}$. We can plug the vectors $\vec{u}_1$ and $\vec{u}_2$ into Formula (6). We get

$$[T]_\mathcal{B}\,[\vec{u}_1]_\mathcal{B} = [T(\vec{u}_1)]_\mathcal{B}$$

On the other hand,

$$[T]_\mathcal{B}\,[\vec{u}_1]_\mathcal{B} = [T]_\mathcal{B}\binom{1}{0}$$

which is the first column of $[T]_\mathcal{B}$.

Similarely, we have

$$\text{second column of } [T]_\mathcal{B} = [T]_\mathcal{B}\binom{0}{1} = [T]_\mathcal{B}\,[\vec{u}_2]_\mathcal{B} = [T(\vec{u}_2)]_\mathcal{B}$$

So the two coluns of $[T]_\mathcal{B}$ are given by $[T(\vec{u}_1)]_\mathcal{B}$ and $[T(\vec{u}_2)]_\mathcal{B}$.

$$[T]_\mathcal{B} = \begin{pmatrix} | & | \\ [T(\vec{u}_1)]_\mathcal{B} & [T(\vec{u}_2)]_\mathcal{B} \\ | & | \end{pmatrix}$$

In words:

> To find the matrix of T in the basis $\mathcal{B} = (\vec{u}_1, \vec{u}_2)$, Find the images of the basis vectors $T(\vec{u}_1)$ and $T(\vec{u}_2)$, and then express these in the basis $\mathcal{B}$, to find the coordinate vectors $[T(\vec{u}_1)]_\mathcal{B}$ and $[T(\vec{u}_2)]_\mathcal{B}$, finally, but the resulting coordiate vectors as columns of the matrix $[T]_\mathcal{B}$.

In the example of Figure 6, the first basis vector $\vec{u}_1$ is on the reflection line, so $T(\vec{u}_1) = \vec{u}_1$. The second basis vector $\vec{u}_2$ is perpendicular to the reflection line, so $T(\vec{u}_2) = -\vec{u}_2$. The corresponding coordinate vectors are $[T(\vec{u}_1)]_\mathcal{B} = [\vec{u}_1]_\mathcal{B} = \binom{1}{0}$ and $[T(\vec{u}_2)]_\mathcal{B} = [-\vec{u}_2]_\mathcal{B} = \binom{0}{-1}$, which give the matrix $[T]_\mathcal{B} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Of course, the same principle works in $\mathbb{R}^n$:

> If $\mathcal{B} = (\vec{u}_1, \ldots, \vec{u}_n)$ is a basis of $\mathbb{R}^n$, and $T : \mathbb{R}^n \to \mathbb{R}^n$ a linear operator, then
>
> $$[T]_\mathcal{B} = \begin{pmatrix} | & & | \\ [T(\vec{u}_1)]_\mathcal{B} & \cdots & [T(\vec{u}_n)]_\mathcal{B} \\ | & & | \end{pmatrix}$$

For another example, suppose E is a plane through the origin in $\mathbb{R}^3$. Suppose $\vec{u}_1$ is a normal vector to E, and that $\vec{u}_2$ and $\vec{u}_3$ are two vectors spanning E. Then $\mathcal{B} = (\vec{u}_1, \vec{u}_2, \vec{u}_3)$ is a basis of $\mathbb{R}^3$. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the orthogonal projection onto E. Then we have

$$T(\vec{u}_1) = \vec{0} \qquad \text{so} \qquad [T(\vec{u}_1)]_\mathcal{B} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$T(\vec{u}_2) = \vec{u}_2 \qquad \text{so} \qquad [T(\vec{u}_2)]_{\mathcal{B}} = [\vec{u}_2]_{\mathcal{B}} = \left(\begin{smallmatrix} 0 \\ 1 \\ 0 \end{smallmatrix}\right)$$

$$T(\vec{u}_3) = \vec{u}_3 \qquad \text{so} \qquad [T(\vec{u}_3)]_{\mathcal{B}} = [\vec{u}_3]_{\mathcal{B}} = \left(\begin{smallmatrix} 0 \\ 0 \\ 1 \end{smallmatrix}\right)$$

And therefore the matrix of T with respect to $\mathcal{B}$ is

$$[T]_{\mathcal{B}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{7}$$

2.2 How to convert between $[T]_\mathcal{B}$ and $[T]_\mathcal{S}$

Let us combine Equation (6) with Equation (3) to obtain:

$$[T]_\mathcal{B}\, P^{-1}\, [\vec{a}]_\mathcal{S} = P^{-1}\, [T(\vec{a})]_\mathcal{S}$$

Multiplying through by P on the left gives

$$\left(P\,[T]_\mathcal{B}\, P^{-1}\right)[\vec{a}]_\mathcal{S} = P\, P^{-1}\, [T(\vec{a})]_\mathcal{S} = [T(\vec{a})]_\mathcal{S} \tag{8}$$

On the other hand, the matrix which converts $[\vec{a}]_\mathcal{S}$ into $[T(\vec{a})]_\mathcal{S}$ is $[T]_\mathcal{S}$:

$$[T]_\mathcal{S}\, [\vec{a}]_\mathcal{S} = [T(\vec{a})]_\mathcal{S} \tag{9}$$

Comparing Equations (8) and (9), we see that the two matrices $P\,[T]_\mathcal{B}\, P^{-1}$ and $[T]_\mathcal{S}$ have the same effect on all vectors $[\vec{a}]_\mathcal{S}$. Therefore, these two matrices have to be equal:

$$\boxed{[T]_\mathcal{S} = P\,[T]_\mathcal{B}\, P^{-1}}$$

Multiplying this equation by P^{-1} on the left and P on the right, we get the other conversion formula

$$\boxed{[T]_\mathcal{B} = P^{-1}\,[T]_\mathcal{S}\, P}$$

Example 1 Returning to Figure 6, recall that $\vec{u}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\vec{u}_2 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}$, so that the transition matrix is $P = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$, whose inverse is $P^{-1} = \frac{1}{5}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$. Thus, we have

$$[T]_\mathcal{S} = P\,[T]_\mathcal{B}\, P^{-1} = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tfrac{1}{5} \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} = \tfrac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

This is the standard matrix of T. So we can deduce the formula for T in standard coordinates:

$$T\begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{5} \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \tfrac{1}{5} \begin{pmatrix} 3x + 4y \\ 4x - 3y \end{pmatrix}$$

Example 2 Let us return to the Example leading to Equation 7. To be specific, say that E has equation $2x + y - 3z = 0$. Then we can take $\vec{u}_1 = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$ as normal vector. Two vectors which span the plane are $\vec{u}_2 = \begin{pmatrix} 0 \\ 3 \\ 1 \end{pmatrix}$ and $\vec{u}_3 = \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$. This gives the transition matrix

$$P = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \\ -3 & 1 & 0 \end{pmatrix}$$

with inverse

$$P^{-1} = \tfrac{1}{14} \begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & 5 \\ -10 & 2 & -6 \end{pmatrix}$$

And hence

$$[T]_{\mathcal{S}} = P\,[T]_{\mathcal{B}}\,P^{-1} = \begin{pmatrix} 2 & 0 & -1 \\ 1 & 3 & 2 \\ -3 & 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \frac{1}{14} \begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & 5 \\ -10 & 2 & -6 \end{pmatrix} = \frac{1}{14} \begin{pmatrix} 10 & -2 & 6 \\ -2 & 13 & 3 \\ 6 & 3 & 5 \end{pmatrix}$$

2.3 Exercises

Exercise 2.1 Find the standard matrix of the reflection across the line $2x+5y=0$.

Exercise 2.2 Find the standard matrix of the orthogonal projection onto the line $x-3y=0$.

Exercise 2.3 Find the standard matrix of the reflection acros the plane $x+y+2z=0$.