Today's Plan

- Integral as area under a curve
- Riemann Sums (approximation and definition of the definite integral)
- Ch 2 in the notes
- Properties of the integral (Ch 5.2)

If \( f \) is a positive continuous function then \( \int_a^b f(x) \, dx \) "definite integral of \( f \) from \( a \) to \( b \)" is the area under the graph of \( f \) e.g. between the graph of \( f \) and the \( x \)-axis.

Ex: \( f(x) = \begin{cases} x + 2 & -2 \leq x < 0 \\ \frac{1}{\sqrt{-x^2 + 1}} & 0 \leq x < 1 \end{cases} \)

The circle \((x, y)\) is \( x^2 + y^2 = 1 \).

\[
\int_{-2}^{1} f(x) \, dx = \int_{-2}^{-1} (x + 2) \, dx + \int_{0}^{1} 1 \, dx + \int_{0}^{1} \frac{1}{\sqrt{-x^2 + 1}} \, dx
\]

\[
= \left[ \frac{1}{2} x^2 + 2x \right]_{-2}^{-1} + 1 + \int_{0}^{1} \frac{1}{\sqrt{-x^2 + 1}} \, dx
\]

\[
= \frac{1}{2} + 1 + \frac{\pi}{4}
\]
Area is \( \int_a^b f(x) \, dx \)
Riemann Sum Approximation

- Approximate the area by a sum of areas of rectangles

1. \( f(x) = x \), 4 segments

\[
\int_{0}^{1} f(x) \, dx \approx \frac{1}{4}(\frac{1}{4}) + \frac{1}{4}(\frac{1}{2}) + \frac{1}{4}(\frac{3}{4}) + \frac{1}{4}(1) \\
= \frac{1}{16}[1 + 2 + 3 + 4] = \frac{10}{16} = \frac{5}{8}
\]

For \( N \) segments

\( \Delta x = \frac{1}{N} \)

- Label the points

\( b = 0, \quad x_1 = \frac{1}{N}, \quad x_2 = \frac{2}{N}, \ldots, \quad x_N = \frac{N}{N} = 1 \)

\( x_i = \frac{i}{N} \)

- The height of the \( i^{th} \) rectangle is

\( f(x_i) = x_i \) [for right endpoint Riemann Sums]
\[ f(x) = x \]

\[ \Delta x = \frac{1}{4} \]

\[ X_0 \ x_1 \ x_2 \ x_3 \ x_4 \]

\[ X_{N-1} \]

\[ \Delta x = \frac{1}{N} \]
\[ \int_0^1 f(x) \, dx \leq \sum_{i=1}^{N} f(x_i) \Delta x \]

\[ \sum_{i=1}^{N} \frac{1}{N} = \frac{1}{N} \sum_{i=1}^{N} i = \frac{1}{N} \frac{N(N+1)}{2} \]

\[ = \frac{N^2 + N}{2N^2} = \frac{1}{2} + \frac{1}{2N} \]

Riemann's definition of the integral is

\[ \int_0^1 f(x) \, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta x \]

where \( \Delta x = \frac{1}{N} \)

\[ x_i = \frac{i}{N} \]

\[ = \lim_{N \to \infty} \sum_{i=1}^{N} f\left( \frac{i}{N} \right) \left( \frac{1}{N} \right) \]
Ex: \( f(x) = e^x \)

The \( N^{th} \) Right-Riemann sum is

\[
\sum_{i=1}^{N} e^{in \left( \frac{1}{N} \right)} = \sum_{i=1}^{N} e^{\frac{in}{N}} \left( \frac{1}{N} \right)
\]

\[
= \frac{1}{N} \sum_{i=1}^{N} (e^{in})^j
\]

Shift to \( j = i - 1 \)

\[
= \frac{1}{N} \sum_{j=0}^{N-1} (e^{in})^j
\]

\[
= \frac{e^{in}}{N} \left[ 1 - (e^{in})^{N-1} \right]
\]

\[
= \frac{1}{N} \left( \frac{1 - e^{in}}{1 - e^{in/N}} \right)
\]

Use L'Hopital's (Taylor expansion) \( e^{-x} = 1 - x + R(x) \) Remainder

\[
\lim_{N \to \infty} \frac{1}{N} \left( \frac{1 - e}{1 - e^{1/N}} \right) = \lim_{N \to \infty} \frac{(1-e)}{-1 + NR(\frac{1}{N})} = e - 1
\]
Taylor Series (and L'Hopital's rule)

Approximating a function \( f(x) \) for small values of \( x \)

Thm: If \( f \) is continuously differentiable at \( x=0 \) then \( f(x) = f(0) + f'(0)x + R(x) \)

where \( \lim_{x \to 0} \frac{R(x)}{x} = 0 \) \( \tag{3} \)

This useful to calculate limits

\[
\lim_{x \to 0} \frac{f(x)}{x} = \lim_{x \to 0} \left[ \frac{f(0) + f'(0)x + R(x)}{x} \right]
\]

if \( f(0) = 0 \)

and \( f \) is cont. diff

\[
= \lim_{x \to 0} f'(0) + \frac{R(x)}{x}
\]

\[
= f'(0)
\]

L'Hopital: If \( f \) and \( g \) are cont. diff, \( f(0) = g(0) = 0 \)

\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)} = \frac{f'(0)}{g'(0)} \quad \text{and} \quad g'(0) \neq 0
\]
$f(x) = e^x$

$\ell(x) = f(0) + f'(0) x$
\[
\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f(0) + f'(0)x + R_1(x)}{g(0) + g'(0)x + R_2(x)}
\]

\[
= \lim_{x \to 0} \frac{f'(0) + \frac{R_1(x)}{x}}{g'(0) + \frac{R_2(x)}{x}}
\]

Since the limit exists on the top and bottom (Taylor's theorem) and the bottom is non-zero, then

\[
= \frac{f'(0)}{g'(0)}
\]
General Domain Riemann Sum

\[ \int_{a}^{b} f(x) \, dx = \lim_{N \to \infty} \sum_{i=1}^{N} f(x_i) \Delta x \]

where \( \Delta x = \frac{b-a}{N} \)

and \( x_i = a + i \Delta x \)

\[ x_0 = a \Delta x \]

N times \( \Delta x \) is the total width \( b-a \)

\[ \Delta x = \frac{b-a}{N} \]

\[ x_i = a + i \Delta x = a + i \frac{(b-a)}{N} \]

\[ x_N = a + N \frac{(b-a)}{N} = b \]

We could also use the left end points

The \( N^{th} \) approximation is

\[ \sum_{i=0}^{N-1} f(x_i) \Delta x \]
Example Riemann Sum Problem

Express \( \lim_{N \to \infty} \sum_{i=1}^{N} \cos \left( \frac{6i}{N} \right) \frac{1}{N} \)

a) as an integral from \( x=0 \) to \( x=1 \)

We need \( \cos \left( \frac{6i}{N} \right) = f(X_i) \) and \( \frac{1}{N} = \Delta x \)
\[ \Delta x = \frac{1}{N} \quad X_i = \frac{i}{N} \]
\[ \cos \left( \frac{6i}{N} \right) \frac{1}{N} = \cos \left( 6X_i \right) \Delta x \]
\[ f(X_i) \]

Answer 13 \( \int_{0}^{1} \cos (6x) \, dx \)

b) as an integral from \( x=0 \) to \( x=6 \)
\[ \Delta x = \frac{6}{N} \quad X_i = \frac{6i}{N} \]
\[ \cos \left( \frac{6i}{N} \right) \frac{1}{N} = \cos \left( X_i \right) \Delta x \]
\[ \frac{1}{6} \]

Answer \( \int_{0}^{6} \cos (x) \, dx \)