Large deviations of SLE and Weil-Petersson Teichmüller space

Lecture 3

— Yilin Wang (MIT)

Loewner energy

SLE$_\kappa$ $\longleftrightarrow$ Weil-Petersson Teichmüller space

Schramm-Loewner evolutions

Probabilistic	Deterministic
• Stochastic analysis	• quasicircles
• Conformally invariant 2D-statistical mechanics model	• Quasiconformal mapping
• Random planar maps	• Geometric function theory
• 2D-quantum gravity	• Complex structures on Riemann surfaces
• Conformal field theory	• Kähler geometry
	• String theory

Monday:

I) Brownian Motion and Dirichlet energy

Schilder's theorem

II) SLE and Loewner energy

$W = \sqrt{\kappa} B$ $\qquad$ $I(\gamma) = \frac{1}{2}\int_0^\infty \dot{W}^2(t)\,dt$

III) SLE_{0+} large deviations

Energy reversibility from SLE reversibility

Tuesday:

I) Loop energy

Generalizes chordal energy

II) Weil-Petersson Teichmüller space

$I^L(\gamma) < \infty \iff \gamma$ is Weil-Petersson

and 25 other equivalent definitions ...

Today:

I) Radial SLE_∞ large deviations

II) Foliations by Weil-Petersson quasicircles

Motivation: SLE duality

For $\kappa \geq 8$

[Dubédat] [Zhan] [Miller-Sheffield]

locally a $SLE_{16/\kappa}$ curve

SLE_κ

Asymptotic behavior of SLE_{0+} $\xleftrightarrow{\text{dual}}$ SLE_∞

LDP $\updownarrow$

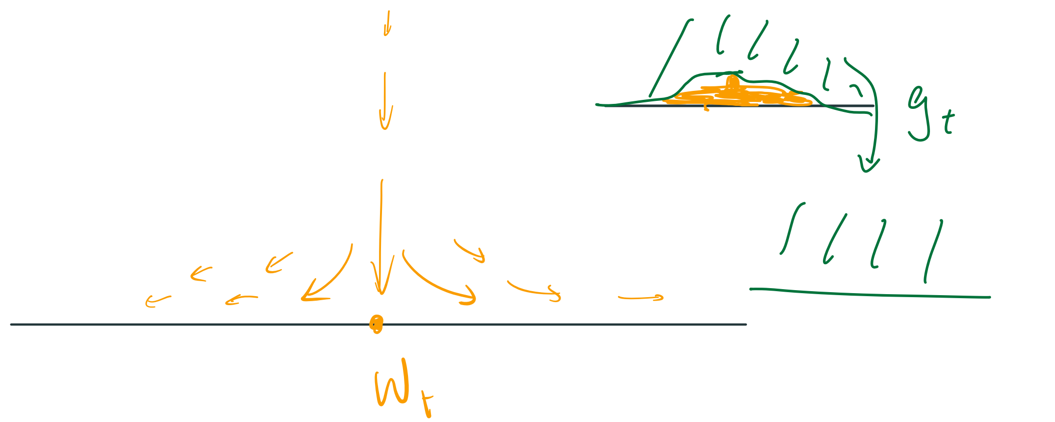
(W-P. quasicircle) Loewner energy $\xleftrightarrow{[V.W.]}$ Loewner-Kufarev energy

Q: What happens when we let $\kappa \to \infty$?

For SLE_κ, $\forall z \in \mathbb{H}$

$$\frac{d}{dt} g_t(z) = \frac{2}{g_t(z) - \sqrt{\kappa} B_t}, \quad g_0(z) = z.$$

$\tau(z) :=$ maximal solution time.



$$H_t := \{ z \in \mathbb{H} \mid \tau(z) > t \}$$

$$= \text{domain of definition of } g_t = \mathbb{H} \setminus \gamma[0,t]$$

Can show: $g_t(z) \xrightarrow{k\to\infty} z$.

Not so interesting.

Issue: normalized at a boundary point (∞)

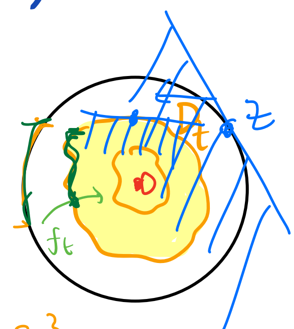
I) Radial SLE_∞ Large deviations.

1) Loewner–Kufarev equation

$$\mathcal{N}_+ := \left\{ (\rho_t)_{t\geq 0} \;\middle|\; \begin{array}{l}\rho_t \in \mathrm{Prob}(S^1) \\ \text{measurable in } t\end{array} \right\}$$

Loewner–Kufarev equation $\quad z \in \mathbb{D}$

(PDE) $\quad \begin{cases} \partial_t f_t(z) = -z f_t'(z) \underbrace{\displaystyle\int_{S^1} \frac{e^{i\theta}+z}{e^{i\theta}-z} \rho_t(d\theta)}_{\mathbb{D} \to \{z \mid \mathrm{Re}(z) > 0\}} \\ f_0(z) = z \end{cases}$



$f_t : \mathbb{D} \to D_t$ with $f_t(0) = 0$, $f_t'(0) = e^{-t}$.

$(\rho_t)_{t\geq 0} \longleftrightarrow (f_t)_{t\geq 0} \longleftrightarrow$ Evolution family $(D_t)_{t\geq 0}$

(The normal velocity of ∂D_t at $f_t(e^{i\theta})$ is $2\pi \rho_t(\theta) |f_t'(e^{i\theta})|$ if $\rho_t(\theta)\, d\theta = \rho_t$)

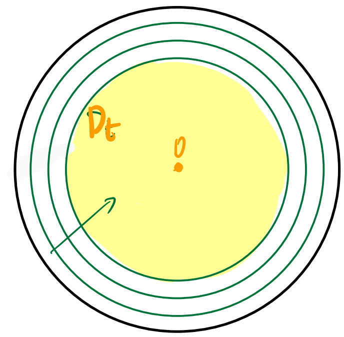
$g_t = f_t^{-1}$ satisfies

(ODE) $$\partial_t g_t(z) = g_t(z) \int_{S^1} \frac{e^{i\theta}+g_t(z)}{e^{i\theta}-g_t(z)} \rho_t(d\theta)$$

Examples:

- $\rho_t(d\theta) = \frac{1}{2\pi}\, d\theta \quad \forall\, t \geq 0$

  $\Rightarrow \; D_t = e^{-t}\,\mathbb{D}$

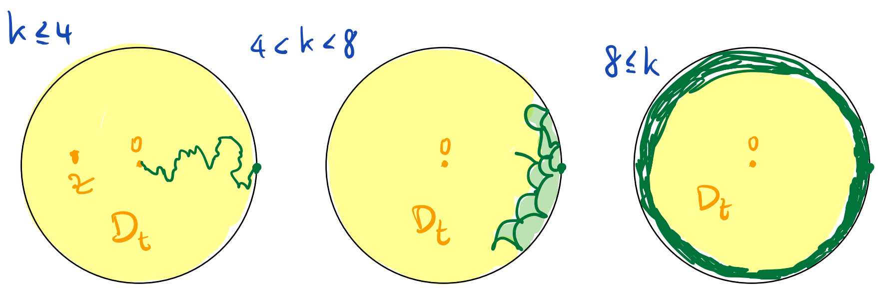


- $\rho_t(d\theta) = \delta_{e^{i\underbrace{B_{\kappa t}}}}$

  Dirac measure

  Brownian motion on $\mathbb{R}$

Radial SLE_κ



2) $\kappa \to \infty$ limit.

Let $\kappa \to \infty$ in $\rho_t(d\theta) = \delta_{e^{iB_{\kappa t}}}$

$t \rightsquigarrow t + \Delta t$

$$\Delta g_t(z) = \int_t^{t+\Delta t} g_t(z) \int_{S^1} \frac{e^{i\theta} + g_t(z)}{e^{i\theta} - g_t(z)}\, \delta_{e^{iB_{\kappa s}}}(d\theta)\, ds$$

$$\simeq g_t(z) \int_{S} \frac{e^{i\theta} + g_t(z)}{e^{i\theta} - g_t(z)} \left(L^k_{t+\Delta t}(\theta) - \underbrace{L^k_t(\theta)}_{\text{occupation measure up to time } t} \right) d\theta$$

$$\xrightarrow{k \to \infty} \Delta t \, g_t(z) \int_{S^1} \frac{e^{i\theta} + g_t(z)}{e^{i\theta} - g_t(z)} \frac{|d\theta|}{2\pi}$$

$$= \Delta t \, g_t(z)$$

$$\Rightarrow \quad \partial_t g_t(z) = g_t(z) \; ; \qquad g_0(z) = z$$

$$\Rightarrow \quad g_t(z) = e^t z : D_t \to \mathbb{D}$$

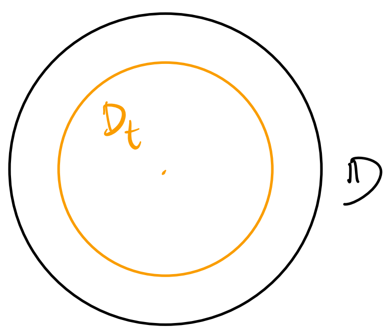
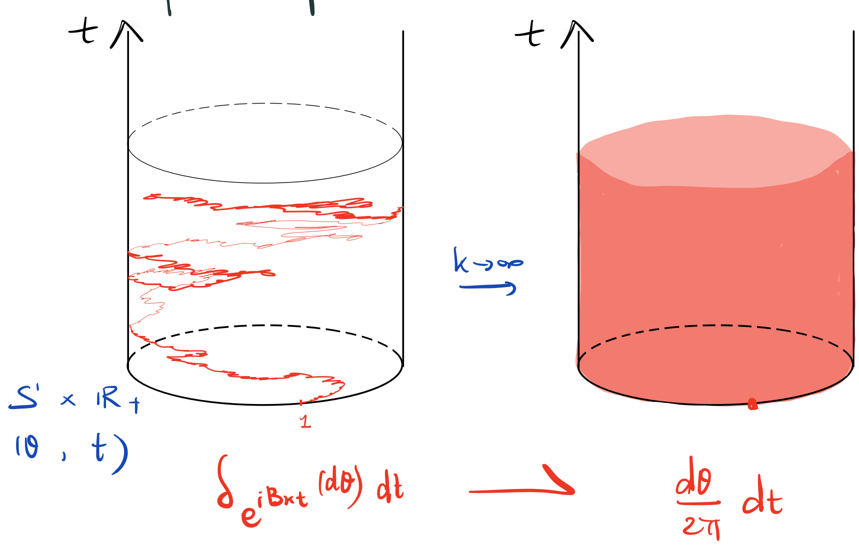


Illustration of occupation measure.



3) Radial SLE_∞ LDP

Thm (Ang - Park - W. '20)

Radial SLE_κ process satisfies the LDP as $\kappa \to \infty$ with rate function S_+ (Loewner - Kufarev energy)

" $\mathbb{P}(SLE_\kappa \simeq (D_t)_{t \geq 0}) \simeq \exp(-\kappa S_+(\rho))$ as $\kappa \to \infty$ "

where

$$S_+(\rho) := \int_0^\infty L(\rho_t)\, dt$$

$$L(\rho_t) := \frac{1}{2} \int_{S^1} |\nu_t'(\theta)|^2 \, d\theta$$

if $\rho_t = \nu_t^2(\theta)\, d\theta$ and $L(\rho_t) = \infty$ otherwise.

(Follows from Donsker - Varadhan theorem on the LDP of occupation measures)

⚠ $S_+(\rho) < \infty$ only for a.c measures on $S^1 \times \mathbb{R}_+$.

More regular than Dirac masses !!!

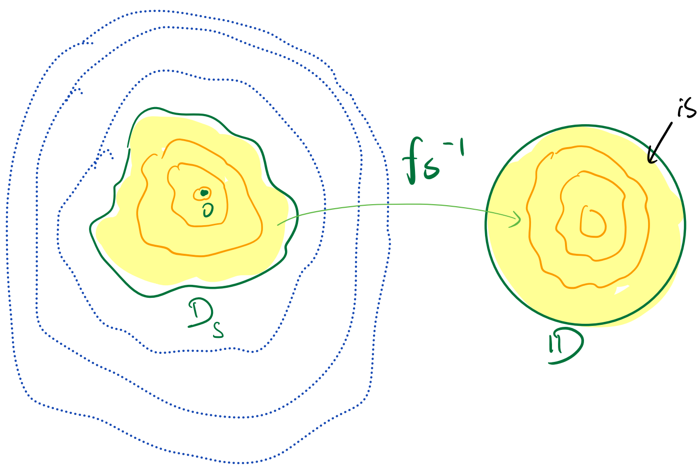
Q: What are the families (D_t) generated by (ρ_t), where $S_+(\rho) < \infty$?

II) Foliations by Weil-Petersson quasi circles

Whole-plane Loewner-Kufarev equation.

$\rho = (\rho_t)_{t\in\mathbb{R}} \rightarrow (D_t)_{t\in\mathbb{R}}$ and $(f_t : \mathbb{D} \to D_t)$ with $f_t(0)=0$, $f_t'(0) = e^{-t}$

such that



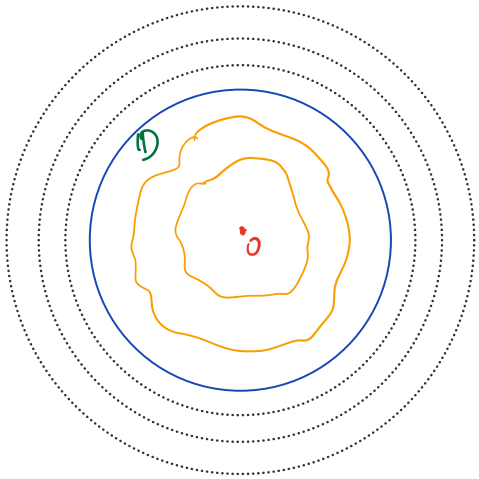
is the Loewner-Kufarev chain generated by $(\rho_{t+s})_{t\geq 0}$

$$S(\rho) := \int_{-\infty}^{\infty} L(\rho_t)\, dt$$

Claim : Loewner-Kufarev chain in $\mathbb{D}$ is a special case of whole plane L-K chain.

Given $(\rho_t)_{t\geq 0}$

Set $\rho_t = \frac{1}{2\pi} d\theta$ for all $t<0$



$D_0 = \mathbb{D}$

- $D_t = e^{-t}\,\mathbb{D}$ for $t<0$
- $(D_t)_{t\geq 0}$ is the family generated by $(\rho_t)_{t\geq 0}$.

Thm (Viklund. W.)

If $S(\rho) < \infty$, then

- ∂D_t is a Weil-Petersson quasicircle $\forall t \in \mathbb{R}$
- $\bigcup \partial D_t = \mathbb{C} \setminus \{0\}$.
- $t \mapsto \partial D_t$ is continuous in the sup-norm.

"Foliation" of $\mathbb{C}\setminus\{0\}$ by Weil-Petersson quasi-circles

- Non-smooth
- Monotone, but not strictly monotone.
- ∂D_t is called a leaf

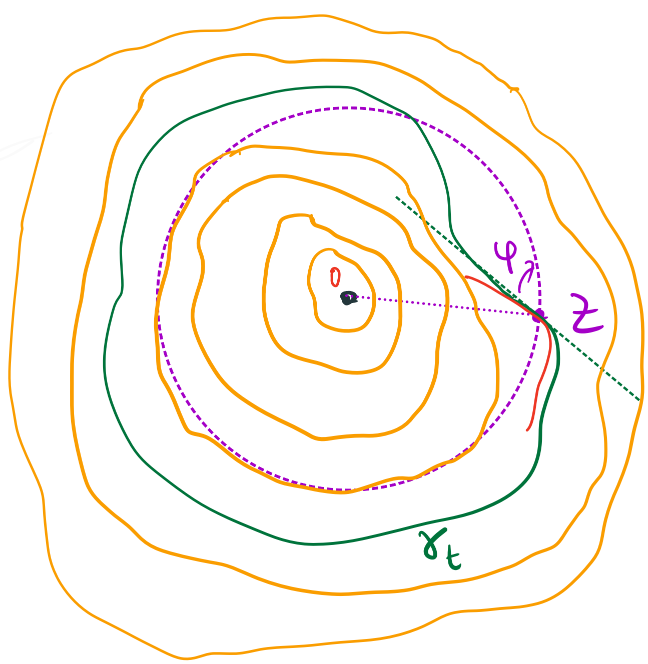
Finite L-K energy foliation has finite Loewner energy leaves.

We will prove it by showing a quantitative result.

Recall $g_t = f_t^{-1} : D_t \to \mathbb{D}$

Define $\varphi(z) := \arg \dfrac{g_t'(z)\, z}{g_t(z)}$

if $z \in \partial D_t$



φ is the winding function of the foliation generated by $(\rho_t)_{t\in\mathbb{R}}$.

Thm (V.-W.)

$$16\, S(\rho) = \frac{1}{\pi}\int_{\mathbb{C}} |\nabla\varphi|^2\, dA(z) =: \mathcal{D}(\varphi)$$

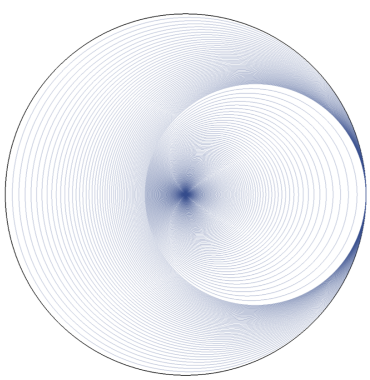
- "16" is consistent with SLE duality

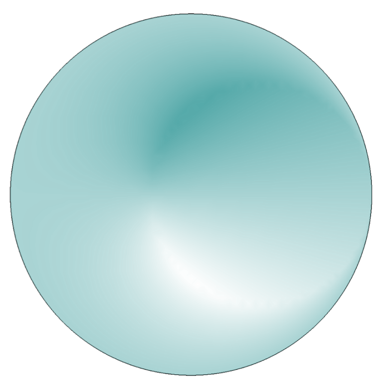
$$\kappa \leftrightarrow \frac{16}{\kappa}$$

Example:

$$\rho_t(d\theta) = \begin{cases} \frac{1}{\pi}\sin^2\left(\frac{\theta}{2}\right)d\theta & \text{for } t\in[0,1]; \\ \frac{1}{2\pi}\, d\theta & \text{otherwise.} \end{cases}$$

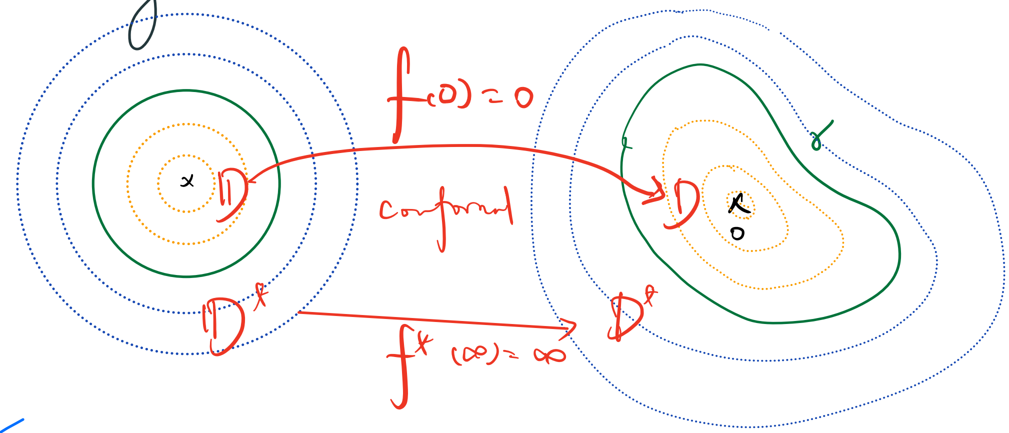
foliation $(\partial D_t)_{t\geq 0}$





φ

Corollary $(S(\rho) \Leftrightarrow I^L(\gamma))$



ρ^γ generates the foliation formed by equipotentials.

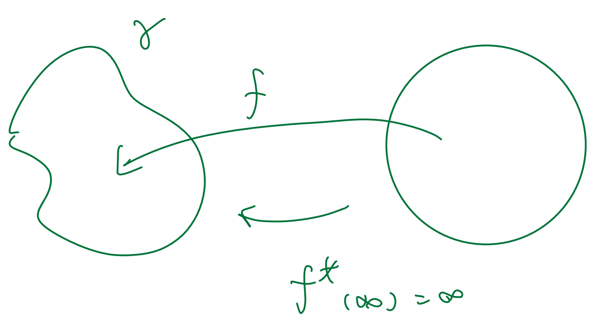
$$16\, S(\rho^\gamma) = I^L(\gamma) + 2\log\left|\frac{f^{*\prime}(\infty)}{f'(0)}\right|$$

φ is harmonic in $\hat{\mathbb{C}} \setminus \gamma$

(We recall $I^L(\gamma) = \frac{1}{\pi}\int_{\mathbb{D}} |\nabla \arg f'|^2 \, dA$

$$+ \frac{1}{\pi}\int_{\mathbb{D}^*} |\nabla \arg f^{*\prime}|^2 \, dA$$

$$+ 4\log\left|\frac{f'(0)}{f^{*\prime}(\infty)}\right| \quad)$$



Question from T. Amaba:

16 from SLE duality?

$$\mathbb{P}(\text{SLE}_\kappa \text{ loop stays close to } \gamma)$$

$$\underset{\kappa\to 0}{\sim} \exp\left(-\frac{I^L(\gamma)}{\kappa}\right)$$

$$\mathbb{S}\|$$

$$\mathbb{P}(\partial(\text{SLE}_{\underbrace{16/\kappa}_{\kappa'}})_t \text{ stays close to } \gamma)$$

$$\mathbb{S}\|$$

$$\exp\left(-\kappa' \inf_{\substack{\rho \\ \partial D_t = \gamma}} S(\rho)\right)$$

$$\|$$

$$\exp\left(-\frac{16}{\kappa} S(\rho^\gamma)\right)$$

Cor. (V.W.) Definition 27

A Jordan curve γ separating 0 and ∞ is Weil-Petersson $\Longleftrightarrow$ γ can be realized as a leaf in the foliation generated by a measure with $S(\rho) < \infty$.

Proof: $\Rightarrow$ follows from previous corollary.

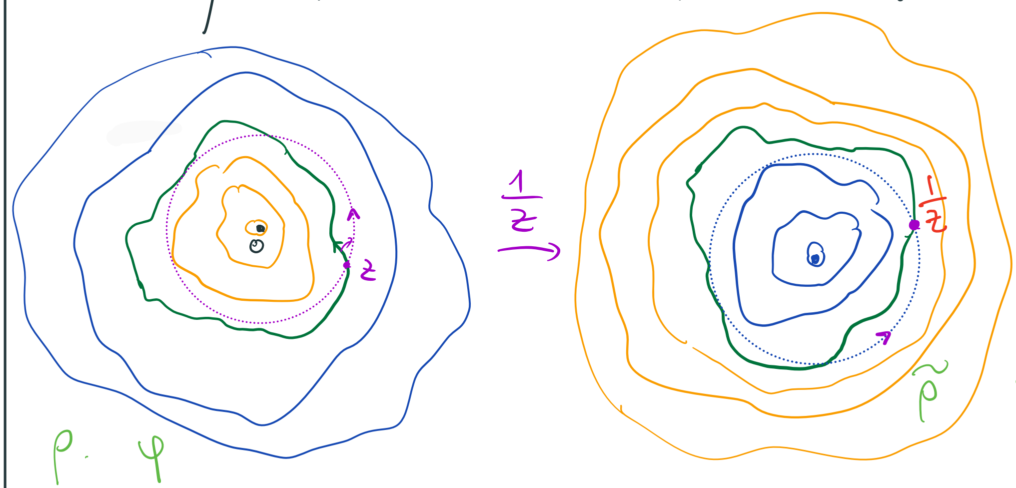
$\Leftarrow$ $\quad I^L(\gamma) \le 16\, S(\rho^\gamma) \le 16\, S(\rho)$

winding function φ^γ $\qquad \varphi$

$\varphi^\gamma|_\gamma = \varphi|_\gamma$ and φ^γ is harmonic in $\mathbb{C} \setminus \gamma$.

since $\log \left| \frac{f^{*\prime}(\infty)}{f'(0)} \right| \ge 0$ true for all Jordan curves separating 0 and ∞. $\square$

Reversibility of Loewner-Kufarev energy



$$S(\rho) = S(\tilde{\rho}).$$

$(\ast)$

Proof: $\tilde{\varphi}(z) = \varphi(\frac{1}{z}) \Rightarrow \mathcal{D}(\tilde{\varphi}) = \mathcal{D}(\varphi)$. ∎

Remark:

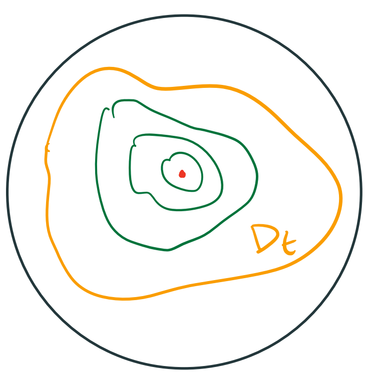
Reversibility of radial SLE_κ for $\kappa > 8$ is not known.

this result suggests it to be true.

Proof sketch of ✶ $\quad 16\, S(\rho) = \mathcal{D}(\varphi)$

- Dirichlet energy is conformally invariant.
- Loewner chain $\longrightarrow$ Explore a conformally invariant object layer by layer.

Assume $(\rho_t)_{t\geq 0}$ generates a foliation of $\mathbb{D}$



ϕ a function on $\mathbb{D}$, s.t. $\mathcal{D}(\phi) < \infty$

$\phi|_{\partial\mathbb{D}} = 0 \qquad (\phi \in W_0^{1,2}(\mathbb{D}))$

$$\phi = \underset{\substack{\uparrow \\ \text{harmonic} \\ \text{in } D_t}}{\phi^{h,t}} + \phi^{0,t} \quad \in W_0^{1,2}(D_t)$$

0 outside of D_t

$\rho = \rho_t(d\theta)\, dt$

Thm (VW) Disintegration isometry

$$\left(W_0^{1,2}(\mathbb{D}),\ \mathcal{D}^{1/2}\right) \longrightarrow L^2(S^1\times\mathbb{R}_+,\ 2\rho)$$

$$\iota:\quad \phi \mapsto \frac{1}{2\pi}\int_{\mathbb{D}} \Delta(\phi\circ f_t)(z)P_{\mathbb{D}}(z,e^{i\theta})\,\mathrm{d}A(z). \text{``}= \partial_n(\phi^{0,t}\circ f_t)\text{''}$$

is an bijective isometry with inverse operator:

$$\varkappa[u](w) = 2\pi\int_0^{\tau(w)} \underbrace{P_{\mathbb{D}}[u_t\rho_t](g_t(w))}_{\text{harmonic function in } D_t}\,\mathrm{d}t, \qquad u_t(\cdot) := u(\cdot,t).$$

A consequence: GFF → White noise decomp. generalizes [Hedenmalm–Nieminen]

Proof of $\star$

If φ = winding function

$\rho = \nu_t^2(\theta)\,d\theta\,dt$

Show $\iota(\varphi)(\theta,t) = \dfrac{-2\nu_t'}{\nu_t}$.

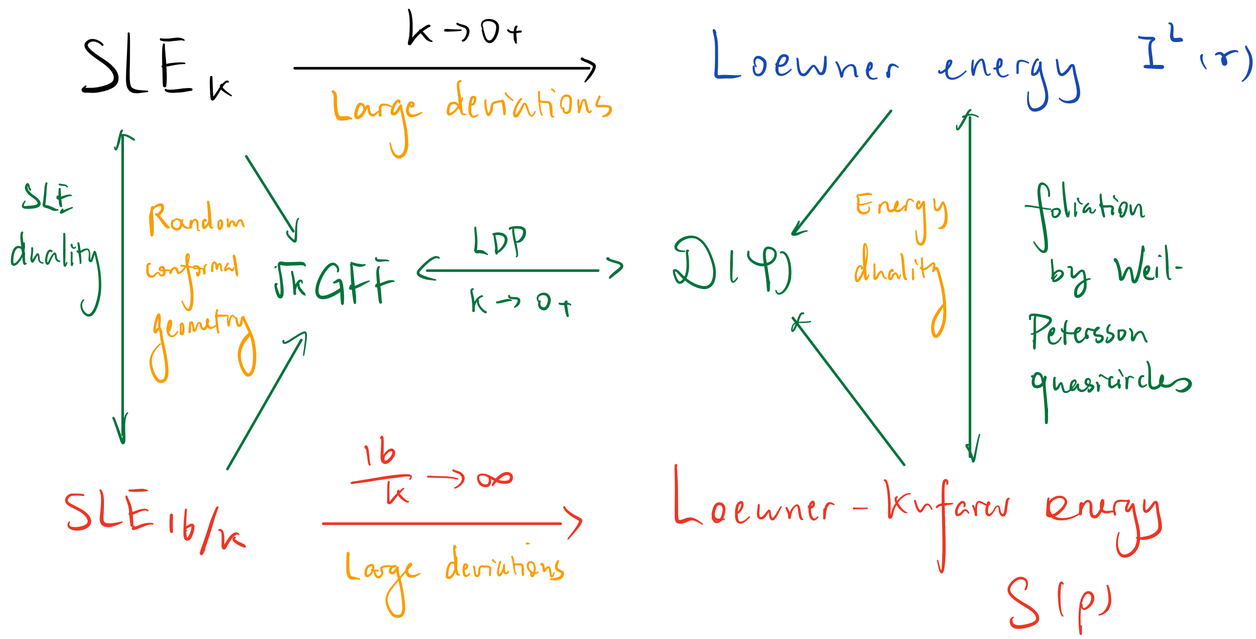
$\mathcal{D}(\varphi)$

$\|$

$$\|\iota(\varphi)\|^2_{L^2(2\rho)} = \int_0^\infty\int_{S^1} \frac{4(\nu_t')^2}{\nu_t^2}\,2\nu_t^2(\theta)\,d\theta\,dt$$

$$= 16\int_0^\infty L(\rho_t)\,dt = 16\,S_+(\rho). \quad \square$$

Conclusion



Wang Large deviations of Schramm-Loewner evolutions: A survey 2021

[VW20b] Fredrik Viklund and Yilin Wang. The Loewner-Kufarev Energy and Foliations by Weil-Petersson Quasicircles. *arXiv preprint: 2012.05771*, 2020.

[APW20] Morris Ang, Minjae Park, and Yilin Wang. Large deviations of radial SLE_∞. *arXiv preprint: 2002.02654*, 2020.

[Wan19b] Yilin Wang. Equivalent descriptions of the Loewner energy. *Invent. Math.*, 218(2):573–621, 2019.