OOPS II

Monday, August 2, 2021 10:02 PM

I. Review

Stationary partition function

$$Z^{\theta}_{N,t} = \int_{-\infty < s_0 < \cdots < s_{N-1} < t} e^{\theta s_0 - B_0(s_0) + B_1(s_1) - B_1(s_0) + \cdots + B_N(t) - B_N(s_{N-1})} \, d\underline{s}.$$

Showed: $\mathrm{Var}(\log Z^{\theta}_{N,t}) \leq C N^{2/3}$

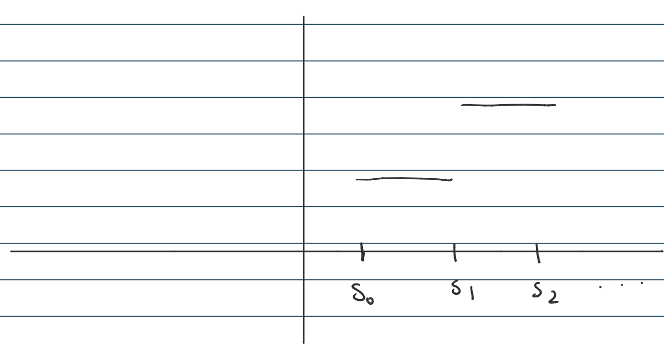
if $\quad |N\psi_1(\theta) - t| \leq C' N^{2/3}$

$$\psi_1(\theta) = \frac{d}{d\theta} \frac{\Gamma'(\theta)}{\Gamma(\theta)}, \quad \Gamma = \text{Gamma function}$$

$$\log Z^{\theta}_{N,t} = \underbrace{\sum_{j=1}^{N} r_j(t)}_{\text{i.i.d. sum}} + B_0(t) - \theta t$$

$$\Longrightarrow \mathrm{Var}(\log Z^{\theta}_{N,t}) = N\psi_1(\theta) - t + 2\mathbb{E}[s_0^+].$$

$$\leq N^{2/3} + 2\,\mathbb{E}[\,E[s_0]^2]^{1/2}$$



II. Comparing $E[s_0]$ and $E[s_0^+]$.

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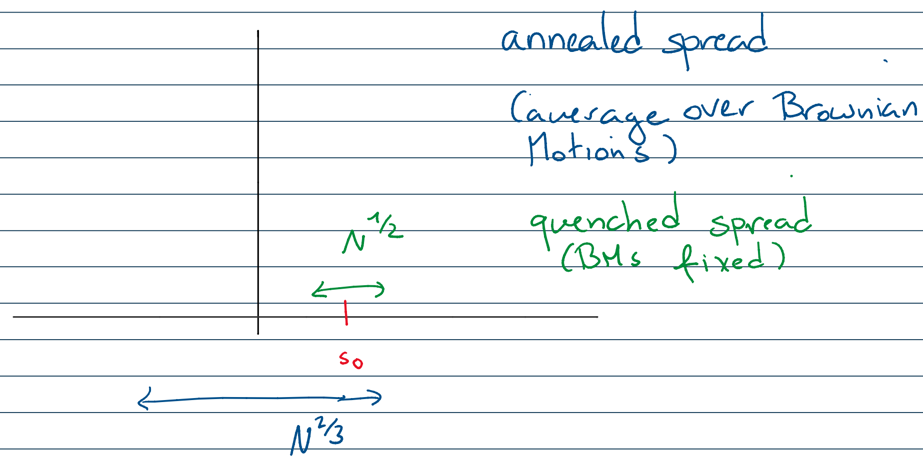
$$E[s_0]^2 = E[s_0^+]^2 + E[s_0^-]^2 - 2E[s_0^+]E[s_0^-]$$

$$s_0^- = \max\{-s_0, 0\}$$

$$\Longrightarrow \quad E[s_0^+]^2 \le E[s_0]^2 + 2E[s_0^+]E[s_0^-]$$

Claim: $\mathbb{E}\big[E[s_0^+]E[s_0^-]\big] = O(N)$

s_0^+ and s_0^- are not simultaneously large under quenched measure.



$$E\left[\left(s_0 - E[s_0]\right)^2\right]$$

$$= E\left[\left(s_0^+ - s_0^- - E[s_0^+] + E[s_0^-]\right)^2\right]$$

$$= E\left[\left(s_0^+ - E[s_0^+]\right)^2\right] + E\left[\left(s_0^- - E[s_0^-]\right)^2\right]$$

$$- 2E\left[\left(s_0^+ - E[s_0^+]\right)\left(s_0^- - E[s_0^-]\right)\right]$$

By disjoint support of s_0^+, s_0^-:

$$E[(s_0^+ - E[s_0^+])(s_0^- - E[s_0^+])]$$

$$= \underbrace{E[s_0^+ s_0^-]}_{=0} - E[s_0^+]E[s_0^-] - E[s_0^+]E[s_0^-] + E[s_0^+]E[s_0^-]$$

$$= - E[s_0^+]E[s_0^-]$$

$$\mathrm{Var}(s_0) = E[(s_0 - E[s_0])^2]$$

$$= \mathrm{Var}(s_0^+) + \mathrm{Var}(s_0^-) + 2E[s_0^+]E[s_0^-]$$

All terms are non negative!

$$E[s_0^+]E[s_0^-] \leq \mathrm{Var}(s_0).$$

Taking expectations:

$$\mathbb{E}[E[s_0^+]E[s_0^-]] \leq \frac{1}{2}\mathbb{E}[E[(s_0 - E[s_0])^2]]$$

$$\mathbb{E}[E[(s_0 - E[s_0])^2]] = \frac{d^2}{d\theta^2}\mathbb{E}[\log Z^{\theta}_{N,t}]$$

$$= \frac{d^2}{d\theta^2}(-N\psi_0(\theta) + \theta t)$$

$$= O(N).$$

III. Variance:

Final ingredient in the proof is the estimate

$$|\mathrm{Var}(\log Z^{\theta+h}) - \mathrm{Var}(\log Z^{\theta})| \leq CNh.$$

$$\mathrm{Var}(\log Z^{\theta+h}) = N\psi(\theta+h) - t + 2\mathbb{E}[E^{\theta+h}[s_0^+]]$$

$$\mathrm{Var}(\log Z^{\theta+h}) = N\psi_1(\theta+h) - t + 2\mathbb{E}\left[E^{\theta+h}[s_0^+] \right]$$

$$\downarrow v_1$$

$$\mathrm{Var}(\log Z^{\theta}) = N\psi_1(\theta) - t + 2\mathbb{E}\left[E^{\theta}[s_0^+] \right]$$

$$\rightarrow \quad \mathrm{Var}(\log Z^{\theta+h}) - \mathrm{Var}(\log Z^{\theta}) \geq N(\psi_1(\theta+h) - \psi_1(\theta))$$

$$\begin{aligned}\mathrm{Var}(\log Z^{\theta+h}) &= N\psi_1(\theta+h) - t + 2\mathbb{E}^{\theta+h}[s_0^+] \\ &= N\psi_1(\theta+h) - t + 2\mathbb{E}[s_0] + 2\mathbb{E}[s_0^-] \\ &= t - N\psi_1(\theta+h) + 2\mathbb{E}^{\theta+h}[s_0^-] \qquad \leq E^{\theta}[s_0^-]\end{aligned}$$

$$\Longrightarrow \quad \mathrm{Var}(\log Z^{\theta+h}) - \mathrm{Var}(\log Z^{\theta}) \leq N(\psi_1(\theta+h) - \psi_1(\theta))$$

III. Higher moments

What about higher moments?

$$\mathbb{E}\left[\overline{\log Z^{\theta}_{N,t}}^{\,p} \right] \leq C N^{2p/3} \, ?$$

Higher order version of:

$$\mathrm{Var}(\log Z^{\theta}_{N,t}) = N\psi_1(\theta) - t + 2\mathbb{E}[s_0^+] \, ?$$

$$\log Z^{\theta}_{N,t} = \underbrace{\sum_{j=1}^{N} r_j(t)}_{= R} + B_0(t) - \theta t$$

Hard to work <u>directly</u> with moments:

$$\mathbb{E}\, \mathcal{R}^4 = \mathbb{E}[\overline{\log Z}^4] + 4\mathbb{E}[\overline{\log Z}\, B_0(t)^3]$$
$$+ 6\,\mathbb{E}[\overline{\log Z}^2 B_0(t)^2] + 4\mathbb{E}[\overline{\log Z}^3 B_0(t)]$$
$$+ 3t^2.$$

Cumulants:

$$K_k(X) = \frac{d^k}{d\lambda^k} \log E[e^{\lambda X}] \Big|_{\lambda=0}$$

$$K(X_1, \dots, X_k) = \frac{d^k}{d\underline{\xi}} \log E[e^{\sum \xi_i X_i}] \Big|_{\underline{\xi}=0}$$

$$K_k(X) = \mathbb{E}[X^k] + \dots$$

Claim:

$$K_k(\log Z) + N(-1)^{k-1}\psi_{k-1}(\theta) + t\,\delta_{k=2}$$

$$= \sum_{\pi \in P} (|\pi|-1)!\,(-1)^{|\pi|} \sum_{j=1}^{k-1} \binom{k}{j} \prod_{B \in \pi} \mathbb{E}[\overline{\log Z}^{a_{j,B}} H_{b_{j,B}}(B_0)]$$

P: partitions of $\{1, \dots, k\}$

B: blocks of the partition

$a_{j,B} = |B \cap \{1, \dots, j\}|$, $\quad b_{j,B} = |B| - a_{j,B}$

$H_{k,t}(B_0)$: Hermite polynomial

$\mathbb{E}[\overline{\log Z}^k] =$ sum of products of

$\mathbb{E}[\overline{\log Z}^{a_B} H \dots (B(t))]$

$\mathbb{E}[\log Z] =$ sum of products of

$$\mathbb{E}[\overline{\log Z}^{a_B} H_{b_B}(B_0(t))]$$

$$a_B < k, \quad \sum a_B + b_B = k.$$

<u>Power counting:</u> $\overline{\log Z} \sim N^{\tilde{\chi}}$ best power we can prove by induction so far

$$H_{b_B}(B_0(t)) \sim t^{\frac{b_B}{2}}$$ not good if $t \sim N$

IV. Gaussian Integration by parts

$$\mathbb{E}[\overline{\log Z} \; H_j(B_0(t))] = \mathbb{E}[\partial^j_{B_0} \overline{\log Z}] = (-1)^j \, \mathbb{E}[\kappa_j(s_0^+)]$$

$\kappa_j(s_0^+) =$ j^{th} cumulant wrt quenched measure.

Follows from Cameron-Martin:

$$e^{\delta B_0(t) - \delta^2 \frac{t}{2}} = \sum_{n=0}^{\infty} H_{n,t}(B_0(t)) \frac{\delta^n}{n!}$$

$$\mathbb{E}[e^{\delta B_0(t) - \frac{\delta^2 t}{2}} F(B_0(s), s \leq t)]$$

$$= \mathbb{E}[F(B_0(s) + \delta s 1[0 < s < t], s < t)]$$

As a result, have:

$\mathbb{E}[\overline{\log Z}^k] =$ sum of products of expectations

$$\mathbb{E}[\overline{\log Z}^a \prod^b \ldots]$$

$$\mathbb{E}\Big[\, \overline{\log Z}^{\,a} \prod_{i=1}^{b} \kappa_{\ell_i}(s_0^+) \Big]$$

$a < k$

cumulants of s_0^+ wrt. quenched measure.

$=$ sum of products

$$\prod_{B \in \pi} \mathbb{E}\Big[\, \underbrace{\overline{\log Z}^{\,a_B}}_{N^{a_B \tilde{\chi}}} \underbrace{H_{b_B}(B_0(t))}_{t^{\frac{b_B}{2}}} \Big]$$

not good.

If we replace $B_0(t)$ by $B_0(\tau)$, $\tau < t$, same IBP argument gives:

$$\mathbb{E}\Big[\, \overline{\log Z}^{\,a} H_{b,\tau}(B_0(\tau)) \Big]$$

$$= (-1)^b \sum_{\ell_1 + \dots + \ell_a = b} \frac{b!}{\ell_1! \dots \ell_a!} \mathbb{E}\Big[\prod_{i=1}^{a} \kappa_{\ell_i}(s_0^+ \wedge \tau) \Big]$$

If we know a priori that $s_0^+ \ll \tau$ with respect to the quenched measure, then

$$|\kappa_{\ell_i}(s_0^+ \wedge \tau) - \kappa_{\ell_i}(s_0)| \ll 1,$$

so

$$\mathbb{E}\Big[\, \overline{\log Z}^{\,a} H_{b,t}(B_0(t)) \Big] = \mathbb{E}\Big[\, \overline{\log Z}^{\,a} H_{b,\tau}(B_0(\tau)) \Big] + \text{small}$$

Using this, we have:

$$\mathbb{E}[(s_0^+)^k] \ll C_k \tau^k.$$

$$\Rightarrow \mathbb{E}[\overline{\log Z}^k] = \sum_B \prod \mathbb{E}[\overline{\log Z}^{a_B} \overbrace{H_{b_B}(B_0(\tau))}^{\tau^{b_B/2}}]$$

$$\leq C\tau^{k/2}$$

Symbolically: if $\mathbb{E}[s_0^+] \lesssim N^{\xi}$

then $\overline{\log Z} \lesssim N^{\xi/2}$

Recall
$$\mathbb{E}[\overline{\log Z}^2] = N\psi_1(\theta) - t + 2\mathbb{E}[s_0^+]$$

$\overline{\log Z} \sim N^{\chi}$

$E[s_0^+] \sim N^{\xi}$

$$\boxed{\chi \leq \xi/2.}$$

V. Bounds on s_0^+.

$$Z = \int_{-\infty < s_0 < \cdots < s_{N-1} < t} e^{\theta s_0} e^{-B_0(s_0) + B_1(s_1) - B_1(s_0) + \cdots + B_N(t) - B_N(s_{N-1})} \, d\underline{s}$$

$\log Z$ is cumulant g.f. for s_0.

$$P(s_0^+ > u) = \int_{u < s_0 < \cdots < t} e^{\lambda s_0} e^{(\theta - \lambda) s_0} \cdots \, d\underline{s}$$

$$P(s_0^+ > u) = \frac{\int_{u < s_0 < \cdots < t} e^{\lambda s_0} e^{(\theta-\lambda) s_0} \cdots \, d\underline{s}}{Z^{\theta}_{N,t}}$$

$$\leq e^{-(\lambda-\theta)u} \frac{Z^{\lambda}_{N,t}}{Z^{\theta}_{N,t}}$$

$$= e^{-(\lambda-\theta)u} e^{\overline{\log Z^{\lambda}_{N,t}}} e^{-\overline{\log Z^{\theta}_{N,t}}}$$

$$\times e^{\mathbb{E}[\log Z^{\lambda}] - \mathbb{E}[\log Z^{\theta}]}$$

Recall: $\left|\mathbb{E}[\log Z^{\lambda}] - \mathbb{E}[\log Z^{\theta}]\right| \sim (\lambda-\theta)^2 N + (\lambda-\theta) N^{2/3}$

Symbolically: $\overline{\log Z} \sim N^{\chi}$

$$N^{\chi} \sim (\lambda-\theta)^2 N$$

Set $u = N^{\xi}$: $(\lambda-\theta) N^{\xi} \sim (\lambda-\theta)^2 N \Rightarrow (\lambda-\theta) = N^{\xi-1}$

$$\Rightarrow \boxed{2\xi \leq \chi + 1}$$

Combined with $\chi \leq \xi/2$, this should give

$$\chi \leq 1/3, \quad \xi \leq 2/3$$

In fact what we prove with Noack is

$$\mathbb{E}[\overline{\log Z}^k] \le C\tau^{k/2}$$

$$\mathbb{E}[(S_0^+)^{2k}] \le C n^{k+\varepsilon}\, \mathbb{E}[\overline{\log Z}^k].$$

Start with $\mathbb{E}[\overline{\log Z}^2] \le Cn^{2/3}$, $\mathbb{E}[\overline{\log Z}^k] \le Cn^{k/2}$

and iterate.

VI. Rains – EJS identity

Emrah, Jianjigian, Seppalainen

EJS give a proof of an identity of Rains for

arXiv 2105.09402

last passage percolation that uses only stationarity, so it can be transposed to O-Y setting.

Define: non-equilibrium partition function

$$Z_{N,t}^{\lambda,\theta} = \int_{-\infty < s_0 < \cdots < t} e^{\lambda s_0^+ - \theta s_0^- - B_0(s_0) + \cdots + B_N(t) - B_N(s_{N-1})} \, d\underline{s}$$

Then:

$$\boxed{\mathbb{E}[e^{(\theta-\lambda)\log Z_{N,t}^{\lambda,\theta}}] = e^{\varphi(\theta)-\varphi(\lambda)}.} \quad (*)$$

where: $\varphi(\lambda) = -N \log \Gamma(\lambda) + \lambda^2 t.$

where: $\varphi(\lambda) = -N \log \Gamma(\lambda) + \frac{\lambda^2}{2} t.$

Multiply $(*)$ by $e^{(\lambda-\theta)\,\mathbb{E}[\log Z_{N,t}^{\theta,\theta}]} = e^{(\lambda-\theta)\varphi'(\theta)}$

$$\mathbb{E}\left[e^{(\theta-\lambda)\left(\log Z^{\lambda,\theta} - \mathbb{E}[\log Z^{\theta,\theta}]\right)} \right]$$

$$= \sum_{k=0}^{\infty} \frac{(\theta-\lambda)^k}{k!} \mathbb{E}\left[\left(\log Z^{\lambda,\theta} - \mathbb{E}[\log Z^{\theta,\theta}]\right)^k\right] = e^{(\lambda-\theta)\varphi'(\theta)}$$

Differentiate in λ p times to obtain a recurrence for moments.

$$\mathbb{E}\left[\overline{\log Z_{N,t}^{\theta}}^{\,p} \right]$$

$$= \sum_{k=0}^{p-1} (-1)^{k-1} \binom{p}{k} \mathbb{E}\left[\partial_\lambda^{p-k} \, \overline{\log Z}^{\,k} \right] + N \psi_{p-1}(\theta)$$

$$= \sum_{k=0}^{p-1} (-1)^{k-1} \binom{p}{k} \mathbb{E}\left[H_{p-k}(B_0(H)) \, \overline{\log Z}^{\,k} \right] + N \psi_{p-1}(\theta).$$

Simpler than the cumulant recursion.