OOPS 1

Sunday, August 1, 2021 10:50 PM

Fluctuations of stationary integrable polymers in 1+1 dimensions

I. Directed polymers

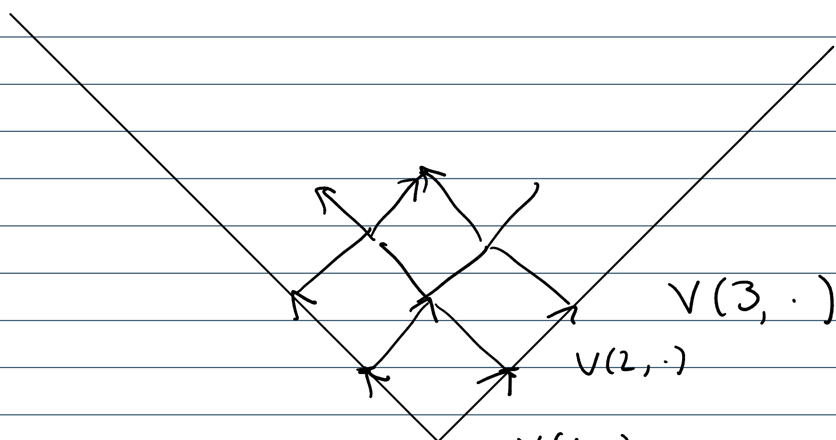
Huse and Henley 1985 (Ising model with random couplings)

$X_1, X_2, \ldots$ random walk on $\mathbb{Z}$

$V(n, x) : \mathbb{N} \times \mathbb{Z} \to \mathbb{R}$ i.i.d., e.g. $\mathbb{E}[e^{\alpha V}] < \infty$.

Weight of path $X_1, \ldots, X_N = \exp(\beta \sum_{i=1}^{N} V(i, X_i))$

Partition function: $Z = \sum_{X.} \exp(\beta \sum_{i=1} V(i, X_i)) P(X.)$



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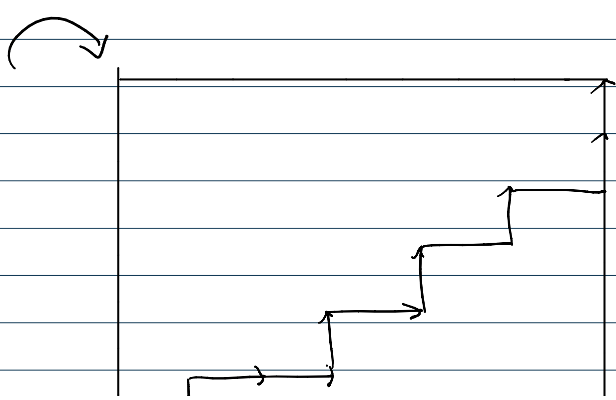
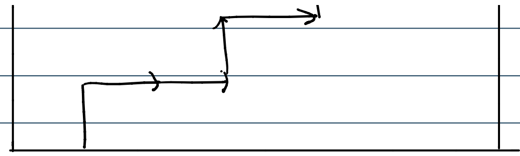


up-right paths



Key object: Quenched measure

$$Q(x_1, \dots x_n) = \frac{1}{Z} e^{\beta \sum_{i=1}^{N} V(i, X_i)}$$

$$E_Q[|X_n|] \ ?$$

II. What is expected

Given sufficiently many moments:

$$\overline{\log Z_n} = \log Z_n - \mathbb{E}[\log Z_n] \sim n^{1/3}$$

$$E_Q[|X_n|] \sim n^{2/3}$$

+ distributional results

Tracy–Widom limit:

$$\frac{\log Z_n - an}{cn^{1/3}} \xrightarrow{d} F_{GUE}$$

Known for some exactly solvable models

(Seppalainen, Borodin–Corwin–Ferrari, . . .)

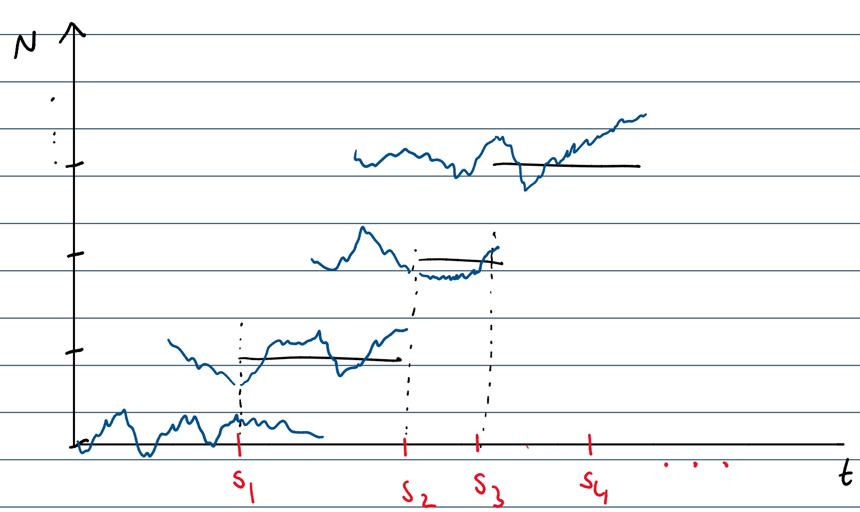
III . A semi-discrete model

O'Connell – Yor (Brownian Analogues of Burke's Theorem)

$B_1, B_2, \ldots, B_N$ Brownian Motions.

Partition function:

$$Z_{N,t} = \int_{0<s_1<\cdots<s_{N-1}<t} e^{B_1(s_1)-B_1(0)+B_2(s_2)-B_2(s_1)+\cdots+B_N(t)-B_N(s_{N-1})} \, d\underline{s}$$



Ensemble of up-right paths

Stationary Version of O'Connell-Yor polymer.

Add Brownian Motion B_0.

Extend all BMs $B_0, B_1, \dots, B_N$ to 2-sided BMs.

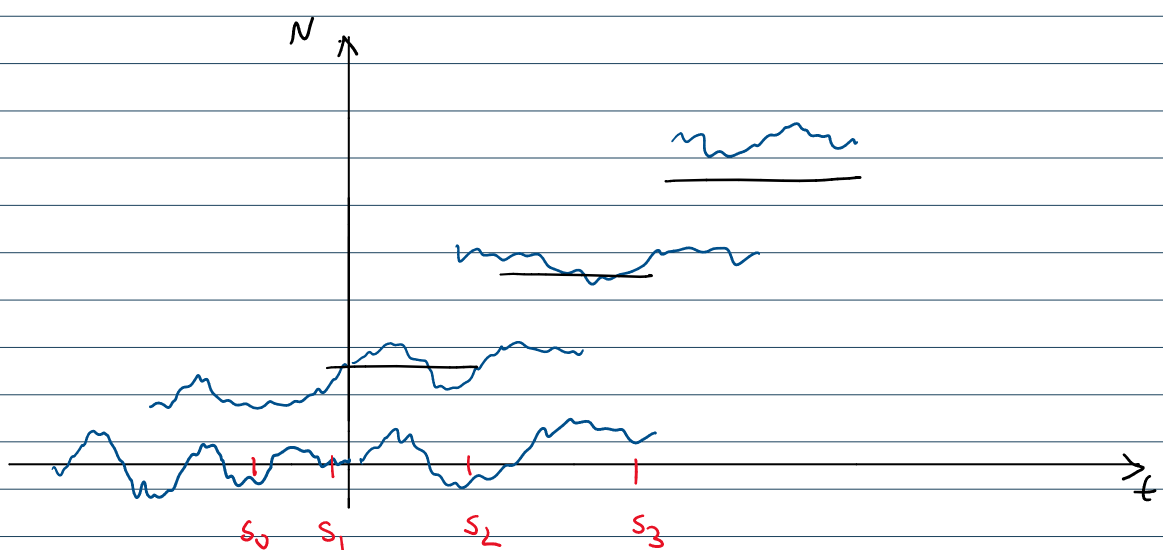
$s < 0$: $B_j(s) := \tilde{B}_j(-s)$,

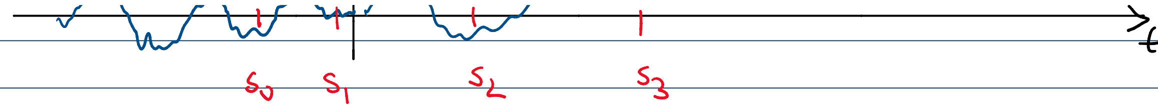
$\{B_j, \tilde{B}_j\}$, $j = 0, \dots, N$ independent.

Choose Parameter $\theta > 0$.

Partition function of stationary model:

$$Z^{\theta}_{N,t} = \int_{-\infty < s_0 < \dots < s_{N-1} < t} e^{-B(s_0) + \theta s_0 + B_1(s_1) - B_1(s_0) + \dots + B_N(t) - B_N(s_{N-1})}$$





IV. Stationarity

O'Connell–Yor: write

$$\log Z^{\theta}_{0,t} := B_0(t) - \theta t$$

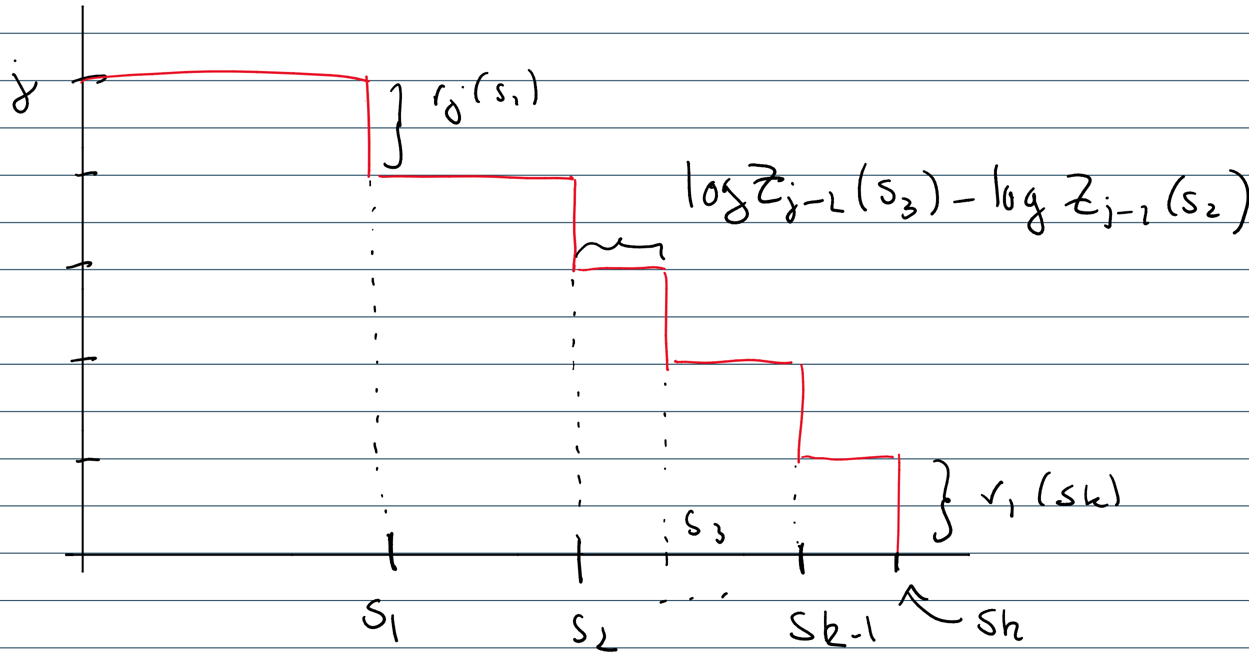
Then: $$\log Z^{\theta}_{N,t} - \log Z^{\theta}_{0,t} = \sum_{j=1}^{N} r_j^{\theta}(t)$$

where $r_j(t) = \log Z^{\theta}_{j,t} - \log Z^{\theta}_{j-1,t}$.

<u>Theorem:</u> For each $t \in \mathbb{R}$, $r_j(t)$ is i.i.d. with distribution:

O'Connell – Yor prove more:

$\log Z_j^{\theta}(t) - \log Z_j^{\theta}(s) - \theta(t-s)$ is a Brownian motion for $s<t$



Dufresne identity:

$$Y_t = \int_{-\infty}^{t} e^{B_0(t) - B_0(s) - (t-s)\theta}\, ds$$ is stationary with

distribution 1/Gamma random variable.

V. Integration by parts and cumulants.

Let $R = \sum_{j=1}^{N} r_j(t)$. Then:

$$\log Z^{\theta}_{N,t} = R + \log Z^{\theta}_{0,t}$$
$$= R - B_0 + \theta t$$

i.i.d. sum

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i.i.d. sum.

Suppose $N, t \gg 1$:

$$\left.\begin{aligned} R &= O(N) \\ B_0(t) - \theta t &= O(t) \end{aligned}\right\} \text{Law of Large Numbers}$$

What about fluctuations?

$$\overline{\log Z^{\theta}_{N,t}} := \log Z^{\theta}_{N,t} - \mathbb{E}[\log Z^{\theta}_{N,t}]$$

$$= \underbrace{\overline{R}}_{O(\sqrt{N})} + \underbrace{B_0(t)}_{O(\sqrt{t})}$$

This is not $O(N^{1/3})$ unless there is some cancellation between $B_0(t)$ and R.

Let us compute the variance:

$$\begin{aligned} \mathrm{Var}(R) &= \mathrm{Var}(\log Z + B_0 - \theta t) \\ &= \mathrm{Var}(\log Z) + \mathrm{Var}(B_0) \\ &\quad + 2\,\mathrm{Cov}(\log Z, B_0) \end{aligned}$$

$$\Rightarrow \operatorname{Var}(\log Z) = N\operatorname{Var}(r_1) - t + 2\,\mathbb{E}[\log Z\, B_0]$$

$$R = \sum_{j=1}^{N} r_j \quad , \quad r_j \sim \log \frac{1}{X} \quad \text{where } X \text{ is } 1/\text{Gamma}(\theta)$$

p.d.f.:

$$r_j \sim e^{-\theta x} e^{-e^{-x}} \frac{dx}{\Gamma(\theta)}$$

$$\Rightarrow \mathbb{E}[r_j] = \int x\, e^{-\theta x} e^{-e^{-x}} \frac{dx}{\Gamma(\theta)}$$

$$= -\frac{d}{d\theta} \int e^{-\theta x} e^{-e^{-x}} \frac{dx}{\Gamma(\theta)}$$

$$= -\frac{\Gamma'(\theta)}{\Gamma(\theta)} =: -\psi_0(\theta)$$

$$\operatorname{Var}(r_j) = \int x^2 e^{-\theta x} e^{-e^{-x}} \frac{dx}{\Gamma(\theta)} - \left(\int x\, e^{-\theta x} e^{-e^{-x}} \frac{dx}{\Gamma(\theta)} \right)^2$$

$$= \frac{d}{d\theta} \frac{\Gamma'(\theta)}{\Gamma(\theta)}$$

$$=: \psi_1(\theta)$$

So: $\operatorname{Var}(\log Z_{N,t}^{\theta}) = N\psi_1(\theta) - t + 2\,\mathbb{E}[\log Z\, B_0]$

function of $B_0(s)$, $s \le t$.

$\mathbb{E}[\log Z\, B_0]$

$= \cap \quad \theta s_0 - B(s_0) + \dots + B_N(t) - B_N(s_{N-1})$

$\mathbb{E}[\log Z \;\; B_0]$

$$Z = \int_{-\infty < s_0 < \cdots < t} e^{\theta s_0 - B(s_0) + \cdots + B_N(t) - B_N(s_{N-1})} \, \underline{ds}$$

If F is smooth and X, Y are joint Gaussian:

$$\mathbb{E}[F(Y) X] = \mathrm{Cov}(X, Y) \, \mathbb{E}[F'(Y)]$$

(Gaussian integration by parts)

Cameron-Martin Theorem: if $\int_0^t h^2(s) \, ds < \infty$

$$\mathbb{E}\left[F\left(B(\tau) + \delta \int_0^\tau h(s) \, ds, \ \tau \le t \right) \right]$$

$$= \mathbb{E}\left[e^{\delta \int_0^t h(s) \, dB_s - \frac{\delta^2}{2} \int_0^t h^2 \, ds} F(B(\tau), \ \tau \le t) \right]$$

$$\xrightarrow{\left.\frac{d}{d\delta}\right|_{\delta=0}} \quad \left.\frac{d}{d\delta}\right|_{\delta=0} \mathbb{E}\left[F\left(B(\tau) + \delta \int_0^\tau h(s) \, ds, \ \tau \le t \right) \right]$$

$$= \mathbb{E}\left[\int_0^t h(s) \, dB_s \; F(B(\tau), \ \tau \le t) \right]$$

Apply this with

$$h = \mathbb{1}[0,t].$$

$$\Rightarrow \mathbb{E}[B_0(t) \log Z^{\theta}_{N,t}]$$

$$= \frac{d}{d\delta} \mathbb{E}\Big[\log \int_{-\infty < s_0 < \cdots < s_{N-1} < t} e^{\theta s_0 + \delta s_0 \cdot \mathbb{1}[0,t] + \cdots + B_N(t) - B_N(s_{N-1})} \Big]$$

$$\frac{d}{d\delta} \log \int e^{\delta X + Y} \Big|_{\delta = 0}$$

$$= \frac{\int X e^Y}{\int e^Y} = E_{\int \cdot e^Y} [X]$$

Here:

$$\frac{d}{d\delta} \log \int_{-\infty < s_0 < \cdots < s_{N-1} < t} e^{\delta s_0 \mathbb{1}[0,t]} \, e^{\theta s_0 - B_0(s_0) + \cdots + B_N(t) - B_N(s_{N-1})}$$

$$= \frac{1}{Z^{\theta}_{N,t}} \int s_0 \mathbb{1}[0,t]\, e^{\theta s_0 - B_0(s_0) + \dots + B_N(t) - B_N(s_{N-1})}$$

$$= E[s_0^+]$$ quenched expectation

$$s_0^+ = \max\{0, s_0\}$$

In summary:

$$\log Z^{\theta}_{N,t} - \log Z^{\theta}_{0,t} = \sum_{j=1}^{N} \underbrace{\log Z_j - \log Z_{j-1}}_{= r_j \quad \text{i.i.d.}}$$

stationarity

$$\mathrm{Var}(\log Z_{N,t}) = N\psi_1(\theta) - t + \mathbb{E}[B_0(t) \log Z^{\theta}_{N,t}]$$

arithmetic

$$\mathbb{E}[B_0(t) \log Z^{\theta}_{N,t}] = \mathbb{E}[E[s_0^+]]$$

integration by parts

$$\boxed{\mathrm{Var}(\log Z) = N\psi_1(\theta) - t + 2\mathbb{E}[s_0^+]}$$ (Seppalainen–Valkio)

Choose θ, N, t s.t.

$$|N\psi_1(\theta) - t| \leq C N^{2/3}.$$

Claim: $\mathrm{Var}(\log Z) \leqslant C N^{2/3}$ for such choice of parameters.

VI Convexity

To estimate $E[s_0^+]$, we will

1. Compare $|E[s_0^+]|$ to $|E[s_0]|$ easier

2. Use convexity:

$$E[s_0] = \frac{d}{d\theta} \log Z_{N,t}^{\theta}$$

1. Claim: $\mathbb{E}[s_0^+] \leq \mathbb{E}[E[s_0]^2]^{1/2} + O(N^{1/2})$

2. $E[s_0]$

$$= \frac{d}{d\theta} \log \int e^{\theta s_0 - B_0(s_0) + B_1(s_1) - B_1(s_0) + \dots + B_N(t) - B_N(s_{N-1})}$$

$$\frac{d^2}{d\theta^2} \log Z_{N,t}^{\theta} = Var(s_0) \geq 0$$

$$\Rightarrow \log Z_{N,t}^{\theta} \text{ is convex.}$$

f convex

$$\Rightarrow |f'(x)| \leq \max\left\{ \left| \frac{f(x+h) - f(x)}{h} \right|, \left| \frac{f(x-h) - f(x)}{h} \right| \right\}$$

$$\left| \frac{d}{d\theta} \log Z_{N,t}^{\theta} \right| = |E[s_0]|$$

$$\leq \left| \frac{\log Z_{N,t}^{\theta+h} - \log Z_{N,t}^{\theta}}{h} \right|$$

$$+ \left| \frac{\log Z_{N,t}^{\theta-h} - \log Z_{N,t}^{\theta}}{h} \right|$$

<u>Assume</u>: $\mathbb{E}[s_0^+] \leq \mathbb{E}[|E[s_0]|^2]^{1/2} + O(N^{1/2})$.

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Then:

$$\mathbb{E}[s_0^+]$$

$$\leq h^{-1}\left(\mathrm{Var}(\log Z^{\theta+h})\right)^{1/2} + \left(\mathrm{Var}(\log Z^{\theta-h})\right)^{1/2}$$

$$+ h^{-1}\left(\mathrm{Var}(\log Z^{\theta})\right)^{1/2}$$

$$+ h^{-1}\left|\mathbb{E}[\log Z^{\theta+h}] - \mathbb{E}[\log Z^{\theta}]\right|$$

$$+ h^{-1}\left|\mathbb{E}[\log Z^{\theta-h}] - \mathbb{E}[\log Z^{\theta}]\right|$$

Two final ingredients:

i) $\left|\mathrm{Var}(\log Z^{\theta \pm h}) - \mathrm{Var}(\log Z^{\theta})\right| \leq C\underline{Nh}$

ii) $\left|\mathbb{E}[\log Z^{\theta \pm h}] - \mathbb{E}[\log Z^{\theta}]\right| \leq C\underline{Nh^2}$

$$\mathbb{E}[s_0^+] \leq h^{-1}\,\mathrm{Var}(\log Z^{\theta})^{1/2} + CN^{1/2}h^{-1/2} + CNh$$

$$\boxed{\begin{aligned} N^{1/2}h^{-1/2} &\sim Nh \\ \Rightarrow \quad h &\sim N^{-1/3} \end{aligned}}$$

$$\mathrm{Var}(\log Z) = \underbrace{N\Psi_1(\theta) - t} + 2\mathbb{E}[s_0^+]$$

$$\lesssim N^{2/3} + N^{1/3}\left(\mathrm{Var}(\log Z)\right)^{1/2}$$

$$V \lesssim N^{2/3} + N^{1/3} V^{1/2}$$

$$\Rightarrow V = O(N^{2/3}) \quad !$$

Left to do:

- Compare $E[s_0^+]$ to $E[s_0]$.
- Estimates

$$\left| \mathbb{E}[\log Z_{N,t}^{\theta+h}] - \mathbb{E}[\log Z_{N,t}^{\theta}] \right| \leq CNh^2$$

$$\left|\mathrm{Var}(\log Z_{N,t}^{\theta+h}) - \mathrm{Var}(\log Z_{N,t}^{\theta})\right| \leq CNh.$$

- Easiest is diff. of expectations:

$$\mathbb{E}[\log Z_{N,t}^{\theta+h}] = \mathbb{E}\Big[\sum_{j=1}^{N} r_j - B_0(t) + (\theta+h)t$$

$$= -N\psi_0(\theta+h) + (\theta+h)t$$

$$\mathbb{E}[\log Z_{N,t}^{\theta+h}] - \mathbb{E}[\log Z_{N,t}^{\theta}]$$

$$= N(\psi_0(\theta) - \psi_0(\theta+h)) + ht$$

$$= -N\psi_1(\theta)h + ht + N\,O(h^2)$$

because $\frac{d}{d\theta}\psi_0(\theta) = \psi_1(\theta)$

$$\lesssim hN^{2/3} + Nh^2$$

- Diff. of variances:

$$\mathrm{Var}(\log Z^{\theta+h}) - \mathrm{Var}(\log Z^{\theta})$$

$$= N\big(\psi_1(\theta+h) - \psi_1(\theta)\big) + 2\,\mathbb{E}\Big[\underbrace{E^{\theta+h}[s_0^+] - E^{\theta}[s_0^+]}_{\geq 0}\Big]$$

$$\geq N\big(\psi_1(\theta+h) - \psi_1(\theta)\big)$$

<u>Also:</u>

$$\mathrm{Var}(\log Z^{\theta}) = N\psi_1(\theta) - t + 2\,\mathbb{E}[s_0^+]$$

$$= N\psi_1(\theta) - t + 2\big(\mathbb{E}[s_0] + \mathbb{E}[s_0^-]\big)$$

$$= t - N\psi_1(\theta) + 2\,\mathbb{E}[s_0^-]$$

$$\mathrm{Var}(\log Z^{\theta+h}) - \mathrm{Var}(\log Z^{\theta})$$

$$= N\big(\psi_1(\theta) - \psi_1(\theta+h)\big) + 2\,\mathbb{E}\Big[\underbrace{E^{\theta+h}[s_0^-] - E^{\theta}[s_0^-]}_{\leq 0}\Big]$$

$$\leq N\big(\psi_1(\theta) - \psi_1(\theta+h)\big)$$

∘ Comparing $E[s_0^+]$ to $E[s_0]$.

$$E[s_0]^2 = E[s_0^+]^2 + E[s_0^-]^2 - 2E[s_0^+]E[s_0^-]$$

<u>Claim:</u> $\mathbb{E}\left[E[s_0^+]E[s_0^-]\right] = O(N)$

$$E\left[(s_0 - E[s_0])^2\right]$$

$$= E\left[\left(s_0^+ - s_0^- - E[s_0^+] + E[s_0^-]\right)^2\right]$$

$$= E\left[\left(s_0^+ - E[s_0^+]\right)^2\right] + E\left[\left(s_0^- - E[s_0^-]\right)^2\right]$$

$$- 2E\left[\left(s_0^+ - E[s_0^+]\right)\left(s_0^- - E[s_0^-]\right)\right]$$

By disjoint support of s_0^+, s_0^-:

$$E\left[\left(s_0^+ - E[s_0^+]\right)\left(s_0^- - E[s_0^+]\right)\right]$$

$$= E[\underbrace{s_0^+ s_0^-}_{=0}] - E[s_0^+]E[s_0^-] - E[s_0^+]E[s_0^-] + E[s_0^+]E[s_0^-]$$

$$= - E[s_0^+]E[s_0^-]$$

$$\mathrm{Var}(s_0) = E[(s_0 - E[s_0])^2]$$

$$= \mathrm{Var}(s_0^+) + \mathrm{Var}(s_0^-) + 2\,E[s_0^+]\,E[s_0^-]$$

All terms are non negative!

$$E[s_0^+]\,E[s_0^-] \leq \frac{1}{2} K_2(s_0)$$

Taking expectations:

$$\mathbb{E}\big[\,E[s_0^+]\,E[s_0^-]\,\big] \leq \frac{1}{2}\,\mathbb{E}\big[\,E[(s_0 - E[s_0])^2]\,\big]$$