Local & global structure of uniform spanning trees

(joint works with subsets of {Noga Alon, Eleanor Archer, Peleg Michaeli, Yuval Peres, Matan Shalev})

OOPS 2021

Setup: G finite, connected, simple graph

T is a spanning tree if $V(T) = V(G)$
T conn.
no cycles.

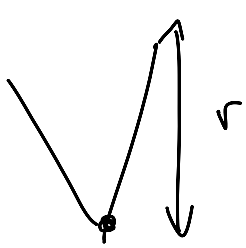
Rick: "The UST is a hero
prob. theory"

Cayley's theorem (1889): # trees on n labelled vertices $= n^{n-2}$

Kirchhoff's thm (matrix-tree)

spanning trees of any G = det (minor the laplacian)

Global structure:



(+) Typical dist between a pair of vertices

(+) Diameter (len longest path)

(+) Height seen from random vertices

(+) ~~The~~ Does the

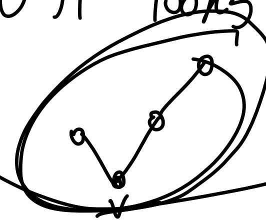
tree, considered as
a metric space, converge.
(Gromov-Hausdorff-Prohorov)

Local structures

(+) # leaves

(+) # dg k vertices

$v \in$ UST
s.t. ball of distance
2 in UST looks like



(+) local limit of the UST

Case study: UST(K_n)

$T = \text{UST}(K_n)$

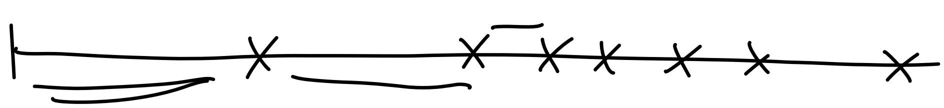
Claim: $x \neq y$ $\quad P\Big(\underbrace{d_T(x,y)}_{\text{tree distance}} \geq x\sqrt{n} \Big) = e^{-x^2/2} + o(1)$

Renyi - Szekeres ('67) $\quad \dfrac{\text{Height}}{\sqrt{n}} \xrightarrow{(d)}$ cont r.v. supp on $[0,\infty)$

Szekeres '82 : $\quad \dfrac{\text{diameter}}{\sqrt{n}} \xrightarrow{(d)}$

Ultimate theorem '90s (Aldous 90-93
Le-Gall)

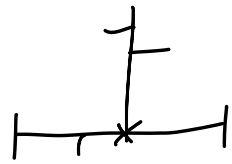
$$\frac{d_T(\cdot,\cdot)}{\sqrt{n}} \xrightarrow[\text{GHP}]{(d)} \text{Continuum Random Tree (CRT)}$$



$\mathbb{R}^+$ Poisson process w/ intensity $t\,dt$

stick

(~~Line~~ Stick breaking



Brownian excursion

Corollary: $\dfrac{\text{Height}}{\sqrt{n}} \xrightarrow{(d)} \sqrt{2} \cdot \max_{t \in [0,1]} e_t$

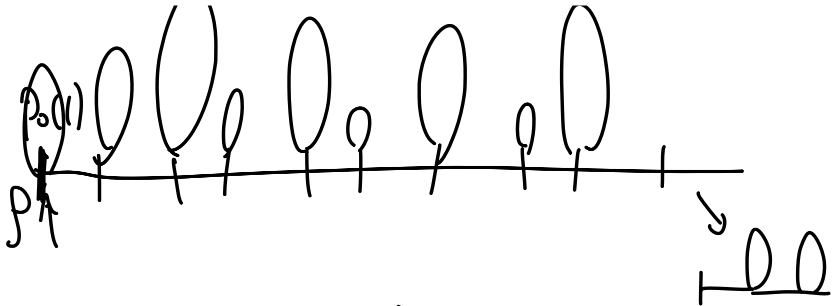
$\dfrac{\text{Diam}}{\sqrt{n}} \xrightarrow{(d)} \sqrt{2} \sup_{0 \le t_1 < t_2 \le 1} e_{t_2} + e_{t_1} - 2 \inf_{t_2 \le t \le t_1} e_t$

Local structure $UST(K_n) = T$

Prüfer codes degree $\stackrel{(d)}{=} \text{Bin}(n-2, \frac{1}{n}) + 1$

Ultimate theorem (Grimmett 81)

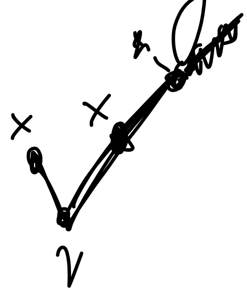
Local limit of T is the $Po(1)$ BP conditioned to survive.

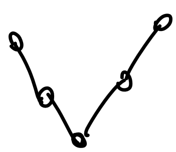


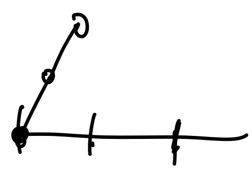
Corollary: $\mathbb{P}(v \text{ leaf}) = \frac{1}{e}$

$$\# \text{ leaves} = \frac{n}{e}(1+o(1)) \quad \text{whp}$$

$$\deg \text{root} \overset{(d)}{=} 1 + \mathrm{Po}(1)$$

$\#$  $= \frac{n}{e^3}(1+o(1))$

$\#$  $= \frac{n}{e^3}(1+o(1))$



All this was for K_n

Other graphs: hypercube $\{0,1\}^n$

$(\mathbb{Z}_n)^d$ expanders ...

$d > 4$

$\cancel{(\mathbb{Z}_n)^2}$

Results Global

Thm (Michaeli, N., Shalev '20)

For any "high dimensional" graph seq.

$$\forall \varepsilon \ \exists A \quad \mathbb{P}\left(A^{-1}\sqrt{n} \le \mathrm{diam}(T) \le A\sqrt{n} \right) \ge 1-\varepsilon$$

$\mathrm{UST}(G_n)$

$n = \# V(G_n)$

Work in progress (Archer, N., Shalev)

Same setup, $\exists\, \beta > 0$ s.t.

$$\beta \frac{\mathrm{diam}(T_n)}{\sqrt{n}} \xrightarrow{(d)} \mathrm{diam}(CRT)$$

$G = (\mathbb{Z}_n)^5$

[in fact $\beta \cdot \frac{d_{T_n}(\cdot,\cdot)}{\sqrt{n}} \xrightarrow[GHP]{(d)} CRT$]

Local

Thm (Peres, N.) G_n a graph seq., regular and deg $\to \infty$

then local limit of $UST(G_n)$ is Poi(1) BP cond. survival

Thm (Kirchhof 1847): G finite conn. graph

$e=(x,y)\in E(G)$, $T = UST(G)$

$$P(e\in T) = R_{eff}(x\leftrightarrow y)$$

Pf: $\theta(\vec{e}) := P\left(\begin{array}{c}\text{the unique } x\to y \\ \text{path in } T \\ \text{uses } \vec{e}\end{array}\right) - P\left(\begin{array}{c}'' \\ \overleftarrow{e}\end{array}\right)$

Claim: θ is unit current flow.

Why cycle law holds, i.e. $\forall\ \vec{e}_1, \dots, \vec{e}_m$ cycle

$$\sum_{i=1}^{m} \theta(\vec{e}_i) \stackrel{?}{=} 0$$

$$\sum_{t} \sum_{i=1}^{m} f_i^{+}(t) - \sum_{t} \sum_{i=1}^{m} f_i^{-}(t)$$

t spanning tree of G

$$f_i^+(t) = \begin{cases} +1 & \text{if } e_i \text{ is used in } \cancel{x} \to y \text{ path in } t \\ 0 & \text{o/w} \end{cases}$$

$$f_i^-(t)$$

$$T_i^+ = \{(t, i) : t \text{ span. tree}, f_i^+(t) = 1\}$$

$$T_i^- = \quad \text{"}$$

$$\left| \dot{\bigcup_{i=1}^m} T_i^+ \right| = \left| \dot{\bigcup_{j=1}^m} T_j^- \right|$$

$(t, i) \in T_i^+$

$t \setminus \{e_i\}$ has 2 conn. comp

let e_j be the 1st edge after $\vec{e_i}$ (in the cycle) that connects those 2 comp.

$$\Rightarrow \quad \underline{(t \setminus \{e_i\} \cup \{e_j\}, j)} \in T_j^-$$

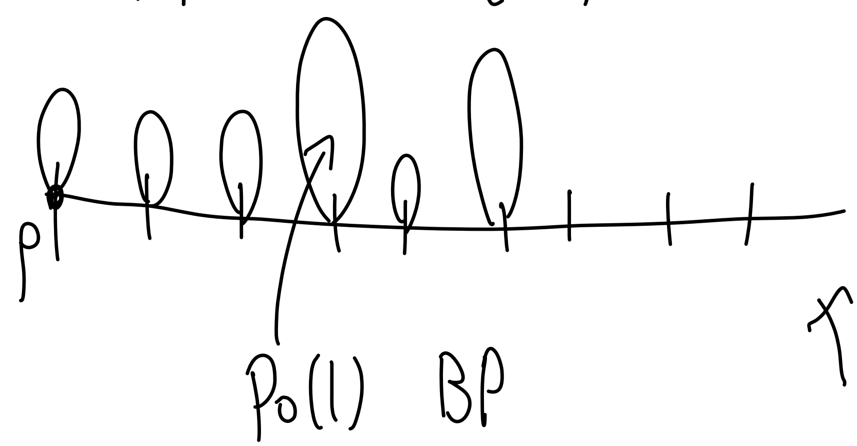
$\square$

The local structure of USTs

Thm (Yuval Peres, N. '21)

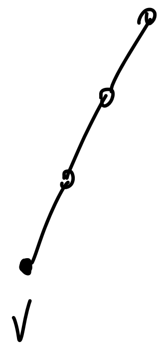
G_n sequence of connected, finite simple, regular with $\deg \to \infty$.

Then $\mathrm{UST}(G_n)$ converges locally Po(1) BP conditioned to survive.



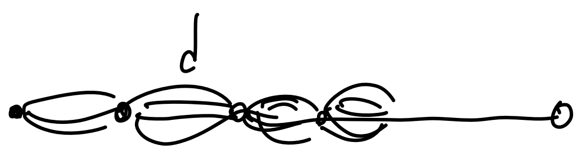
I.e. X uniform vertex of G_n

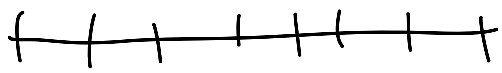
Then $\lim_{n\to\infty} P\left(B_{T_n}(X, r) = T \right) = P\left(\begin{matrix} B(X,r) = T \\ Po(1) \\ BP \end{matrix} \right)$



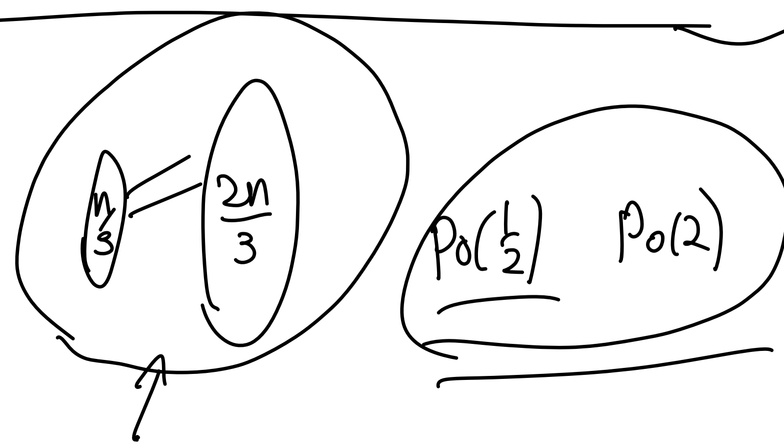
$T_n = UST(G_n)$

$\frac{n}{e^3}(1+o(1))$





Example



Reminders about electric networks

$c(i,j)$ = conductance on (i,j)

$\pi(i) = \sum_j c(i,j)$

$j \in V$

$$R_{eff}(a \leftrightarrow z) = \frac{1}{\pi_a P_a(\tau_z < \tau_a^+)}$$

Commute time identity (in unit cond)

$$2E(G) R_{eff}(a \leftrightarrow z) = \underline{\underline{E_a \tau_z}} + E_z \tau_a$$

<u>Kirchhoff</u> (1847): $e \in E$ $e = (x, y)$

$$P(e \in T) = R_{eff}(x \leftrightarrow y)$$

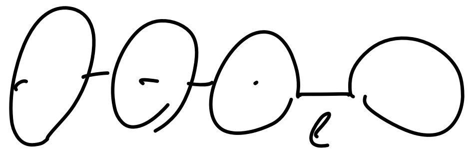
<u>Foster's theorem</u>: $\displaystyle\sum_{e \in (x,y)} R_{eff}(x,y) = n-1$

immediate

<u>Alternatively</u>: $\boxed{E\, R_{eff}(X_0, X_1) = \frac{n-1}{E(G)} = \frac{2}{d}(1+o(1))}$

G d-regular

$d \to \infty$



<u>Observation:</u> In any graph, $x \neq y$

$$R_{eff}(x \leftrightarrow y) \geq \frac{1}{\deg(x)+1} + \frac{1}{\deg(y)+1}$$

$\frac{1}{d}$ $\qquad$ $\frac{1}{d}$ $\qquad$ $\frac{2}{d}$

<u>Corollary</u> $N_\varepsilon = \left| \{ e = (x,y) : R(x,y) \geq \varepsilon \} \right|$

$$|N_\varepsilon| \leq \frac{n}{\varepsilon d - 2} \qquad \varepsilon > \frac{2}{d}$$

A random edge (X_0, X_1) whp $R(X_0, X_1) = \frac{2}{d}(1+o(1))$

X_0 - random vertex X_1 - random nbr

$(X_0, \dots, X_k)$ - SRW k-steps

Goal: whp $\underbrace{R(X_0, X_k)}_{\text{whp}} = \frac{2}{d}(1+o(1))$

Proof of

Fix $x_0 \in V$

$$\mathbb{E}_{x_0} \tau_{x_0}^+ = n = \frac{1}{d} \sum_{x_1 \sim x_0} (\mathbb{E}_{x_1} \tau_{x_0} + 1)$$

$\frac{1}{d^k} \sum_{x_k, x_{k-1}, \dots, x_1} \dots$

$$n - 1 = \frac{1}{d} \sum_{x_1 \sim x_0} \mathbb{E}_{x_0} \tau_{x_1} = \mathbb{E}\, \mathbb{E}_{X_1} \tau_{x_0}$$

random nbr of x_0

X_0 - uniform vertex

$$\underset{\substack{\| \\ n-1}}{\mathbb{E}\,\mathbb{E}_{X_1}\tau_{X_0}} + \mathbb{E}_{X_0}\tau_{X_1} = 2(n-1) = 2|E(G)|\, R_{eff}(X_0 \leftrightarrow X_1)$$

Similarly $\quad \mathbb{E}\, R_{eff}(X_0 \leftrightarrow X_k) = \frac{2}{d}(1+\underset{\substack{\uparrow \\ O(d^{-2})}}{o(1)})$

Remark: again whp on the RW $\square$

$$R_{eff}(X_0, X_k) = \frac{2}{d}(1+o(1))$$

Thm $\quad X_1, \dots, X_k \quad k \geq 2$

whp

$$R_{eff}(X_k \leftrightarrow \{X_1, \dots, X_{k-1}\}) = \frac{k}{(k-1)d}(1+o(1))$$

We know $\forall\, i \neq j \in [k]$ $\quad R_{eff}(X_i \leftrightarrow X_j) = \frac{2}{d}(1+o(1))$

Does this imply the Thm,

Or: If we know the $R(x,y)$ $\forall x \neq y$ in a network, can we reconstruct the network? Is this stable?

Answer: Yes, up to loops

Lemma $k \geq 3$, $x > 0$ fixed, $\varepsilon > 0$ small

A network on $\{1, \dots, k\}$

If $|R(i,j) - x| < \varepsilon \quad \forall i \neq j \in [k]$

then $$R(1, \{2, \dots, k\}) = \frac{kx}{2(k-1)} + O(\varepsilon)$$

Pf: Step 1: Knowing pairwise eff resistance we can recover all voltages, stable

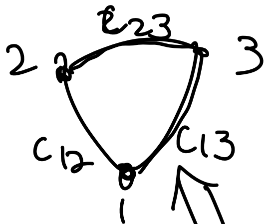
Indeed

$V_a(i,j) :=$ voltage at i when $a \to 0$, $j \to 1$

$$= P_i(\tau_j < \tau_a)$$

$$V_a(i,j) = \frac{R_{eff}(a \leftrightarrow j) + R_{eff}(a \leftrightarrow i) - \cancel{R_{eff}(i,j)}}{2\,R_{eff}(a \leftrightarrow j)}$$

Suffices to consider a network w/ 3 vertices

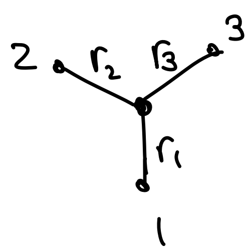


$$R_{eff}(1,2) = \frac{1}{c_{12} + \frac{1}{\frac{1}{c_{13}} + \frac{1}{c_{23}}}}$$

$$\_\_\_\_ =$$

$$\_\_\_\_ =$$

$Y-\Delta$



$$R(1,2) = r_1 + r_2$$
$$R(2,3) = r_2 + r_3$$
$$R(1,3) = r_1 + r_3$$

$i = 3$

$a, j = (1,2)$

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

$$= r_3 = \frac{-R_{12} + R_{13} + R_{23}}{2}$$

Step 2: Knowing the voltages allows to recover the network.

$$\Delta(i,j) = \begin{cases} -C(i,j) & i \neq j \\ \pi(i) & i = j \end{cases}$$

$a \in V$ $\quad \Delta[a] = \begin{array}{c|c} & a \\ \hline a & \left(\Delta\right) \end{array}$

then $\quad \underline{\Delta[a]} = \underline{\left(V_a(\cdot,\cdot)\right)}^{-1}$

Easy fact $\quad \frac{1}{1-x} = 1 + x + x^2 \ldots$

If A is invertible, E matrix with small entries then $A+E$ is invertible and close coordinate with [illegible] A^{-1}

to A

$$V_a(\cdot,\cdot) \begin{pmatrix} x & & \frac{x}{2} \\ & \ddots & \\ \frac{x}{2} & & x \end{pmatrix}$$

$$(V_a)^{-1} = \begin{pmatrix} \frac{2(k-1)}{kx} & \cdots & -\frac{2x}{k} \\ & \ddots & \\ & & \ddots \end{pmatrix}$$

$$R_{eff}(1, \{\underline{\underline{2, \dots, k}}\}) = \frac{1}{\sum_{j=2}^{k} c(1,j)} = \frac{1}{(k-1)\cdot\frac{2}{kx}}$$