Disordered systems and Hamilton-Jacobi equations (part 2)

We need to think about

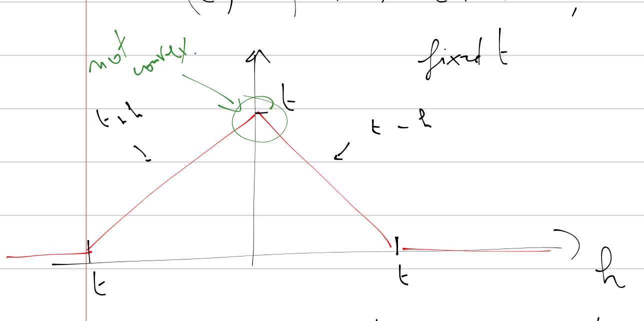
$$\partial_t f - (\partial_h f)^2 = 0 \qquad \text{(HJ)}$$

What about we ask that f be Lipschitz, and solves the eq. a.e.?

Pb: No uniqueness..

Ex: 0 is a solution.

$(t,h) \mapsto t+h$; $t-h$



Def: We say that a Lipschitz function $f: \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is a weak solution of (HJ) if

1) (HJ) is satisfied almost everywhere.
2) For every $t \geq 0$, the mapping $h \mapsto f(t,h)$ is convex.

Prop: (uniqueness) if f and g are two weak solutions of (HJ) with $f(0,\cdot) = g(0,\cdot)$, then $f = g$.

Sketch of proof: Denote $w = f - g$.

Almost everywhere:

$$\partial_t w = (\partial_h f)^2 - (\partial_h g)^2$$

$$= \underbrace{(\partial_h f + \partial_h g)}_{=: b} \partial_h w.$$

$$\partial_t w - b\, \partial_h w = 0. \qquad \text{(a. e.)}$$

Rough idea: $I(t) = \int w(t,h)\, dh.$

$$\partial_t I(t) = \int \partial_t w = \int b\, \partial_h w$$

$$= -\int \underbrace{\partial_h b}_{\geq 0}\, w$$

Let $\phi(x) = \frac{x^2}{1+x^2}$, $\quad v = \phi(w) = \phi(f-g).$

$$\partial_t v - b\, \partial_h v = 0.$$

Denote $L = \|\partial_h f\|_{L^\infty} + \|\partial_h g\|_{L^\infty} + 1$

Fix $T < +\infty$, and study

$$J(t) := \int_{-L(T-t)}^{L(T-t)} v(t,h)\, dh$$

$$= \int_{-R_t}^{R_t} v$$

$$\partial_t J = \int_{-R_t}^{R_t} \partial_t v - L\big(v(t,R_t) + v(t,-R_t)\big)$$

$$\int_{-R_t}^{R_t} b\, \partial_h v = -\int_{-R_t}^{R_t} (\partial_h b)\, v + \underbrace{\big[b\, v\big]_{-R_t}^{R_t}}$$

$$\partial_t J \leq -\int_{-R_t}^{R_t} \underbrace{(\partial_h b)}_{\geq 0} \underbrace{v}_{\geq 0} \leq 0.$$

$$\boxed{\partial_t J \leq 0} \qquad J(0) = 0, \quad J \geq 0$$

We conclude that $J \equiv 0$.

$v = 0$ a.e. (+ Lipschitz)

$\boxed{v = 0}$ $\boxed{f = g}$ $\triangle$

Ex: make this proof rigorous (credit). (Evans PDE)

$$\partial_t f - \underbrace{H}_{\uparrow \text{ convex.}}(\nabla f) = 0.$$

Prop: (Hopf - Lax formula):

The function:

$$f(t,h) = \sup_{h' \in \mathbb{R}} \left(\psi(h - h') - \frac{(h')^2}{4t} \right)$$

is the weak solution of

$$\begin{cases} \partial_t f - (\partial_h f)^2 = 0 \\ f(0,\cdot) = \psi \end{cases}$$

$p \mapsto p^2$, $q \mapsto \frac{q^2}{4}$. (Evans)

Ex: show that for $t > 0$ small: $\partial_h f(t,0) = 0$.

For $t < +\infty$ large: $\partial_h^+ f(t,0) > 0 > \partial_h^- f(t,0)$.

III/ Back to CW.

Prop: if (t,h) is a point of differentiability (in h) of f, and if $F_N \to f$, then

$$\partial_h F_N(t,h) \longrightarrow \partial_h f(t,h).$$

Proof: F_N convex in h.

$$F_N(t,h') \geq F_N(t,h) + \underbrace{\partial_h F_N(t,h)}_{\text{bounded.}} (h'-h)$$

Take a subsequence along which $\partial_h F_N(t,h) \to p$.

$$f(t,h') \geq f(t,h) + p\,(h'-h)$$

p must be $\partial_h f(t,h)$. $\square$

Convergence of F_N.

$$\partial_t F_N - (\partial_h F_N)^2 = \frac{1}{N} \partial_h^2 F_N.$$

$w = F_N - f$ $\qquad \partial_t w - b\, \partial_h w = \frac{1}{N} \partial_h^2 F_N$

where $b = \partial_h F_N + \partial_h f$.

$$v = \phi(w)$$

$$\partial_t v - b\, \partial_h v = \phi'(w) \frac{1}{N} \partial_h^2 F_N.$$

$$J(t) = \int_{-R_t}^{R_t} v(t,h)\, dh$$

$$\partial_t J \leq 0 + \int_{-R_t}^{R_t} \underbrace{\phi'(w)}_{\leq 1} \frac{1}{N} \underbrace{\partial_h^2 F_N(t,h)}_{\geq 0}\, dh$$

$$\partial_t J \leq \frac{1}{N} \int_{-R_t}^{R_t} \partial_h^2 F_N(t,h)\, dh = \frac{1}{N} \big[\partial_h F_N\big]_{-R_t}^{R_t} \leq \frac{2}{N}.$$

Recall $J(0) = 0$.

So $J(t) \leq \frac{2t}{N}$. $\square$

Ex: clean up!

Key point: if there are "error terms on the right hand side", we need to estimate it in L^1 in the h variable, uniformly in t (locally).

"local $L^\infty_t L^1_h$ estimate".

For every $\delta > 0$, there exists $C_\delta < +\infty$ such that $\forall t \geq \delta$: $h \mapsto f(t,h) + C_\delta h^2$ is convex.