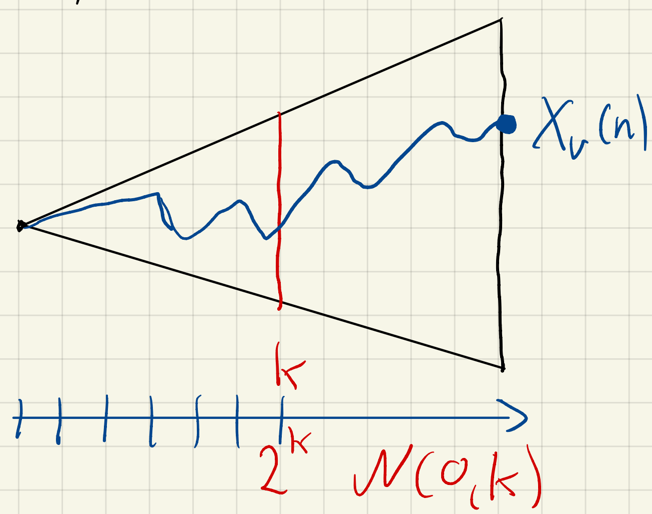
Why is there a 3 ?

Aug 24, 2021

binary BRW Gaussian increments



$X_v(k)$ = sum of weights up to level k

$\{X_v(k),\ k \in \{1,\dots,n\}\}$ path / RW with Gaussian increments.

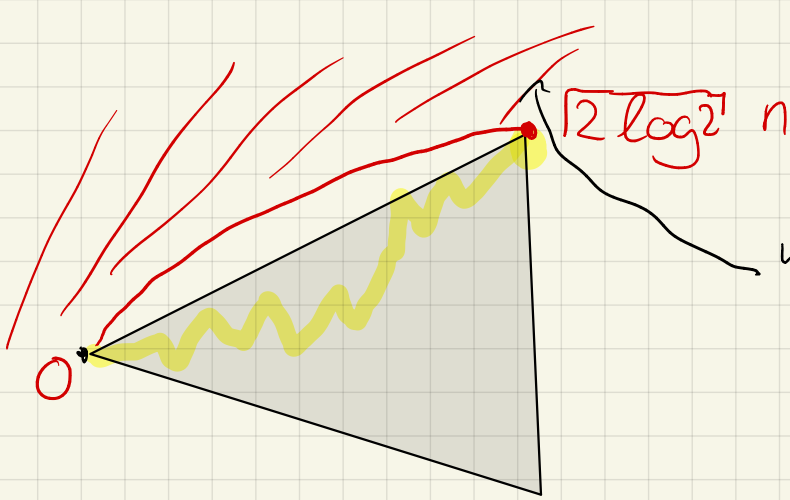
So at level k:

Maximal value should not be larger than

$$\sqrt{2\log 2}\, k \qquad k \in \{1,\dots,n\}$$

$\longrightarrow$ W hp there no "particle" above

$$k \longmapsto \sqrt{2\log 2}\, k + \left(C \log(k \wedge n-k) \vee C \right)$$

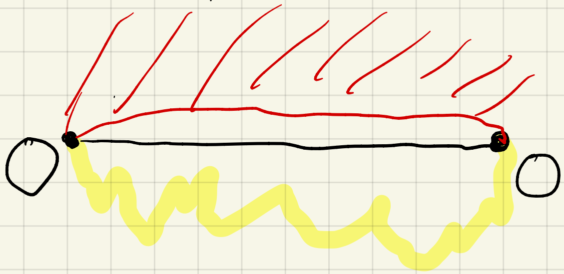


with high probability: There is no particle!

Imagine: Particle at n: reaches $\approx \sqrt{2\log 2}\, n$.

RW path $0 \to$ ● that never enters the forbidden region.

Flip picture



RW leading to the maximal particle: Gaussian RW that is more or less forbidden to positive

$\hat{=}$ Discrete time Brownian bridge $0 \to 0$ in time n

$\mathbb{P}($ Discrete time Brownian bridge $0 \to 0$ in time n stay below 0 (away from endpoints)$) \neq O(1)$

$\approx \frac{1}{n}$ (Ballot estimates)

Let us try to combine this with upper bound.

$$\mathbb{P}\Big(\{\max_{v \in V_n} X_v(n) > \underbrace{\sqrt{2\log 2}\, n - \frac{3}{2\sqrt{2\log 2}} \log n}_{:= m(n)} + y\} \cap \{\text{no particle } \}\Big)$$

$$\approx \mathbb{P}\Big(\sum_{v \in V_n} \mathbb{1}_{\{X_v(n) > m(n)+y,\ X_v(k) < \sqrt{2\log 2}\, k\}}$$

$$\geq 1 \Big)$$

Markov

$$\leq \mathbb{E}\Big(\sum_{v \in V_n} \dots \Big)$$

$$= 2^n \, \mathbb{P}\big(X_v(n) > m(n) + y, \; X_v(k) < \sqrt{2\log 2}\, k \big)$$

"Take out the linear drift"

$$X_v(k) - \underbrace{\frac{k}{n} X_v(n)}_{\approx \frac{k}{n}\sqrt{2\log 2}\, n}$$ 'discrete time Brownian Bridge'

indep. of endpoint $X_v(n)$.

$$\approx 2^n \, \mathbb{P}\big(X_v(n) > m(n) + y \big) \underbrace{\mathbb{P}\Big(X_v(k) - \frac{k}{n} X_v(n) < 0 \Big)}_{\mathbb{P}\left[\text{Brownian bridge} < 0\right)}$$

$$\approx 2^n \underbrace{\frac{\sqrt{n}}{m(n)+y}}_{\approx \frac{1}{\sqrt{n}}} e^{-\frac{(m(n)+y)^2}{2n}} \; \frac{1}{n} = \left(\frac{1}{\sqrt{n}}\right)^2$$

Yesterday: Needed $-\frac{1}{2\sqrt{2\log 2}} \log n$ to cancel $\frac{1}{\sqrt{n}}$.

Now: Need: $-\frac{3}{2\sqrt{2\log 2}}\log n$!

This the correct answer (for order of max).

$\longrightarrow$ This gives upper bound.

<u>Lower bound:</u> Second moment

$$\frac{\left(\mathbb{E}\left[\sum_{v\in V_n} \mathbb{1}_{\{\ldots\}}\right]\right)^2}{\mathbb{E}\left(\left(\sum_{v\in V_n} \mathbb{1}_{\{\ldots\}}\right)^2\right)} \overset{\text{C-S}}{\leq} \mathbb{P}\left(\sum_{v\in V_n} \mathbb{1}_{\{\ldots\}} \geq 1\right)$$

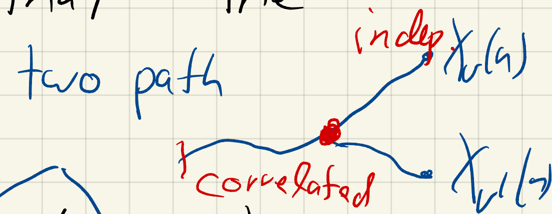
Paley - Zygmund.

Useful: order 1 / close to one

<u>Goal</u> To put extra constraint in $\mathbb{1}_{\{\ \}}$ such that the second moment

$$\mathbb{E}\left(\sum_{v,v'\in V_n} \mathbb{1}_{\{v\ldots\}} \mathbb{1}_{\{v'\ldots\}}\right)$$

is of order $\left(\mathbb{E}\left[\sum_{v\in V_n} \mathbb{1}_{\{\ldots\}}\right]\right)^2$!



We used "forbidden region".

Finding good extra conditions to put in $\mathbb{1}_{\{\ldots\}}$ can be difficult.

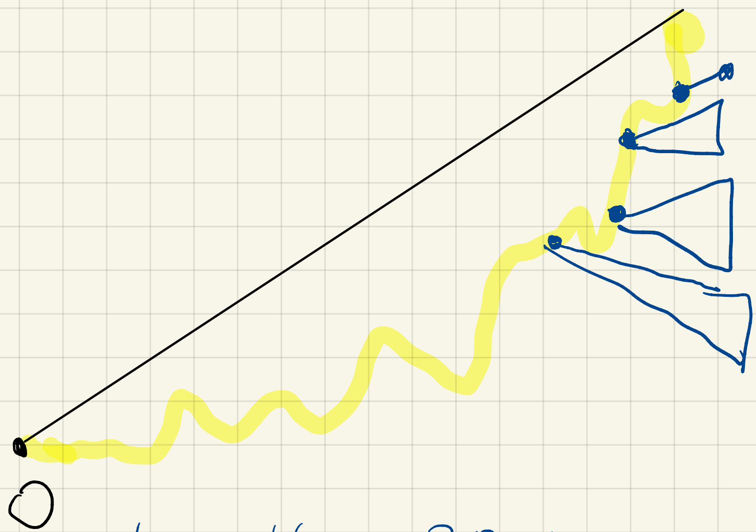
Sometimes : In several steps.

Order max of BRW is $m(n)$.

→ Add $\{\max X_v(n) \leq m(n) + C\}$ ← large

to the indicator function to find more.

→ Describe extremes better...



path to max + collection of smaller BRW's branch of.

Describe the BRV seen from maximal particle!

maximum of the atted BRW + Strating value ≤ overall max.

↳ Obtain quantitative estimates on extremal level sets

(Cortines, H., Louidor for BBM).

Refrence: first + second moment method
N. Kistler:

In other model e.g. Ex 4 and Ex 5:

1. Need to find a good notion 'path' / scales.
2. How to compute the second moment?

One option: Compare your model to a BRW (with right number levels)!

Gaussian comparison

Berry-Essen

→ Thursday.