# Invariant measures for KdV and Toda-type discrete integrable systems

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joint with
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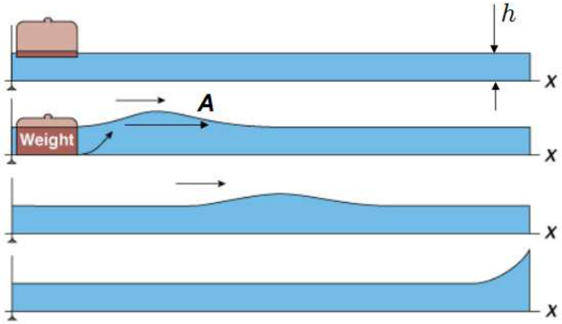
# 1. KDV AND TODA-TYPE DISCRETE INTEGRABLE SYSTEMS

# KDV AND TODA LATTICE EQUATIONS

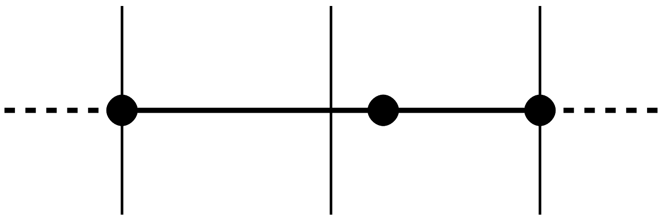



*Source: Shnir*

Korteweg-de Vries (KdV) equation:

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0,$$

where $u = (u(x,t))_{x,t\in\mathbb{R}}$.

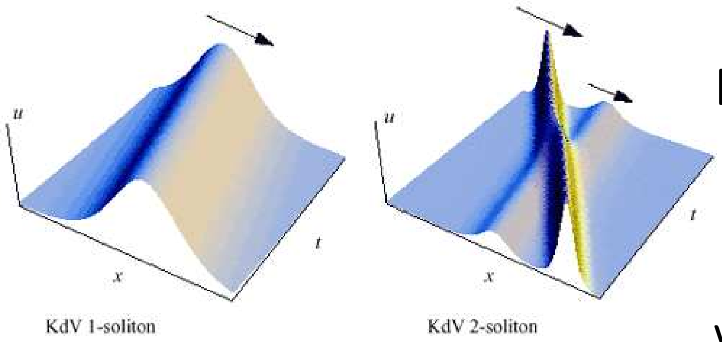


Toda lattice equation:

$$\begin{cases} \frac{d}{dt}p_n & = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1}-q_n)}, \\ \frac{d}{dt}q_n & = p_n, \end{cases}$$

where $p_n = (p_n(t))_{t\in\mathbb{R}}$, $q_n = (q_n(t))_{t\in\mathbb{R}}$.

# KDV AND TODA LATTICE EQUATIONS




*Source: Brunelli*

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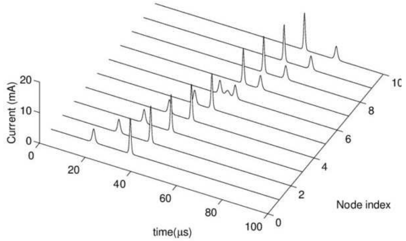



*Source: Singer et al*

Toda lattice equation:

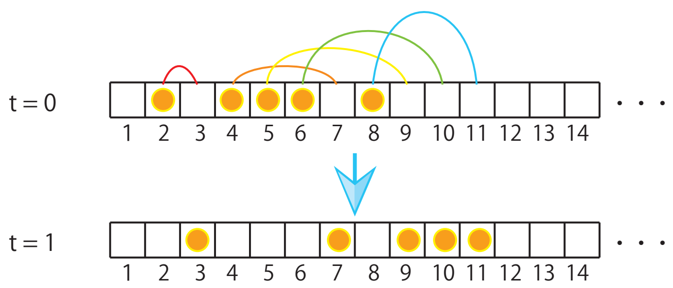
$$\begin{cases} \frac{d}{dt}p_n & = e^{-(q_n - q_{n-1})} - e^{-(q_{n+1} - q_n)}, \\ \frac{d}{dt}q_n & = p_n, \end{cases}$$

where $p_n = (p_n(t))_{t\in\mathbb{R}}$, $q_n = (q_n(t))_{t\in\mathbb{R}}$.

# BOX-BALL SYSTEM (BBS)

Discrete time deterministic dynamical system (cellular automaton) introduced in 1990 by Takahashi and Satsuma. In original work, configurations $(\eta_x)_{x\in\mathbb{Z}}$ with a finite number of balls were considered. (NB. Empty box: $\eta_x = 0$; ball $\eta_x = 1$.)

- Every ball moves exactly once in each evolution time step
- The leftmost ball moves first and the next leftmost ball moves next and so on...
- Each ball moves to its nearest right vacant box




Dynamics $T : \{0,1\}^{\mathbb{Z}} \to \{0,1\}^{\mathbb{Z}}$:

$$(T\eta)_n = \min\left\{1-\eta_n, \sum_{m=-\infty}^{n-1}(\eta_m - (T\eta)_m)\right\},$$

where $(T\eta)_n = 0$ to left of particles.

# BBS CARRIER

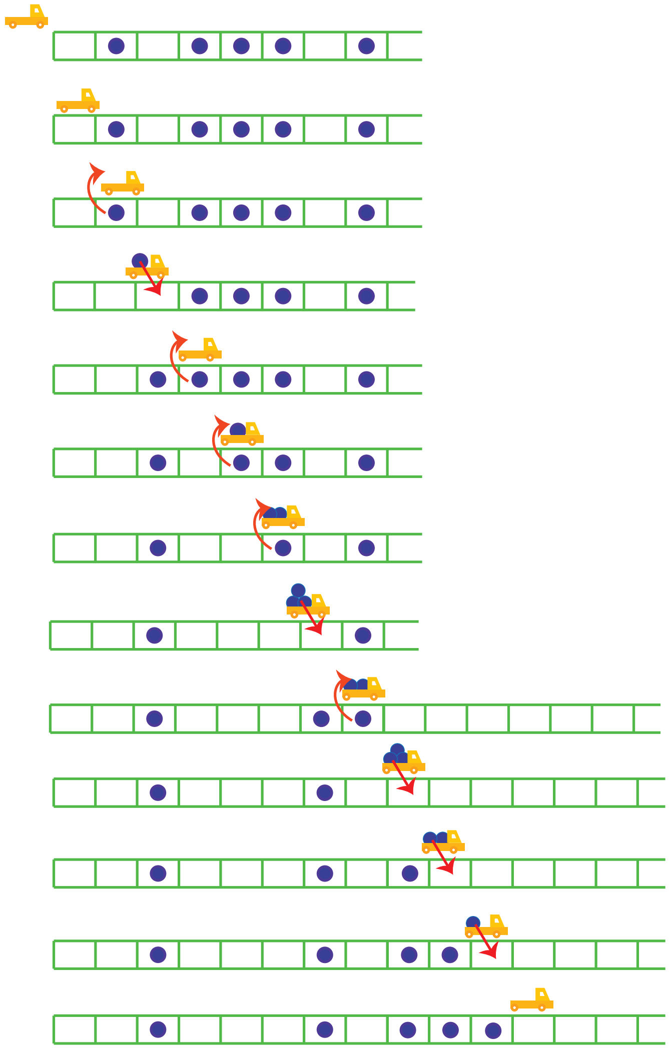
- Carrier moves left to right
- Picks up a ball if it finds one
- Puts down a ball if it comes to an empty box when it carries at least one ball

Set $U_n$ to be number of balls carried from $n$ to $n+1$, then

$$U_n = \begin{cases} U_{n-1}+1, & \text{if } \eta_n = 1, \\ U_{n-1}, & \text{if } \eta_n = 0, U_{n-1} = 0, \\ U_{n-1}-1, & \text{if } \eta_n = 0, U_{n-1} > 0, \end{cases}$$

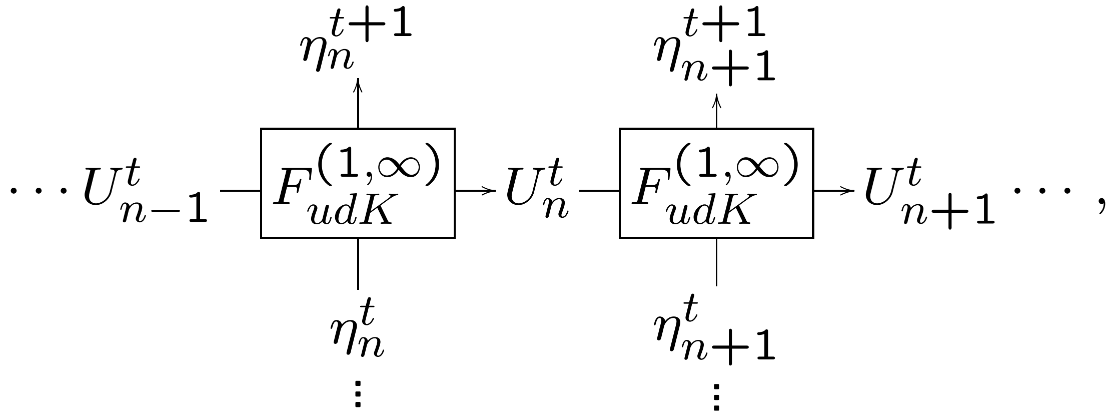
and

$$(T\eta)_n = \min\{1-\eta_n, U_{n-1}\}.$$



# LATTICE EQUATIONS

The local dynamics of the BBS are described via a system of lattice equations:




where $F_{udK}^{(1,\infty)}$ is an involution, as given by:

$$F_{udK}^{(1,\infty)}(\eta, u) := (\min\{1-\eta, u\}, \eta + u - \min\{1-\eta, u\}) \, .$$

This is (a version of) the *ultra-discrete KdV equation (udKdV)*. Can generalise to box capacity $J \in \mathbb{N} \cup \{\infty\}$ and carrier capacity $K \in \mathbb{N} \cup \{\infty\}$.

## BASIC QUESTIONS

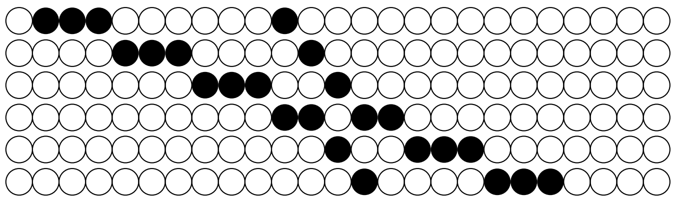
In today's talk, I will address two main topics for the BBS (and related systems):

- Existence and uniqueness of solutions to initial value problem for (udKdV) with infinite configurations?

- I.i.d. invariant measures on initial configurations?

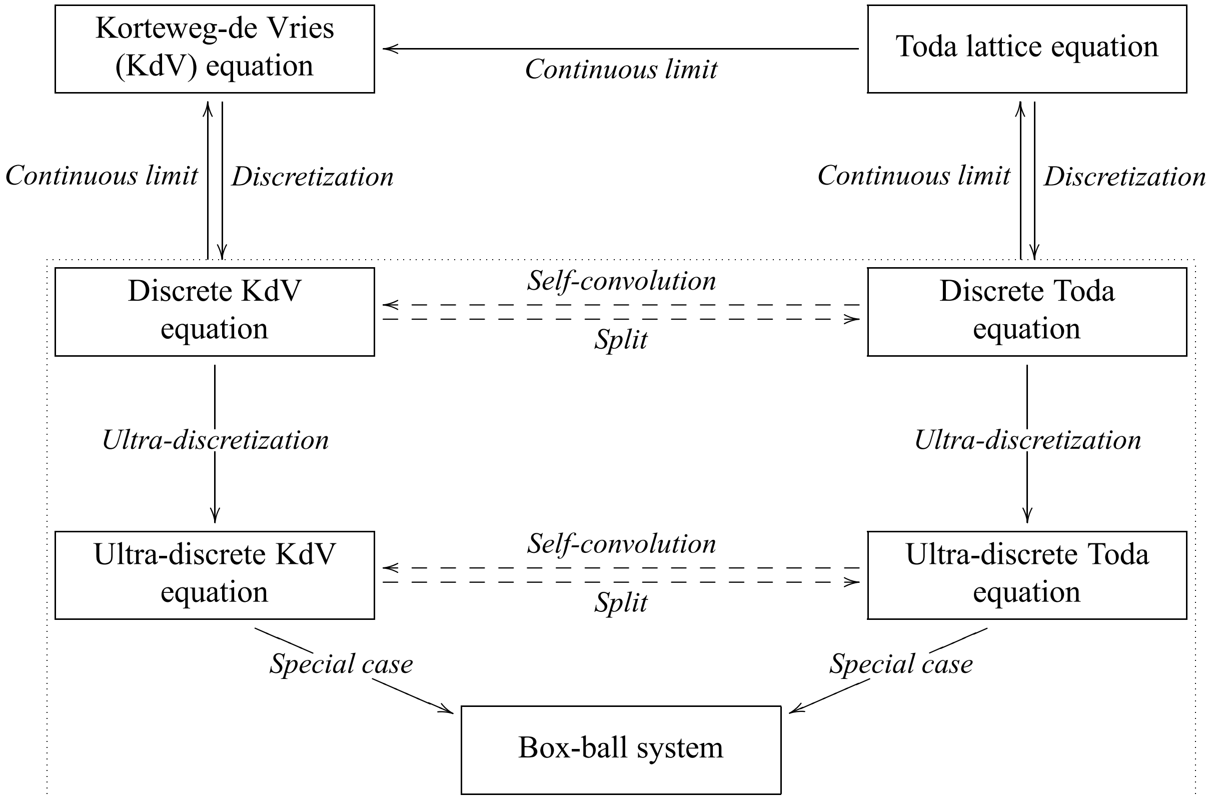
Other recent developments in the study of the BBS that I will not talk about:

- Invariant measures based on solitons, e.g. [Ferrari, Nguyen, Rolla, Wang]. See also [Levine, Lyu, Pike], etc.

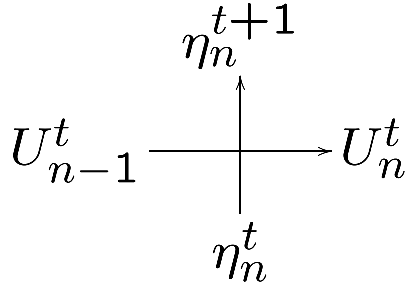
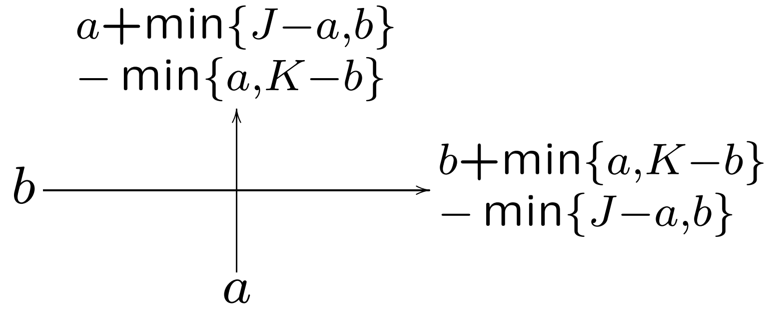


- Generalized hydrodynamic limits, e.g. [C., Sasada], [Kuniba, Misguich, Pasquier].

# INTEGRABLE SYSTEMS DERIVED FROM THE KDV AND TODA EQUATIONS



# ULTRA-DISCRETE KDV EQUATION (UDKDV)

<table>
<tr><th>Model</th><th>Lattice structure</th><th>Local dynamics: $F_{udK}^{(J,K)}$</th></tr>
<tr><td>udKdV</td><td>
</td><td>
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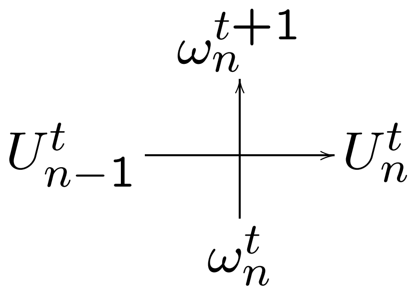
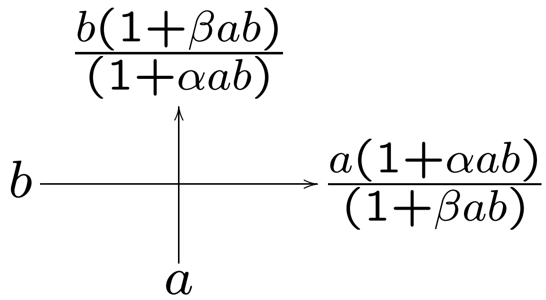
Variables are $\mathbb{R}$-valued. Parameter $J$ represents box capacity, $K$ represents carrier capacity. Multi-coloured version of BBS/ UDKDV also studied [Kondo].

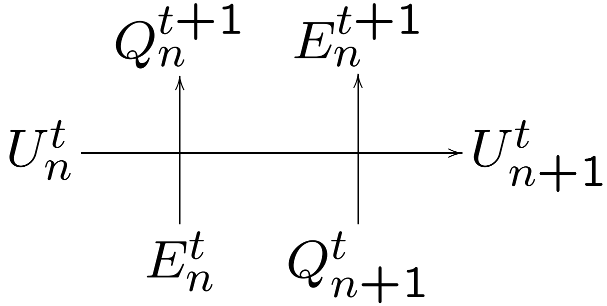
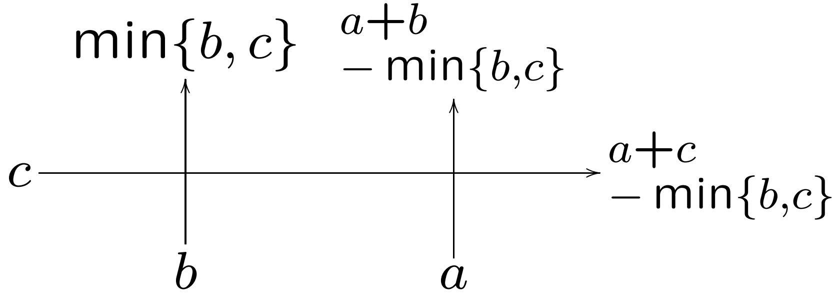
# DISCRETE KDV EQUATION (DKDV)

| *Model* | *Lattice structure* | *Local dynamics:* $F_{dK}^{(\alpha,\beta)}$ |
|---|---|---|
| dKdV |   |   |

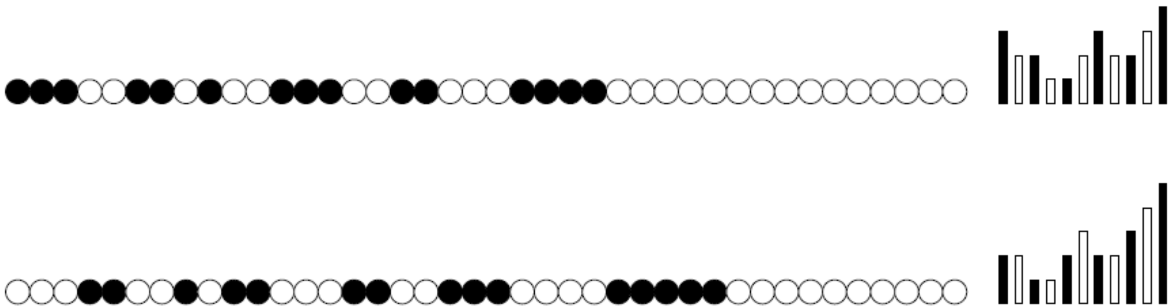
Variables are $(0,\infty)$-valued. UDKDV is obtained as ultra-discrete/ zero-temperature limit by making change of variables:

$$\alpha = e^{-J/\varepsilon}, \qquad \beta = e^{-K/\varepsilon}, \qquad a = e^{a/\varepsilon}, \qquad b = e^{b/\varepsilon}.$$

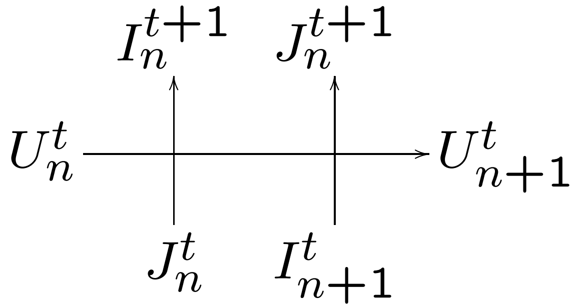
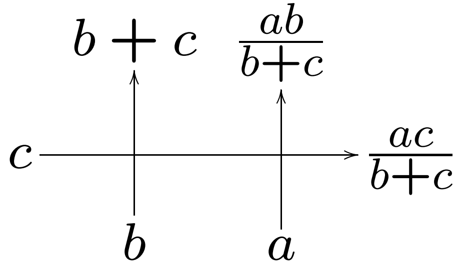
# ULTRA-DISCRETE TODA EQUATION (UDTODA)

| *Model* | *Lattice structure* | *Local dynamics:* $F_{udT}$ |
|---|---|---|
| udToda |   |   |

Variables are $\mathbb{R}$-valued. For BBS(1,$\infty$), can understand $(Q_n^t, E_n^t)_{n\in\mathbb{Z}}$ as the lengths of consequence ball/empty box sequences.



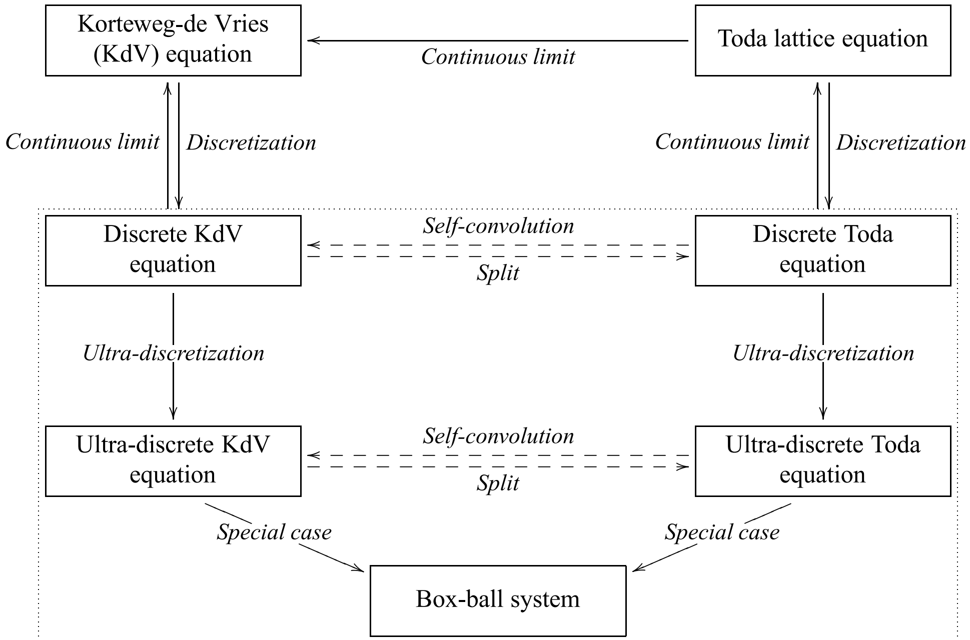
# DISCRETE TODA EQUATION (DTODA)

| *Model* | *Lattice structure* | *Local dynamics:* $F_{dT}$ |
|---|---|---|
| dToda |  |  |

Variables are $(0,\infty)$-valued. UDTODA is obtained as ultra-discrete/ zero-temperature limit by making change of variables:

$$a = e^{-a/\varepsilon}, \qquad b = e^{-b/\varepsilon}, \qquad c = e^{-b/\varepsilon}.$$

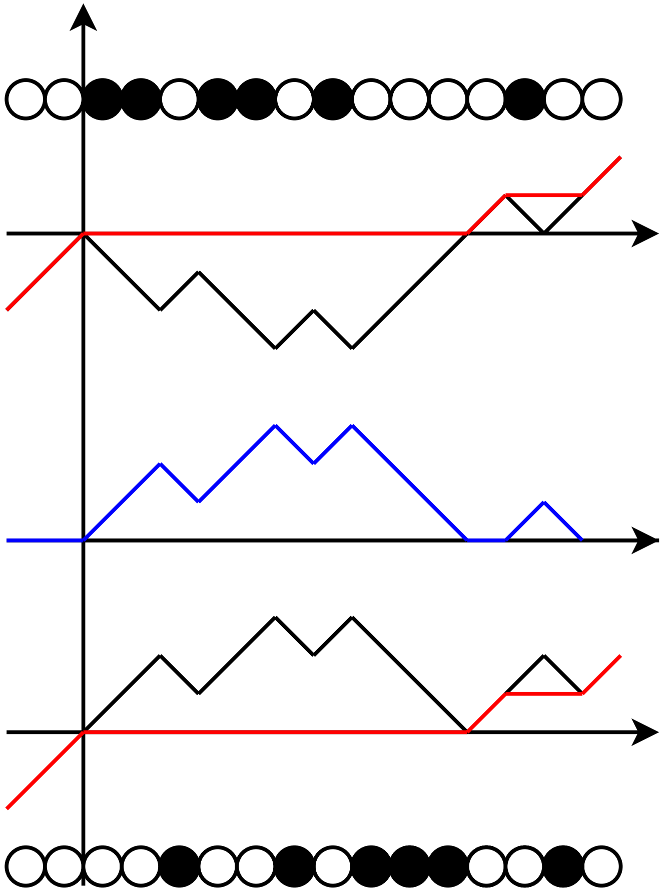
# INTEGRABLE SYSTEMS DERIVED FROM THE KDV AND TODA EQUATIONS




NB. [Quastel, Remenik 2019] connected the KPZ fixed point to the Kadomtsev-Petviashvili (KP) equation. Both dKdV and dToda can be obtained from the discrete KP equation.

# 2. GLOBAL SOLUTIONS
# BASED ON PATH ENCODINGS

# PATH ENCODING FOR BBS AND CARRIER



Let $\eta$ be a finite configuration.

Define $(S_n)_{n\in\mathbb{Z}}$ by $S_0 = 0$ and

$$S_n - S_{n-1} = 1 - 2\eta_n.$$

Let

$$U_n = M_n - S_n,$$

where $M_n = \max_{m\le n} S_m$.

Can check $(U_n)_{n\in\mathbb{Z}}$ is a carrier process, and the path encoding of $T\eta$ is

$$TS_n = 2M_n - S_n - 2M_0.$$

## PITMAN'S TRANSFORMATION

The transformation

$$S \mapsto 2M - S$$

is well-known as Pitman's transformation. (It transforms one-sided Brownian motion to a Bessel process [Pitman 1975].)

Given the relationship between $\eta$ and $S$, and $U = M - S$, the relation $TS = 2M - S - 2M_0$ is equivalent to:

$$(T\eta)_n + U_n = \eta_n + U_{n-1},$$

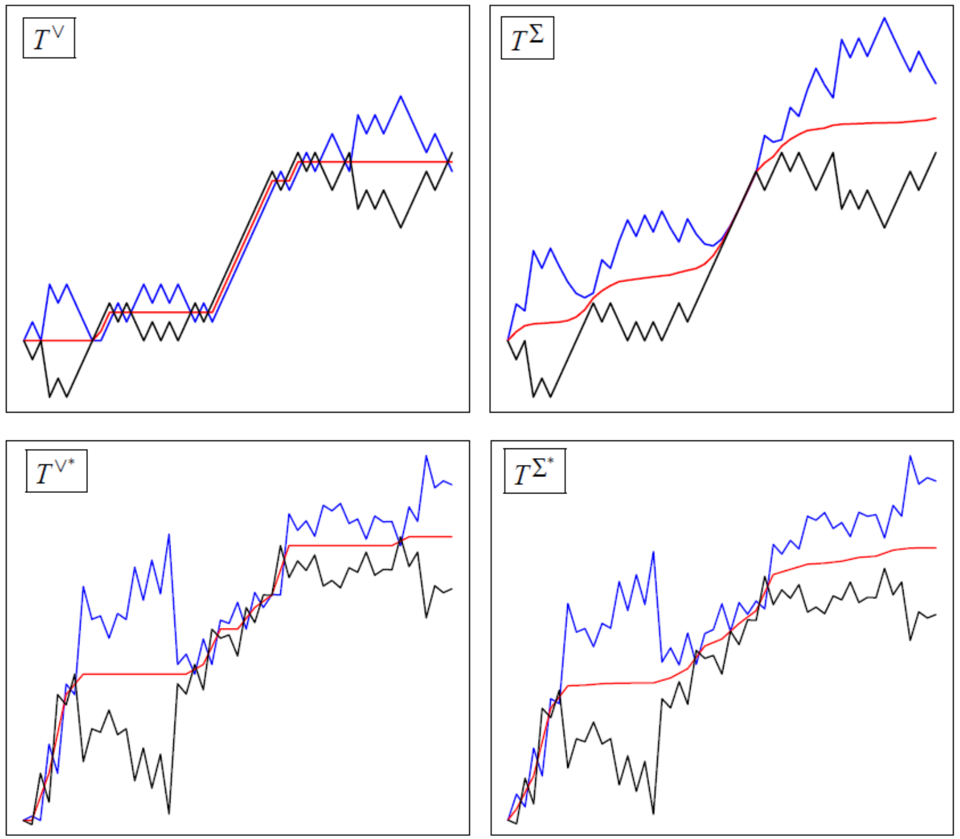
i.e. conservation of mass.

# 'PAST MAXIMUM' OPERATORS

| *Model* | *'Past maximum'* | *Path encoding dynamics* |
|---|---|---|
| udKdV | $M^{\vee}(S)_n := \sup_{m\leq n}\left(\frac{S_m+S_{m-1}}{2}\right)$ | $T^{\vee}(S)=2M^{\vee}(S)-S$ |
| dKdV | $M^{\Sigma}(S)_n := \log\left(\sum_{m\leq n}\exp\left(\frac{S_m+S_{m-1}}{2}\right)\right)$ | $T^{\Sigma}(S)=2M^{\Sigma}(S)-S$ |
| udToda | $M^{\vee^*}(S)_n := \begin{cases} \sup_{m\leq\frac{n-1}{2}} S_{2m}, & n \text{ odd,} \\ \frac{M^{\vee^*}(S)_{n+1}+M^{\vee^*}(S)_{n-1}}{2}, & n \text{ even,} \end{cases}$ | $\mathscr{T}^{\vee^*}(S)=\theta\circ T^{\vee^*}(S)$, where $T^{\vee^*}(S):=2M^{\vee^*}(S)-S$ |
| dToda | $M^{\Sigma^*}(S)_n := \begin{cases} \log\left(\sum_{m\leq\frac{n-1}{2}}\exp\left(S_{2m}\right)\right), & n \text{ odd,} \\ \frac{M^{\Sigma^*}(S)_{n+1}+M^{\Sigma^*}(S)_{n-1}}{2}, & n \text{ even} \end{cases}$ | $\mathscr{T}^{\Sigma^*}(S)=\theta\circ T^{\Sigma^*}(S)$, where $T^{\Sigma^*}(S):=2M^{\Sigma^*}(S)-S$ |

Above corresponds to udKdV($J$,$\infty$) and dKdV($\alpha$,0); parameters appear in path encoding. More novel 'past maximum' operators for udKdV($J$,$K$), $J\leq K$ [C., Sasada]. Spatial shift $\theta$ needed for Toda systems.

# ‘PAST MAXIMUM’ OPERATORS




$T^{\vee}$ =udKdV, $T^{\sum}$=dKdV, $T^{\vee *}$ =udToda, $T^{\sum^*}$=dToda.

# GENERAL APPROACH

Aim to change variables $a_n^t := \mathcal{A}_n(\eta_n^t)$, $b_n^t := \mathcal{B}_n(u_n^t)$ so that $(a_{n-m}^{t+1}, b_n^t) = K_n(a_n^t, b_{n-1}^t)$ satisfies

$$K_n^{(1)}(a,b) - 2K_n^{(2)}(a,b) = a - 2b.$$

Path encoding given by

$$S_n - S_{n-1} = a_n.$$

Existence of carrier $(b_n)_{n\in\mathbb{Z}}$ equivalent to existence of 'past maximum' satisfying

$$M_n = K_n^{(2)}\left(S_n - S_{n-1}, M_{n-1} - S_{n-1}\right) + S_n.$$

Dynamics then given by $S \mapsto T^M S := 2M - S - 2M_0$.

Advantage: $M$ equation can be solved in examples. Moreover, can determine uniquely a choice of $M$ for which the procedure can be iterated. Gives existence and uniqueness of solutions.

## APPLICATION TO BBS($J$,$\infty$)

Given $\eta = (\eta_n)_{n\in\mathbb{Z}} \in \{0, 1, \dots, J\}^{\mathbb{Z}}$, let $S$ be the path given by setting $S_0 = 0$ and $S_n - S_{n-1} = J - 2\eta_n$ for $n \in \mathbb{Z}$. If $S$ satisfies

$$\lim_{n\to\infty} \frac{S_n}{n} > 0, \qquad \lim_{n\to-\infty} \frac{S_n}{n} > 0$$

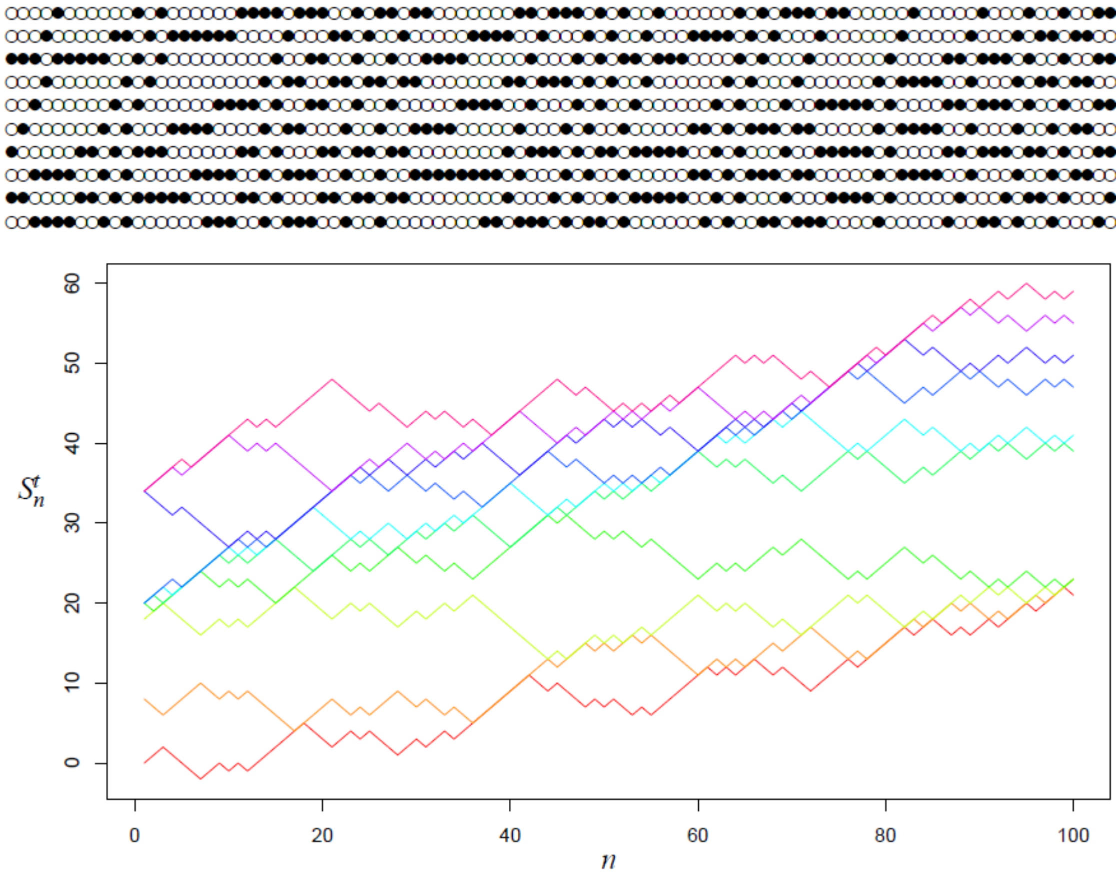
then there is a unique solution $(\eta_n^t, U_n^t)_{n,t\in\mathbb{Z}}$ to udKdV that satisfies the initial condition $\eta^0 = \eta$. This solution is given by

$$\eta_n^t := \frac{J - S_n^t + S_{n-1}^t}{2}, \qquad U_n^t := M^{\vee}(S^t)_n - S_n^t + \frac{J}{2}, \qquad \forall n, t \in \mathbb{Z},$$

where $S^t := (T^{\vee})^t(S)$ for all $t \in \mathbb{Z}$.

[Essentially similar results hold for other systems.]

# APPLICATION TO BBS($J$,$\infty$)




[Simulation with $J = 1$. For configurations, time runs upwards.]

# 3. INVARIANT MEASURES VIA DETAILED BALANCE

## APPROACHES TO INVARIANCE

1. Ferrari, Nguyen, Rolla, Wang: *BBS soliton decomposition*.

2. C., Kato, Tsujimoto, Sasada - *Three conditions theorem for BBS* (later generalized). Any two of the three following conditions imply the third:

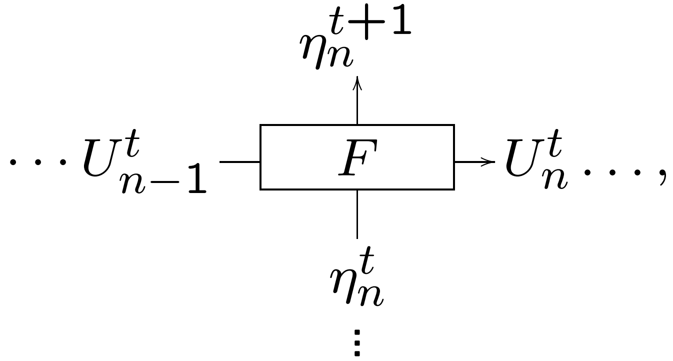
$$\overleftarrow{\eta} \stackrel{d}{=} \eta, \qquad \bar{U} \stackrel{d}{=} U, \qquad T\eta \stackrel{d}{=} \eta,$$

where $\overleftarrow{\eta}$ is the reversed configuration, and $\bar{U}$ is the reversed carrier process given as $\overleftarrow{\eta}_n = \eta_{-(n-1)}$, $\bar{U}_n = U_{-n}$.

3. C., Sasada - *Detailed balance for locally-defined dynamics*.

# DETAILED BALANCE (HOMOGENEOUS CASE)

Consider homogenous lattice system




Suppose $\mu$ is a probability measure such that $\mu^{\mathbb{Z}}(\mathcal{X}^*) = 1$, where $\mathcal{X}^*$ are those configurations for which there exists a unique global solution.

It is then the case that $\mu^{\mathbb{Z}} \circ T^{-1} = \mu^{\mathbb{Z}}$ if and only if there exists a probability measure $\nu$ such that

$$(\mu \times \nu) \circ F^{-1} = \mu \times \nu.$$

Moreover, when this holds, $U_n^t \sim \nu$ (under $\mu^{\mathbb{Z}}$).

# KDV-TYPE EXAMPLES

**udKdV** Up to trivial measures and technical conditions, i.i.d. invariant measures are either:

- shifted, truncated exponential, or;
- scaled, shifted, truncated, bipartite geometric.

Carrier marginal is of same form.

**dKdV($\alpha$,0)** I.i.d. invariant measures are given by:

- $\mu = GIG(\lambda, c\alpha, c)$ with $2\int \log(x)\mu(dx) < -\log\alpha$.

Carrier marginal of form $\nu = IG(\lambda, c)$.

*Duality gives dKdV(0,$\beta$) invariant measures.*

*NB. GIG=generalised inverse Gaussian, IG=inverse gamma.*

**Remark** Can check ergodicity of the relevant transformations.

# CHARACTERISATION THEOREMS

[Kac 1939] If $X$ and $Y$ are independent, then $X+Y$, $X-Y$ are independent if and only if $X$ and $Y$ are normal with a common variance.

[Matsumoto, Yor 1998], [Letac, Wesolowski 2000] If $X > 0$ and $Y > 0$ are independent, then

$$(X+Y)^{-1}, \qquad X^{-1} - (X+Y)^{-1}$$

are independent if and only if $X$ has a generalised inverse Gaussion (GIG) distribution and $Y$ has a gamma distribution.

*NB. Appears in study of exponential version of Pitman's transformation, and random infinite continued fractions.*

## CONJECTURE

**dKdV($\alpha$,$\beta$)** Detailed balance solution:

$$\mu \times \nu = GIG(\lambda, c\alpha, c) \times GIG(\lambda, c\beta, c).$$

**Conjecture** These are only solutions to detailed balance for $F_{dK}^{(\alpha,\beta)}$. In particular, can [Letac, Wesolowski 2000] be generalised to

$$(X,Y) \mapsto \left(\frac{Y(1+\beta XY)}{1+\alpha XY}, \frac{X(1+\alpha XY)}{1+\beta XY}\right)$$

with $\alpha\beta > 0$?

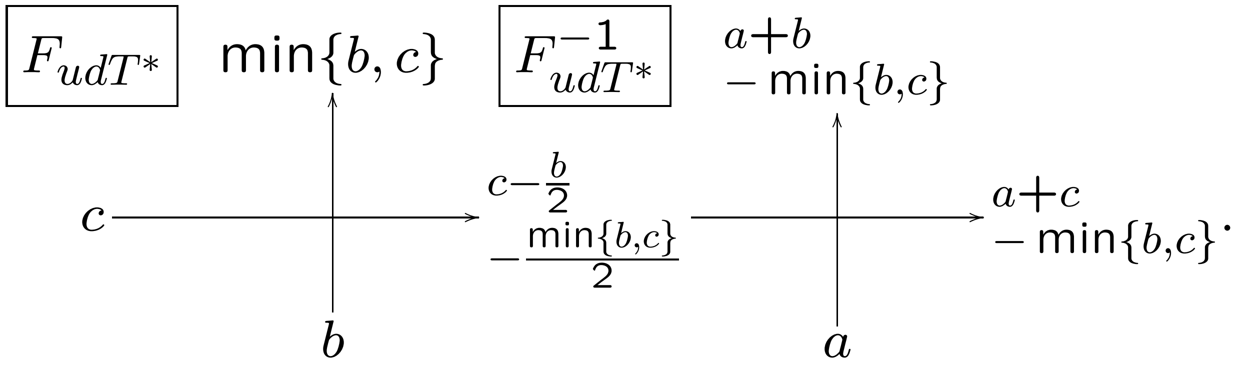
**Remark** Our result for udKdV solves (up to technicalities) the 'zero temperature' version based on the map:

$$(X,Y) \mapsto (Y - \max\{X+Y-J, 0\} + \max\{X+Y-K, 0\},$$

$$X - \max\{X+Y-K, 0\} + \max\{X+Y-J, 0\}).$$

# SPLITTING TODA-TYPE EXAMPLES

Decompose the map $F_{udT}$ into $F_{udT^*}$ and $F^{-1}_{udT^*}$:




[Can do similarly for $F_{dT}$.] Invariance of $(\tilde{\mu}\times\mu)^{\mathbb{Z}}$ for udToda can be related to the existence of $(\tilde{\nu},\nu)$ such that

$$(\mu\times\nu)\circ F^{-1}_{udT^*} = (\tilde{\mu}\times\tilde{\nu}),$$

NB. This is also equivalent to local invariance of $\tilde{\mu}\times\mu\times\nu$ under $F_{udT}$, cf. *Burke's property*, or to

$$(\tilde{\mu}\times\mu\times\nu)\circ(F^{(2,3)}_{udT})^{-1} = (\mu\times\nu).$$

# TODA-TYPE EXAMPLES

**udToda** Up to trivial measures and technical conditions, alternating i.i.d. invariant measures are either:

- shifted exponential, or;
- scaled, shifted geometric.

**dToda** Alternating i.i.d. invariant measures are given by:

- gamma distributions.

NB. Can completely characterise detailed balance solutions in these cases using classical results:

- $(X, Y) \mapsto (\min\{X, Y\}, X - Y)$ [Ferguson, Crawford 1964-1966];
- $(X, Y) \mapsto (X + Y, X/(X + Y))$ [Lukacs 1955].

Ergodicity is an open question.

# LINKS BETWEEN DETAILED BALANCE SOLUTIONS

Discrete KdV $(\alpha,\beta)$:
$\mathrm{GIG}(\lambda, c\alpha, c)\times$
$\mathrm{GIG}(\lambda, c\beta, c)$

*Self-convolution:*
$\beta = 0,$
$(\lambda, c\sqrt{\alpha}) \leftrightarrow (\lambda_2, c)$

Discrete Toda:
$\mathrm{Gam}(\lambda_1+\lambda_2, c)\times$
$\mathrm{Gam}(\lambda_1, c)\times\mathrm{Gam}(\lambda_2, c)$

*Ultra-discretization:*
$\lambda(\varepsilon) = \lambda\varepsilon$
$c(\varepsilon) = e^{c/\varepsilon}$
$\alpha(\varepsilon) = e^{-J/\varepsilon}$
$\beta(\varepsilon) = e^{-K/\varepsilon}$

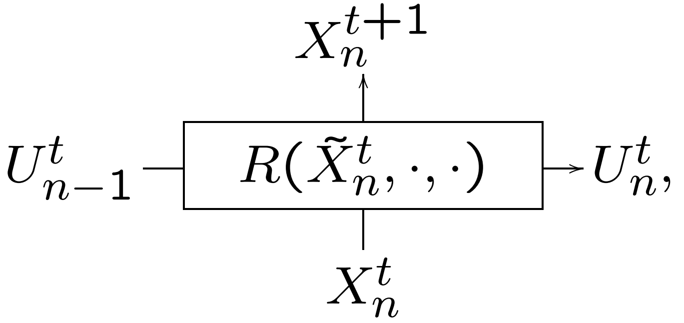
*Ultra-discretization:*
$\lambda_1(\varepsilon) = \lambda_1\varepsilon$
$\lambda_2(\varepsilon) = \lambda_2\varepsilon$
$c(\varepsilon) = e^{c/\varepsilon}$

Ultra-discrete KdV $(J,K)$:
$\mathrm{stExp}(\lambda, c, J-c)\times$
$\mathrm{stExp}(\lambda, c, K-c)$

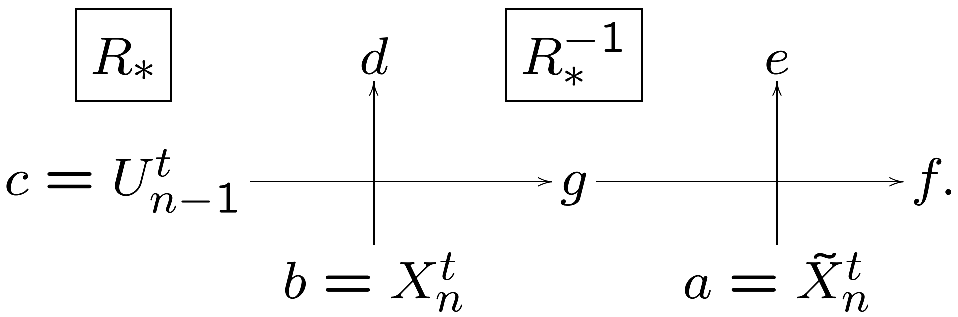
*Self-convolution:*
$K = \infty,$
$(\lambda, c-\frac{J}{2}) \leftrightarrow (\lambda_2, c)$

Ultra-discrete Toda:
$\mathrm{sExp}(\lambda_1+\lambda_2, c)\times$
$\mathrm{sExp}(\lambda_1, c)\times\mathrm{sExp}(\lambda_2, c)$

# RELATED STOCHASTIC INTEGRABLE SYSTEMS
# cf. [CHAUMONT, NOACK 2018]




$$(\tilde{\mu} \times \mu \times \nu) \circ R^{-1} = \mu \times \nu$$




$$(\mu \times \nu) \circ R_*^{-1} = \tilde{\mu} \times \tilde{\nu}$$

- Directed LPP: $R = F_{udT}^{(2,3)}$.
- Directed polymer (site weights): $R = F_{dT}^{(2,3)}$.

Directed polymer (edge weights), higher spin vertex models...