**Problem 1.** (a) Prove that a Markov chain has an eigenvalue \( \lambda \neq 1 \) with \(|\lambda| = 1\) if and only if it is periodic.
(b) Prove that a Markov chain has an eigenvalue \( \lambda = -1 \) if and only if it is bi-partite.

**Problem 2.** Prove that \( P^{2t}(x, x) \) is decreasing in \( t \) for any reversible Markov chain.

**Problem 3.** Consider a reversible Markov chain with a sink state \( \omega \) (so that \( P(\omega, \omega) = 1 \). Let \( A \) be the transition probability matrix on the remaining states, and assume that \( A \) is irreducible (i.e. for every \( x, y \) there is an \( n \) so that \( A^n(x, y) > 0 \)). Note that \( A \) is not stochastic, since there is a probability of jumping to \( \omega \) from some states. Suppose also that the chain is lazy \( (P(x, x) \geq 1/2) \).
(a) Prove that all eigenvalues of \( A \) are in \([0, 1)\) (excluding 1).
(b) Let \( \tau \) be the hitting time of \( \omega \), and let \( \lambda_1 \) be the largest eigenvalue of \( A \) with corresponding eigenfunction \( f_1 \). Prove that
\[
\frac{1}{n} \log \mathbb{P}(\tau \neq n) \xrightarrow{n \to \infty} \log \lambda_1.
\]
(c) If \((X_n)\) is the Markov chain conditioned to have \( \tau > n \), prove that
\[
\mathbb{P}(X_n = y|\tau > n) \xrightarrow{n \to \infty} \frac{f_1(y)}{\sum_z f_1(z)}.
\]

**Problem 4.** Consider two electrical networks on the same graph with conductances \( c, c' \) so that \( \frac{c(x, y)}{c'(x, y)} \in [1, 2] \) for every edge \((x, y)\). Let \( \gamma = 1 - \lambda_2 \) and \( \gamma' = 1 - \lambda'_2 \) be the spectral gaps. Prove that \( \gamma / \gamma' \leq 2 \).

**Problem 5.** Consider a transitive Markov chain (i.e. for any states \( x, y \) there is an automorphism mapping \( x \) to \( y \)). Let \( \hat{P} \) be the dual Markov chain. Prove that \( \|P^t \delta_x - \pi\|_{TV} = \|\hat{P}^t \delta_x - \pi\|_{TV} \) and does not depend on the vertex \( x \).

**Problem 6.** We model the overhand shuffle as follows. Given are a deck of cards of size \( n \), and a parameter \( p \). Hold the deck in your right hand. Repeat the following until going through the deck: take a \( \text{Geom}(p) \) number of cards from the top and move them to the left hand, on top of the previous cards. A step of the shuffle is one such pass over the deck. For example, if \( n = 13 \) and the original order is \((A, 2, 3, 4, 5, 6, 7, 8, 9, T, J, Q, K)\) and the first few geometric variables are 6, 4, 1, 5, the new order is \((Q, K, J, 7, 8, 9, T, A, 2, 3, 4, 5, 6)\). Note that when reaching the 4th group, only two cards are left (the Q and K).
(a) Show that this Markov chain is reversible.
(b) Show that \( t_{mix} \geq C/p \).
(c) Show that \( t_{mix} \geq C(np)^2 \).
(d) Give upper bounds on \( t_{mix} \). (The smaller the bound, the better!)
Problem 7. Consider the lamplighter on a graph $G$. The vertices are $(\psi, x)$ where $x \in G$ and $\psi \in \{0, 1\}^G$ has finitely many 1s, and $(\psi, x) \sim (\psi', x')$ if $x \sim x'$ and $\psi(z) = \psi'(z)$ for all $z \not\in \{x, x'\}$.

If $G$ is the cycle of length $n$, prove that $t_{mix} \asymp n^2$ (give lower and upper bounds).