Hand in assignment 6

Today: Repeated games
Escaping Nash Cooperation! F. III.2

Smallest unique integer (again!)
Prisoner’s Dilemma

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<th>L</th>
<th>D</th>
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<tbody>
<tr>
<td>C</td>
<td>(6, 6)</td>
<td>(0, 8)</td>
</tr>
<tr>
<td>D</td>
<td>(8, 0)</td>
<td>(2, 2)</td>
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Unique NE is (D, D)

If play fixed number of rounds $N$, then unique NE is to always Defect.

Proof: In last round only NE is to D.
With probab. 1.
By induction, on first $N-1$ rounds also D.

Assume after each round, game ends w.p. $1-\beta$.

$P$(at least $k$ rounds) = $\beta^k$
If payoff in round \( i \) is \( X_i \), total expected payoff is \( \sum_{n=0}^{\infty} \beta^n X_n \)

encodes either interest or decay of significance of far future outcomes.

**Tit-for-tat strategy**: Cooperate at first, then mirror opponent’s previous move.

If both use TFT strategy, payoff is \((6,6)\) in every round.

\[
\sum_{n=0}^{\infty} \beta^n 6 = \frac{6}{1-\beta}
\]

If P1 uses TFT, P2 always D:

\((0,8)\) \((2,2)\) \((2,2)\) \(\ldots\)
P1 total is $0 + 2 \beta + 2 \beta^2 + \ldots = \frac{2 \beta}{1 - \beta} < \frac{6}{1 - \beta}$

P2 total is $8 + 2 \beta + 2 \beta^2 + \ldots = 8 + \frac{2 \beta}{1 - \beta} = \frac{8 - 6 \beta}{1 - \beta}$

\[\frac{8 - 6 \beta}{1 - \beta} < \frac{6}{1 - \beta}\] if $\beta \geq \frac{1}{3}$.

Claim: If $\beta \geq \frac{1}{3}$, then both players using TFT is a NE.

Proof: Show that for $\beta \geq \frac{1}{3}$, every seq. of actions gives payoff $\leq \frac{6}{1 - \beta}$ against TFT.

Seq: C C C C C D D D C C C C D D D D D C C C C D

TFT: C C C C C C D D D D D D D D D D

Payoff: 6 6 6 6 6 6 8 2 2 2 0 6 6 6 8 2

Compare to always C: 6 6 6 6 6

Replace a segment of 6's by 8 2 2 2 2 2 2 0.
If change is in rounds $n, \ldots, n+k$, compare

$$6(\beta^n - \beta^{n+k}) \text{ to } 8\beta^n + 2(\beta^{n+1} - \beta^{n+k-1}) + O$$

$$6\beta^n \left(\frac{1 - \beta^{k+1}}{1 - \beta}\right) \quad 8\beta^n + 2\beta^n \left(\frac{1 - \beta^k}{1 - \beta}\right)$$

If $\beta \geq \frac{1}{3}$, first is larger.

Another NE: C at first, switch to D forever after opponent first D.

NE: You: C C C C C C C C C C C C C
Me: D D D D D D D D D D D

If you ever choose D, I switch to D.
Folk Theorem: possible values in a NE for a repeated game are \((u, v)\) where

- \(u \geq\) safety value for P.1.
- \(v \geq\) " " P.2.
- \((u, v)\) is theoretically possible.

Note: Value here is \(\lim_\limits{T \to \infty} \frac{1}{T} \sum_\limits{n=0}^{T} X_n\)

Convex hull of
**Threat:** Do x, otherwise I do y

Proposal: use strict $(x, y)$

P.2 threatens to use y if P.1 does not use x.

P.1 can threaten to use x unless P.2 uses y.

**Credible threat:** hurts opponent more than yourself.

<table>
<thead>
<tr>
<th>$(1, 3)$</th>
<th>$(0, 0)$</th>
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<tbody>
<tr>
<td>$(6, 0)$</td>
<td>$(3, 1)$</td>
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